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The Groups of Order p^6 (p an Odd Prime)

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Abstract. A complete list of the groups of order p^6, where p denotes an odd prime number, is given using P. Hall's concept of isoclinism.

1. **Introduction.** In 1940, P. Hall [7] provided a method for classifying groups (especially p-groups, where p is a prime) into families and in 1964, M. Hall and J. Senior [6] used this method to produce a complete list of all the groups of order 2^n ($n \leqslant 6$). This paper extends their work to odd primes by providing a complete list of all groups of order p^6 ($p > 2$). The number of non-abelian groups of order 3^6 is found to be 491, and the number of non-abelian groups of order p^6 ($p > 3$) is found to be

$$\tfrac{1}{4}\{13p^2 + 145p + 1338 + 80(p-1, 3) + 45(p-1, 4) + 8(p-1, 5) + 8(p-1, 6)\},$$

where $(p - 1, n)$ denotes the greatest common divisor of $p - 1$ and the integer n.

Blackburn [3] includes a list of the groups of order p^6 and class 5, and Leong [11] and Miech [12] have lists of certain subclasses of the class of p-groups with cyclic derived groups, all of which agree with the present list where they overlap it. A list of the groups of order p^5 ($p > 3$) appears in [1], [2], [4] and [13], the first three of which also include the groups of order 3^5. The present list for $p > 3$ agrees with those of Bender and Schreier, who in turn claim to agree with Bagnera and de Séguier. For $p = 3$, Bender has corrected two errors by Bagnera but has omitted the group $\Delta_{10}(2111)a_2$, which is included by Blackburn and de Séguier. The present list of families (as defined by P. Hall) agrees with that of Easterfield [5], whose ordering I have followed.

This work is a summary of my Ph.D. Thesis [9] which may be consulted for most of the detailed calculations, although errors in that thesis have been corrected and an uncompleted family in the thesis has been completed. As well as the acknowledgements in my thesis, I would also like to thank Richard Keane, who pointed out errors in the original list of 3-groups, and Dr. M. F. Newman, who has hounded me into publishing the present paper and given me a great deal of helpful advice and encouragement.

2. **A Guide to the Table and List of Groups.** Information concerning the groups of order p^m ($m \leqslant 6$), with p an odd prime, is given in Table 4.1, and the list of groups in Sections 4.2 to 4.6. Each group of order p^m in the list is presented in terms of

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certain generators and relations, and given a designation of the form $\Phi_s(m_1, m_2, \ldots, m_r)x_t$, where r, s, t, $m_1, m_2, \ldots, m_r$, are positive integers with $m_1 + m_2 + \cdots + m_r = m$ $(m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r)$, and x is a letter.

The presentation $\langle \alpha_1, \alpha_2, \ldots, \alpha_n | w_1 = w_2 = \cdots = w_k = 1 \rangle$ for the group G means that G is the largest group generated by the symbols $\alpha_1, \alpha_2, \ldots, \alpha_n$ subject to the conditions $w_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = w_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = \cdots = w_k(\alpha_1, \alpha_2, \ldots, \alpha_n) = 1$, where the w's represent words in at most n variables. In particular, $[\alpha, \beta]$ is the word $\alpha^{-1}\beta^{-1}\alpha\beta$ and $\alpha_{i+1}^{(p)}$ will denote the word $\alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}} \cdots \alpha_{i+k}^{\binom{p}{k}} \cdots \alpha_{i+p}$ where i is a positive integer and $\alpha_{i+2}, \ldots, \alpha_{i+p}$ are suitably defined. (Note that, for large enough p, this will often be just α_{i+1}^{p}.) For economy of space, *all relations of the form* $[\alpha, \beta] = 1$ (*with* α, β *generators*) *have been omitted from the list* and should be assumed when reading the list. No attempt has been made to find minimal presentations, and those chosen are designed to accentuate the groups' structures.

