Inductive Formulae for General Sum Operations

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Abstract. In this note we report some computer generated formulae for the sum of powers of numbers with nonunitary increments; these reduce to the well-known cases when the increment is one.

1. Introduction. Inductive formulae for sums of powers of consecutive integers are well known; the left side of Table 1, based on [1], depicts such formulae for powers up to 10. These are usually derived directly from the fundamental theorem of sum calculus, or via the Bernoulli polynomials as

$$\sum_{i=0}^{m-1} (1 + i)^n = \frac{1}{n+1} [B_{n+1}(m + 1) - B_{n+1}].$$

Formulae for the case where the increment is not one, cannot apparently be found explicitly in the literature; [4]–[10]. Conceptually these formulae are simple to obtain; however, the algebraic manipulations required tend to be overwhelming. In this note we present the first ten formulae, as obtained on a computer by formal string manipulations.

2. Approach. Let

$$S^n(1, d, m) = \sum_{i=0}^{q} (1 + id)^n$$

where $q = (m - 1)/d$ is an integer, $d > 0$. We desire a formal closed-form expression for $S^n(1, d, m)$. Clearly

$$S^n(1, d, m) = 1 + \sum_{k=0}^{n} \binom{n}{k} d^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (2d)^{n-k} + \cdots + \sum_{k=0}^{n} \binom{n}{k} (qd)^{n-k}$$

$$= \frac{m - 1 + d}{d} + \sum_{k=0}^{n-1} \binom{n}{k} [d^{n-k} + 2^{n-k}d^{n-k} + \cdots + q^{n-k}d^{n-k}]$$

$$= \frac{m - 1 + d}{d} + \sum_{k=0}^{n-1} \binom{n}{k} d^{n-k} \sum_{j=1}^{q} j^{n-k}.$$

The individual terms in the second expression are indeed the entries of the left side of Table 1. The only remaining task is obtaining a formal expression for the first summation, by collecting appropriate terms; this is a rather long and tedious task, particularly for high values of n.

TABLE 1

Inductive formulae for general sum operations

$$s^1(1,1,m) = \frac{m}{2}(m+1)$$

$$s^2(1,1,m) = \frac{m}{6}(m+1)(2m+1)$$

$$s^3(1,1,m) = \frac{m^2}{4}(m+1)^2$$

$$s^4(1,1,m) = \frac{m}{30}(m+1)(2m+1)(3m^2+3m-1)$$

$$s^5(1,1,m) = \frac{m^2}{12}(m+1)^2(2m^2+2m-1)$$

$$s^6(1,1,m) = \frac{m}{42}(m+1)(2m+1)(3m^4+6m^3-3m+1)$$

$$s^7(1,1,m) = \frac{m^2}{24}(m+1)^2(3m^4+6m^3-m^2-4m+2)$$

$$s^8(1,1,m) = \frac{m}{90}(m+1)(2m+1)(5m^6+15m^5+5m^4-15m^3-m^2-9m-3)$$

$$s^9(1,1,m) = \frac{m^2}{20}(m+1)^2(2m^6+6m^5+m^4-8m^3+m^2+6m-3)$$

$$s^{10}(1,1,m) = \frac{m}{66}(m+1)(2m+1)(3m^8+12m^7+8m^6-18m^5-10m^4+24m^3+2m^2-15m+5)$$

$$s^1(1,d,m) = \frac{1}{2d}(m-1+d)(m+1)$$

$$s^2(1,d,m) = \frac{1}{6d}[m(m+d)(2m+d) - (d-1)(d-2)]$$

$$s^3(1,d,m) = \frac{1}{4d}[(m-1+d)(m+1)(m^2+dm+1-d)]$$

$$s^4(1,d,m) = \frac{m^5-1}{5d} + \frac{m^4+1}{2} + \frac{d}{3}(m^3-1) - \frac{d^3}{30}(m-1)$$

$$s^5(1,d,m) = \frac{m^6-1}{6d} + \frac{m^5+1}{2} + \frac{5d}{12}(m^4-1) - \frac{d^3}{12}(m^2-1)$$

$$s^6(1,d,m) = \frac{m^7-1}{7d} + \frac{m^6+1}{2} + \frac{d}{2}(m^5-1) - \frac{d^3}{6}(m^3-1) + \frac{d^5}{42}(m-1)$$

$$s^7(1,d,m) = \frac{m^8-1}{8d} + \frac{m^7+1}{2} + \frac{7d}{12}(m^6-1) - \frac{7d^3}{24}(m^4-1) + \frac{d^5}{12}(m^2-1)$$

$$s^8(1,d,m) = \frac{m^9-1}{9d} + \frac{m^8+1}{2} + \frac{2d}{3}(m^7-1) - \frac{7d^3}{15}(m^5-1) + \frac{2d^5}{9}(m^3-1) - \frac{d^7}{30}(m-1)$$

$$s^9(1,d,m) = \frac{m^{10}-1}{10d} + \frac{m^9+1}{2} + \frac{3d}{4}(m^8-1) - \frac{7d^3}{10}(m^6-1) + \frac{d^5}{2}(m^4-1) - \frac{3d^7}{20}(m^2-1)$$

$$s^{10}(1,d,m) = \frac{m^{11}-1}{11d} + \frac{m^{10}+1}{2} + \frac{5d}{6}(m^9-1) - d^3(m^7-1) + d^5(m^5-1) - \frac{d^7}{2}(m^3-1) + \frac{5d^9}{66}(m-1)$$

The algebraic manipulations have been carried out by a computer program. CPU time on a dedicated DEC PDP 11/70 was 2 hours; the code consisted of about 400 statements. The results are depicted on the right-hand side of Table 1. We now have closed-form expressions for summations such as $\Sigma_j(1 + j\sqrt{A}\,)^2$ or $\Sigma_j(1 + j\pi)^3$.

3. Related Facts.

Fact 1. Besides brute force computation, the results of Table 1 may be proved by induction on q, for a fixed d, and n.

The following facts can also be proved.

Fact 2. For all n, d,

$$S^n(1, d, m) \approx \frac{m^{n+1} - 1}{(n + 1)d} + \frac{m^n + 1}{2} + \frac{nd}{12}(m^{n-1} - 1),$$

which is exact for $n = 1$.

Fact 3. For sufficiently small d, $f(x)$ Riemann integrable, $m \equiv 1 \bmod d$, and

$$\Lambda^f(1, d, m) = \sum_{i=0}^{(m-1)/d} f(1 + id),$$

there exists a δ such that

$$\left| \Lambda^f(1, d, m) - \frac{1}{d} \int_1^m f(x)\, dx \right| < \delta.$$

In particular, if $f(x) = x^n$

$$\left| S^n(1, d, m) - \frac{m^{n+1} - 1}{(n + 1)d} \right| < \delta.$$

This is related to the Euler-Maclaurin sum formula [2], and a result on the generalized factorial, [3].

Fact 4. The sum of the odd integers up to m is equal to the sum of the cubes of all integers up to m, divided by m^2; namely, $S^3(1, 1, m)/m^2 = S^1(1, 2, m)$.

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