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On Stieltjes' Continued Fraction for the Gamma Function

By Bruce W. Char*

Abstract. The first forty-one coefficients of a continued fraction for $\ln \Gamma(z) + z - (z - \frac{1}{2}) \ln z - \ln\sqrt{2\pi}$, are given. The computation, based on Wall's algorithm for converting a function's power series representation to a continued fraction representation, was run on the algebraic manipulation system MACSYMA.**

1. Introduction. Recall Stirling's formula for the gamma function:

$$\ln \Gamma(z) = -z + (z - \tfrac{1}{2}) \ln z + \ln\sqrt{2\pi} + J(z)$$

where, for real$(z) > 0$,

$$J(z) = \frac{1}{\pi} \int_0^\infty \ln \frac{1}{1 - e^{-2\pi u}} \cdot \frac{z}{z^2 + u^2} du.$$

Furthermore, asymptotically

$$J(z) = \sum_{p=0}^{\infty} (-1)^p \frac{c_p}{z^{2p+1}}, \tag{1}$$

where

$$c_p = \frac{B_{2p+2}}{(2p + 1)(2p + 2)}, \qquad p = 0, 1, 2, \ldots,$$

and $B_2 = 1/6, B_4 = 1/30, B_6 = 1/42, \ldots$, are the Bernoulli numbers. Henrici [2] refers to $J(z)$ as the *Binet function*, and gives the details for the derivation of the above formulae.

Wall [6, pp. 192–202] gives an algorithm for constructing a continued fraction development of power series such as (1), which we summarize below:

Using the symbolic operation on polynomials of *formal integration* with respect to a variable u, and an infinite sequence of numbers $c_0, c_1, c_2, \ldots$, in which the ith

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power of u is replaced by c_i:

$$\int (k_0 + k_1 u + \cdots + k_n u^n)\, d\phi_c(u) \equiv k_0 c_0 + k_1 c_1 + \cdots + k_n c_n,$$

one computes a_i, $i = 0, 1, 2, \ldots$, of

$$\sum_{p=0}^{\infty} \frac{c_p}{z^{p+1}} = \cfrac{a_0}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdots}}}$$

by defining the auxiliary polynomials

(2) $$q_{-1}(u) = 0, \qquad q_0(u) = c_0,$$

initializing

(3) $$a_0 = c_0$$

and using the recurrence for $p = 1, 2, 3, \ldots$:

(4) $$q_p(u) = u q_{p-1}(u) - a_{p-1} q_{p-2}(u),$$

(5) $$a_p = \frac{\int u^p q_p(u)\, d\phi_e(u)}{\prod_{i=0}^{p-1} a_i},$$

where

(6) $$e_p = \begin{cases} \text{if } p \text{ is even} & c_{p/2}, \\ \text{if } p \text{ is odd} & 0, \end{cases} \qquad p = 1, 2, \ldots .$$

Stieltjes [5, pp. 520–521] gives the first five a_i for $J(z)$, noting that "Le calcul des $[a_i]$ est trés pénible . . . la loi de ces nombres paraît etrêmement compliqué." However, advances of the last decade in the power of algebraic manipulation languages and systems have made it easy to use the recurrence (2)–(6) as the basis for a computer program. The MACSYMA system [3], [4] was used to compute the first forty-one a_i coefficients.

2. The Coefficients. The first seven coefficients computed via MACSYMA agree with those given by Stieltjes, and by Wall [6, p. 365]:

$$a_0 = \frac{1}{12}, \quad a_1 = \frac{1}{30}, \quad a_2 = \frac{53}{210}, \quad a_3 = \frac{195}{371}, \quad a_4 = \frac{22999}{22737},$$

$$a_5 = \frac{29944523}{19733142}, \quad a_6 = \frac{109535241009}{48264275462}.$$

TABLE 1

i	a_i
0	8.3333333333333333333333333333333333333333E-2
1	3.3333333333333333333333333333333333333333E-2
2	2.5238095238095238095238095238095238095240E-1
3	5.2560646900269541778975741239892183288410E-1
4	1.0115230681268417117473721247306152966530E0
5	1.5174736491532873984284915194954761892460E0
6	2.2694889742049599609091506722098775841770E0
7	3.0099173832593981700731407342077157273740E0
8	4.0268871923439012261688759531814428363270E0
9	5.0027680807540300516885024122761885748120E0
10	6.2839113708157821800726631549516472642780E0
11	7.4959191223840339297523547082675054657460E0
12	9.0406602343677266995311393602604817493360E0
13	1.0489303654509482277188371304592629522100E1
14	1.2297193610386205863989437140091917659740E1
15	1.3982876953992430188259760651278730085910E1
16	1.6053551416704935469715616365006260178350E1
17	1.7976607399870277592569472307671554399320E1
18	2.0309762027441653743805414720494896893700E1
19	2.2470471639933132495517941571507922109000E1
20	2.5065846548945972029163400322506305368240E1
21	2.7464451825029133609175558982646222673230E1
22	3.0321821231673047126882599306405786994490E1
23	3.2958533929972987219994066451412088206960E1
24	3.6077698931299242645153320900855452336780E1
25	3.8952706682311555734544390410481046299160E1
26	4.2333490043576957211381853948856097339910E1
27	4.5446960850061621014424175737541451082850E1
28	4.9089203129012597708164883350275087292450E1
29	5.2441288751415337312569856046996108427150E1
30	5.6344845345341843538420365947476113542130E1
31	5.9935683907165858207852583492752112110130E1
32	6.4100422755920354527906611892237917752910E1
33	6.7930140788018221186367702745198535816520E1
34	7.2355940555211701969680052963236217910750E1
35	7.6424654626829689752585090422287526403570E1
36	8.1111403237247965484814230985683460974500E1
37	8.5419221276410972614585638717348682726990E1
38	9.0366814723864108595513574581683377780790E1
39	9.4913837100009887953076231291986927458770E1
40	1.0012217846392919748899074683446683497990E2