The groups in the list are collected together in *isoclinism* families. Two groups G, H with centers $Z(G)$, $Z(H)$ and derived groups G_2, H_2 are said to be *isoclinic* (written $G \sim H$) if there exist isomorphisms

$$\theta : G/Z(G) \longrightarrow H/Z(H),$$

$$\phi : G_2 \longrightarrow H_2,$$

such that $\phi([\alpha, \beta]) = [\alpha', \beta']$ for all $\alpha, \beta \in G$, where $\alpha' Z(H) = \theta(\alpha Z(G))$ and $\beta' Z(H) = \theta(\beta Z(G))$. It is easy to show that this relation is well defined and is in fact an equivalence relation. The pair (θ, ϕ) of isomorphisms is called an *isoclinism* (or autoclinism if $G = H$), and the equivalence classes are called (isoclinism) *families*. A family of p-groups will be denoted by Φ_s if p is an arbitrary prime and Δ_s if $p = 3$, where s is some positive integer. P. Hall [7] has shown that every family Φ has groups of minimal order $p^{m(0)}$, called the *stem* groups of Φ, and the set $p^{m(0)+k}\Phi$ of groups in Φ of order $p^{m(0)+k}$ is called the *kth branch* of Φ. Thus, the number $m(0)$ is an invariant for the family, called the *rank* of Φ. Some of the family invariants are tabulated in Table 4.1, and these include: nilpotency class, lower central series,* $p^{-k}q_i(G)$ and $p^{-k}r_i(G)$, where G is in the kth branch of a family and $q_i(G)$, $r_i(G)$ denote, respectively, the number of conjugacy classes of G with precisely p^i members and the number of irreducible complex representations of G of degree p^i.

In the designation $\Phi_s(m_1, m_2, \ldots, m_r)x_t$ for a group, the symbol Φ_s denotes the isoclinism family containing the group (in Easterfield's ordering). The numbers $m_1, m_2, \ldots, m_r$ are the type invariants of the group G when p is large enough for G to be regular (and used analogously when p is small) and are defined as follows:

If G is a regular group, write

$$G^{p^i} = \{x^{p^i} | x \in G\}, \qquad i = 0, 1, 2, \ldots \quad \text{(a group!)},$$

*In this paper, two groups will be regarded as the same if they are isomorphic.

and

$$p^{w(i)} = \text{the order of the factor group } G^{p^{i-1}}/G^{p^i} \qquad (i = 1, 2, \ldots);$$

then

$$m_j = \text{number of } w(i)\text{'s with } w(i) \geqslant j \qquad (j = 1, 2, \ldots, r)$$

(these numbers may be thought of as describing the power structure of G). For simplicity, we will write $\mathbf{m}$ for the partition $(m_1, m_2, \ldots, m_r)$ and say that G has type $\mathbf{m}$. Finally, the symbol x_n means that the group is the nth group of genus x, where two groups G and H of the same family and having the same type are said to have the same genus if there is a bijection b: $M(G) \longrightarrow M(H)$ such that M and $b(M)$ are in the same family and have the same type and genus for all $M \in M(G)$, with $M(G)$ = the set of all maximal subgroups of G. In particular, two abelian groups have the same genus if and only if they have the same type (when this happens the symbol x will be omitted).

The list $p^m\Phi_s$ of groups in Φ_s of order p^m is ordered as follows:**

(1) $\Phi_s(m_1, m_2, \ldots, m_r)$ occurs before $\Phi_s(m'_1, m'_2, \ldots, m'_{r'})$ if $m_1 = m'_1$, $\ldots, m_i = m'_i$, $m_{i+1} > m'_{i+1}$ for some $i < \min(r, r')$. When this occurs, we will write $\mathbf{m} < \mathbf{m}'$ where $\mathbf{m}$ represents the partition $(m_1, m_2, \ldots, m_r)$ and $\mathbf{m}'$ represents the partition $(m'_1, m'_2, \ldots, m'_{r'})$.

(2) To describe the ordering of the groups in Φ_s with type $\mathbf{m}$, we introduce for any group G the set $M_i(G)$ of maximal subgroups of G in Φ_i, the set $M_{i,\mathbf{n}}(G)$ of maximal subgroups of G in Φ_i having type $\mathbf{n}$ and the set $M_{i,\mathbf{n},x}(G)$ of maximal subgroups of G in $M_{i,\mathbf{n}}(G)$ having genus x. If G is a group of the form $\Phi_s(m_1, m_2, \ldots, m_r)x$ and H is a group of the form $\Phi_s(m_1, m_2, \ldots, m_r)x'$ with $x \neq x'$, then we may assume that for some Φ_i there is a bijection $b_j: M_j(G) \longrightarrow M_j(H)$ which preserves type and genus for all $j > i$ but no such bijection for $j = i$. G occurs before H (and we write $x < x'$) if

(a) $M_i(G)$ has more elements than $M_i(H)$; or

(b) for some partition $\mathbf{n}$ of $m - 1$, $M_{i,\mathbf{n}'}(G)$ and $M_{i,\mathbf{n}'}(H)$ have the same number of elements for all partitions $\mathbf{n}' < \mathbf{n}$, and $M_{i,\mathbf{n}}(G)$ has more elements than $M_{i,\mathbf{n}}(H)$; or

(c) for some partition $\mathbf{n}$ of $m - 1$ and genus y, $M_{i,\mathbf{n}',z}(G)$ and $M_{i,\mathbf{n}',z}(H)$ have the same number of elements for all partitions $\mathbf{n}' < \mathbf{n}$ and all genera z, $M_{i,\mathbf{n},y'}(G)$ and $M_{i,\mathbf{n},y'}(H)$ have the same number of elements for all genera $y' < y$, and $M_{i,\mathbf{n},y}(G)$ has more elements than $M_{i,\mathbf{n},y}(H)$.

(3) The groups in Φ_s with type $\mathbf{m}$ and genus x are ordered in the way suggested by parameters in the defining relations.

Other notation used is as follows:

**Despite this ordering, the groups which are direct products are listed together for convenience. Also, if the above ordering varies depending on the value of p, the most favorable value is taken in each case.

(1) If G is a group, the *center* of G is

$$Z(G) = \{\alpha \in G : \alpha\beta = \beta\alpha \text{ for all } \beta \in G\}$$

and the *lower central series* is $G_2 = [G, G]$, $G_{i+1} = [G_i, G]$ for $i > 2$, where

$$[H, K] = \text{the largest subgroup of } G \text{ containing } \{[\alpha, \beta] : \alpha \in H, \beta \in K\}.$$

The *class* of G is the number c such that $G_c \neq 1$, $G_{c+1} = 1$.

(2) If G, H are groups, their *direct product* is

$$G \times H = \{(\alpha, \beta) : \alpha \in G, \beta \in H\},$$

where multiplication of ordered pairs is component-wise.

(3) In keeping with usual practice, the symbol Φ_1 will be omitted from all abelian groups (the class Φ_1), which will be designated by their types.

(4) Throughout, ν denotes the smallest positive integer which is a non-quadratic residue (mod p) and g denotes the smallest positive integer which is a primitive root (mod p).

As an illustration of the above, the last five groups in $p^5\Phi_3$ are $\Phi_3(2111)b_\nu = \Phi_3(211)b_\nu \times (1)$ (the direct product of the (cyclic) group (1) of order p and the group $\Phi_3(211)b_\nu$), $\Phi_3(2111)c$, $\Phi_3(2111)d$, $\Phi_3(2111)e$ and $\Phi_3(1^6)$. The first of these has p^2 maximal subgroups in Φ_3 of type (211), the second has $p^2 - 1$ maximal subgroups in Φ_3 of that type, and the others (except the last) have no maximal subgroups in Φ_3. The third of these groups has p maximal subgroups in Φ_2 of type (211), whereas the fourth group only has $p - 1$ such maximal subgroups.

3. Families of p-Groups of Rank at Most 6. In this section, we outline the method used in [9] to obtain a complete list of the groups of order p^m ($1 \leqslant m \leqslant 6$) given in the next section.

In finding these groups, we make use of the well-known list of abelian groups of order p^m, each one corresponding uniquely to a partition of the number m. In particular, since the groups of order p and p^2 are abelian, the only group of order p is the cyclic group (1) and the only two groups of order p^2 are (2) and (11).