The next few numbers in the sequence are:

$$a_7 = \frac{29404527905795295658}{9769214287853155785}$$

$$a_8 = \frac{45537703042011343221011691 4702}{11308412892367501453788572 5485}$$

$$a_9 = \frac{263708125693977190019319929456455 78779849}{52712442679179808019665536491476 04697542}$$

$$a_{10} = \frac{15253749670905480988163889747298599086675385 3122697839}{242742915531051284382973981089021953653738 79212227726}$$

$$a_{11} = \frac{100043420063777451042472529806266909090824649341814868347109676190691}{13346384670164266280033479022693768890138348905413621178450736182873}$$

Table 1 is a list of the a_i, $i = 0, \ldots, 40$, rounded to 40 significant digits, computed from the exact rational coefficients using MACSYMA's "bigfloat" facilities (see [1]).

TABLE 2

i	b_i	c_i
0	.40000000000000000000000000000E0	
1	9.3358490566037735849056603773 6E0	1.2114285714285714285714285714 3E0
2	3.0347960607361549322110363730 7E1	7.6559481814020795864868488578 0E1
3	6.3352876289572297571787496877 0E1	4.9592011901759301980127318309 9E2
4	1.0835586327717517533428854038 6E2	1.7453660775351177576190593167 4E3
5	1.6535796591839779331790021435 9E2	4.5269209768614475199677233981 6E3
6	2.3435956666652650772063412797 8E2	9.7586003418874508770625517369 7E3
7	3.1536084677254363262699037349 6E2	1.8574415694379394456254053613 5E4
8	4.0836190579890255674742106407 2E2	3.2324376115350139742720200505 4E4
9	5.1336280400849743487188027550 4E2	5.2574489042704128652980855754 3E4
10	6.3036358048770126766006751166 2E2	8.1106760737296899390461133706 6E4
11	7.5936426193975241216251998909 4E2	1.1991919657887791295213262178 0E5
12	9.0036486736332958055637253573 6E2	1.7122580126533098270133015487 4E5
13	1.0533654107236629387096723562 4E3	2.3745657895286541896669418513 8E5
14	1.2183659025651352202488168727 7E3	3.2125753335815394533191488340 8E5
15	1.3953663510300924209552753932 8E3	4.2549066783488333468791647603 7E5
16	1.5843667625272629085712917756 5E3	5.5323398543249430188537426089 1E5
17	1.7853671421844967006671817206 3E3	7.0778148894216118267533069678 3E5
18	1.9983674941639072571927984364 4E3	8.9264318093340356174138467454 4E5
19	2.2233678218864879585830776704 8E3	1.1115450637836741239030654449 6E6
20		1.3684291397025831474921126972 8E6

Table 2 above is a similar table of coefficients for an alternative representation of $J(z)$

$$J(z) = \cfrac{z}{12z^2 + b_0 - \cfrac{c_1}{12z^2 + b_1 - \cfrac{c_2}{\;}}}$$

where

$$b_0 = .4, \quad b_i = 12(a_{2i+1} + a_{2i}), \quad c_i = 144(a_{2i}a_{2i-1}), \qquad i = 1, 2, 3, \ldots .$$

3. **Details of the Computation.** The computation was carried out on the MACSYMA Consortium Decsystem 10 (KL model) computer, located at the Massachusetts Institute of Technology's Laboratory for Computer Science in Cambridge, Massachusetts. The programming was done in the MACSYMA language [4], whose mathematical features allowed a few lines of code to completely specify the computation. The computation of the coefficients a_0 through a_{40}, as well as their 40-digit approximations, took approximately twenty minutes of central processing unit time (including the time spent on list "garbage collection"). The limiting constraint to continuing the computation is that the space requirements of the simple data representations used in the program exhaust available memory (approximately 1 million bytes) after the computation of a_{40}.

A listing of the MACSYMA program and output is available upon request from the author.

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Computer Science Division
Electrical Engineering and Computer Science Department
University of California, Berkeley
Berkeley, California 94720

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