If G is a group, we will write $|G|$ for the order of G, $Z(G)$ for the center of G and $G_2, G_3, \ldots, G_c$ for the lower central series of G (i.e. $G_2 = [G, G]$ and $G_{i+1} = [G_i, G]$ for $i > 2$). If $|G| = p^m$ and the groups of order $p, p^2, \ldots, p^{m-1}$ are already listed then, since $|G/Z(G)| < p^m$, the factor group $G/Z(G)$ is listed. Thus, the process consists of determining all those p-groups H, with $|H| < p^m$, capable of being central quotients $G/Z(G)$ for some G and then constructing families of groups G with $G/Z(G)$ isomorphic to H. An example of this may be found in Hall and Senior [6] where the groups of order 2^n ($n \leqslant 6$) are determined. Similarly, P. Hall [7] used the groups of order p^2, p^3, p^4 to determine the families of rank 5 and, hence, the groups of order p^5. This was then used by Easterfield [5], and later James, to find all the families of rank $\leqslant 6$ in the Table 4.1. Details of this may be found in [9].

The method of finding the isomorphism classes of groups for an individual family is essentially the same as that used by Blackburn [3] in determining p-groups of maximal class and is described in [10]. By definition of isoclinism we may suppose that all commutator relations of any group of the family are known, and all relations modulo the center of the group are known. Thus, we only need consider the values of the remaining relations, namely

(a) the structure of the center of the group;
(b) the relationship between the center and commutator subgroup of the group;
(c) the precise value of all words forced to be in the center of the group by the relations modulo the center.

As described in [9] and [10], the isomorphism problem in the family for groups with the relations (a) and (b) specified is equivalent to the determination of equivalence classes of certain matrices (over the field $\mathbf{Z}_p$ with p elements) for a certain equivalence relation. The actual calculations were carried out in [9] for all families except Φ_{21}, although there are errors in some of the calculations.

To illustrate the method used, we shall find the groups of order p^6 in this family Φ_{21} for $p > 3$. If $G \in \Phi_{21}$, then Table 4.1 shows that we may suppose $G = \langle(\alpha_1, \alpha_2, \alpha, Z(G)\rangle$, where $\beta = [\alpha_1, \alpha_2]$, $\beta_i = [\beta, \alpha_i]$, $[\alpha, \alpha_1] = \beta_2$, $[\alpha, \alpha_2] = \beta_1^\nu$, $\beta^p = \beta_i^p = 1$ and α^p, $\alpha_i^p \in Z(G)$ for $i = 1, 2$. Thus $G_2 = \langle\beta, G_3\rangle$ and $G_3 = \langle\beta_1, \beta_2\rangle$ which is in $Z(G)$. If $|G| = p^6$, then $G_3 = Z(G)$ and so

$$\text{(1)} \qquad \alpha^p = \beta_1^{a(1)}\beta_2^{a(2)}, \quad \alpha_i^p = \beta_1^{a(1,i)}\beta_2^{a(2,i)} \qquad (i = 1, 2)$$

for some $a(1), a(2) \in \mathbf{Z}_p$ and some matrix

$$A = \begin{pmatrix} a(1, 1) & a(1, 2) \\ a(2, 1) & a(2, 2) \end{pmatrix}$$

over $\mathbf{Z}_p$. Since the centralizer of G_2 contains α but not α_1 or α_2, an autoclinism of G will map α_i to α_i^*, α to α^*, where

$$\text{(2)} \qquad \alpha^* \equiv \alpha^z, \quad \alpha_i^* \equiv \alpha_1^{x(1,i)}\alpha_2^{x(2,i)}\alpha^{x(i)} \pmod{G_2} \qquad (i = 1, 2)$$

for some $x(1), x(2), z \in \mathbf{Z}_p$, and some matrix

$$X = \begin{pmatrix} x(1, 1) & x(1, 2) \\ x(2, 1) & x(2, 2) \end{pmatrix}$$

over $\mathbf{Z}_p$. Since $[\alpha^*, \alpha_1^*] = \beta_2^*$ and $[\alpha^*, \alpha_2^*] = \beta_1^{*\nu}$, we have

$$\text{(3)} \qquad x(1, 1) = \epsilon x(2, 2), \quad x(1, 2) = \epsilon\nu x(2, 1), \quad z = x(2, 2)^2 - \nu x(1, 1)^2 \neq 0,$$

where $\epsilon = \pm 1$. This autoclinism yields the relations

$$\text{(1*)} \qquad \alpha^{*p} = \beta_1^{*a(1)^*}\beta_2^{*a(2)^*}, \quad \alpha_i^{*p} = \beta_1^{*a(1,i)^*}\beta_2^{*a(2,i)^*} \qquad (i = 1, 2);$$

and, substituting (2) into (1*), we obtain

$$\text{(4)} \qquad \begin{cases} \mathrm{a} = \epsilon X \mathrm{a}^*, \\ AX + (x(1)\mathrm{a}, x(2)\mathrm{a}) = \epsilon z X A^*, \end{cases}$$

where $\mathbf{a} = \binom{a(1)}{a(2)}$ and A^*, $\mathbf{a}^*$ are defined analogously to A, $\mathbf{a}$, respectively. If $\mathbf{a} \neq \mathbf{0}$, it can easily be seen that (by choice of X) we may suppose $\alpha^p = \beta_1$, $\alpha_1^p = \beta_2^a$, $\alpha_2^p = \beta_2^b$, where the only restriction on a, b is that $b = 0, 1, 2, \ldots, \frac{1}{2}(p-1)$. If $\mathbf{a} = \mathbf{0}$, Eq. (4) becomes

$$\epsilon z A^* = X^{-1} A X.$$

Writing $x = x(2, 1)$, $y = x(2, 2)$, $a = \frac{1}{2}(a(1, 1) + a(2, 2))$, $b = \frac{1}{2}(a(1, 2) + \nu a(2, 1))$, $c = \frac{1}{2}(a(1, 1) - a(2, 2))$ and $d = \frac{1}{2}(a(1, 2) - \nu a(2, 1))$, this becomes

$$\epsilon z a^* = a,$$

$$z b^* = b,$$

$$z^2 c^* = \epsilon(y^2 + \nu x^2)c + 2xyd,$$

$$z^2 d^* = 2\epsilon\nu xyc + (y^2 + \nu x^2)d,$$

forcing $z^2(d^{*2} - \nu c^{*2}) = d^2 - \nu c^2$. Thus, by considering the cases $d^2 - \nu c^2 = 0$, a quadratic residue and a non-quadratic residue, we may determine canonical values for a, b, c, d (and hence A) as described in the list of groups $p^6\Phi_{21}$ in Section 4.6 (21).

4. List of Results.

4.1. *Summary of families of rank* $\leqslant 6$.

Family	Rank	Class	G/Z(G)	G_2	G_3	G_4	G_5	q_0^*	q_1^*	q_2^*	q_3^*	q_4^*	r_0^*	r_1^*	r_2^*
Φ_2	3	2	(11)	(1)				p	p^2-1	0	0	0	p^2	$p-1$	0
Φ_3	4	3	$\Phi_2(1^3)$	(11)	(1)			p	p^2-1	p^2-p	0	0	p^2	p^2-1	0
Φ_4	5	2	(111)	(11)				p^2	p^3-p	p^3-p^2	0	0	p^3	p^3-p	0
Φ_5	5	2	(1^4)	(1)				p	p^4-1	0	0	0	p^4	0	$p-1$
Φ_6	5	3	$\Phi_2(1^3)$	(111)	(11)			p^2	0	p^3-1	0	0	p^2	p^3-1	0
Φ_7	5	3	$\Phi_2(1^4)$	(11)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	$p-1$
Φ_8	5	3	$\Phi_2(22)$	(2)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	$p-1$
Φ_9	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	p^3-1	0	p^2-p	0	p^2	p^3-1	0
Φ_{10}	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	$p-1$	p^2-1	p^2-p	0	p^2	p^2-1	$p-1$
Φ_{11}	6	2	(111)	(111)				p^3	0	p^4-p	0	0	p^3	p^4-p	0
Φ_{12}	6	2	(1^4)	(11)				p^2	$2p^3-2p$	p^4-2p^2+1	0	0	p^4	$2p^3-2p^2$	p^2-2p+1
Φ_{13}	6	2	(1^4)	(11)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-p^2	p^2-p
Φ_{14}	6	2	(22)	(2)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-p^2	p^2-p
Φ_{15}	6	2	(1^4)	(11)				p^2	0	p^4-1	0	0	p^4	0	p^2-1
Φ_{16}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^4-p	0	p^3-p^2	0	p^3	p^4-p	0
Φ_{17}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{18}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{19}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{20}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{21}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	0	p^2-1	p^3-p	0	p^3	p^2-p	p^2-1
Φ_{22}	6	3	$\Phi_2(1^5)$	(11)	(1)			p	p^3+p^2-p-1	p^4-p^2-p+1	0	0	p^4	p^3-p^2	p^2-p
Φ_{23}	6	4	$\Phi_3(1^4)$	(1^4)	(111)	(1)		p^2	p^2-p	p^3-p	p^3-p^2	0	p^2	$2p^3-p^2-1$	p^2-2p+1
Φ_{24}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{25}	6	4	$\Phi_3(221)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{26}	6	4	$\Phi_3(221)b_\nu$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p

Family	Rank	Class	G/Z(G)	G_2	G_3	G_4	G_5	q_0^*	q_1^*	q_2^*	q_3^*	q_4^*	r_0^*	r_1^*	r_2^*
Φ_{27}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{28}	6	4	$\Phi_3(221)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{29}	6	4	$\Phi_3(221)b_\nu$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{30}	6	4	$\Phi_7(1^5)$	(111)	(11)	(1)		p	$p-1$	$2p^2-p-1$	p^3-2p+1	0	p^3	p^2-p	p^2-1
Φ_{31}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{32}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{33}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{34}	6	3	$\Phi_4(221)b$	(21)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{35}	6	5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	p	p^4-1	0	0	p^2-p	p^2	p^4-1	0
Φ_{36}	6	5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	p	p^2-1	p^3-p	0	p^2-p	p^2	p^3-1	p^2-p
Φ_{37}	6	5	$\Phi_9(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	p	$p-1$	p^3-1	p^2-p	p^2-2p+1	p^2	p^3-1	p^2-p
Φ_{38}	6	5	$\Phi_{10}(1^5)$	(1^4)	(111)	(11)	(1)	p	$p-1$	p^2-1	p^2-p	p^2-p	p^2	p^2-1	p^2-1
Φ_{39}	6	5	$\Phi_{10}(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	p	$p-1$	$p-1$	$2p^2-p-1$	p^2-2p+1	p^2	p^2-1	p^2-1
Φ_{40}	6	4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{41}	6	4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{42}	6	4	$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	(211)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{43}	6	4	$\Phi_6(221)d_0$	(211)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p

4.2. *The groups of order p, p^2.* The only group of order p is (1), and the groups of order p^2 are (2) and (11).

4.3. *The groups of order p^3*

(1) Abelian: (3), (21) and (111).

(2) Non-abelian: $\Phi_2(21) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$.

$\Phi_2(111) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$.

4.4. *The groups of order p^4.*

(1) Abelian: (4), (31), (22), (211), (1^4).

(2) $\Phi_2(211)a = \Phi_2(21) \times (1)$, $\Phi_2(1^4) = \Phi_2(111) \times (1)$,

$\Phi_2(31) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$

$\Phi_2(211)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle$.

(3) $\Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$

$\Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle$

for $r = 1$ or ν

$\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_3^p = 1 \ (i = 1, 2) \rangle$.

4.5. The groups of order p^5.

(1) Abelian: (5), (41), (32), (311), (221), (2111), (1^5)

(2) $\Phi_2(311)a = \Phi_2(31) \times (1)$, $\Phi_2(221)a = \Phi_2(22) \times (1)$, $\Phi_2(221)b = \Phi_2(21) \times (2)$,

$\Phi_2(2111)a = \Phi_2(211)a \times (1)$, $\Phi_2(2111)b = \Phi_2(211)b \times (1)$, $\Phi_2(2111)c = \Phi_2(211)c \times (1)$,

$\Phi_2(2111)d = \Phi_2(111) \times (2)$, $\Phi_2(1^5) = \Phi_2(1^4) \times (1)$.

$\Phi_2(41) = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha^{p^3} = \alpha_2,\ \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(32)a_1 = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha^{p^2} = \alpha_2,\ \alpha_1^{p^2} = \alpha_2^p = 1\rangle$

$\Phi_2(32)a_2 = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_1^p = \alpha_2,\ \alpha^{p^3} = \alpha_2^p = 1\rangle$

$\Phi_2(311)b = \langle \alpha,\alpha_1,\alpha_2,\gamma \mid [\alpha_1,\alpha] = \gamma^{p^2} = \alpha_2,\ \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(311)c = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_2,\ \alpha^{p^3} = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(221)c = \langle \alpha,\alpha_1,\alpha_2,\gamma \mid [\alpha_1,\alpha] = \gamma^p = \alpha_2,\ \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(221)d = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_2,\ \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = 1\rangle$

(3) $\Phi_3(2111)a = \Phi_3(211)a \times (1)$, $\Phi_3(2111)b_r = \Phi_3(211)b_r \times (1)$ for $r = 1$ or ν ,

$\Phi_3(1^5) = \Phi_3(1^4) \times (1)$,

$\Phi_3(311)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha] = \alpha_2,\ [\alpha_2,\alpha] = \alpha^{p^2} = \alpha_3,\ \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$

$\Phi_3(311)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha] = \alpha_2,\ [\alpha_2,\alpha]^r = \alpha_1^{p^2} = \alpha_3,\ \alpha^p = \alpha_2^p = \alpha_3^p = 1\rangle$

for $r = 1$ or ν

$\Phi_3(221)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha] = \alpha_2,\ [\alpha_2,\alpha] = \alpha^p = \alpha_3,\ \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1\rangle$

$\Phi_3(221)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha] = \alpha_2,\ [\alpha_2,\alpha]^r = \alpha_1^{(p)} = \alpha_3^r\ ,\ \alpha^{p^2} = \alpha_2^p = \alpha_3^p = 1\rangle$

for $r = 1$ or ν

$\Phi_3(2111)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\gamma \mid [\alpha_1,\alpha] = \alpha_2,\ [\alpha_2,\alpha] = \gamma^p = \alpha_3,\ \alpha^p = \alpha_i^{(p)} = 1\ (i=1,2,3)\rangle$

$\Phi_3(2111)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_i,\alpha] = \alpha_{i+1},\ \alpha^{p^2} = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1,2)\rangle$

$\Phi_3(2111)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_i,\alpha] = \alpha_{i+1},\ \alpha^p = \alpha_1^{p^2} = \alpha_{i+1}^p = 1\ (i = 1,2)\rangle$

(4) $\Phi_4(221)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha^p = \beta_2,\ \alpha_1^p = \beta_1,\ \alpha_2^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(221)b = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha^p = \beta_2,\ \alpha_2^p = \beta_1,\ \alpha_1^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(221)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i = \alpha_i^p,\ \alpha^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(221)d_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_1^p = \beta_1^k,\ \alpha_2^p = \beta_2,\ \alpha^p = \beta_i^p = 1\ (i = 1,2)\rangle$

where $k = g^r$ for $r = 1,2,\ \ldots,\frac{1}{2}(p-1)$.

$\Phi_4(221)e = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_1^p = \beta_2^{-\frac{1}{4}},\ \alpha_2^p = \beta_1\beta_2,\ \alpha^p = \beta_i^p = 1\ (i=1,2)\rangle$

$\Phi_4(221)f_0 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_1^p = \beta_2,\ \alpha_2^p = \beta_1^\nu,\ \alpha^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(221)f_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_1^p = \beta_2^k,\ \alpha_2^p = \beta_1\beta_2,\ \alpha^p = \beta_i^p = 1\ (i = 1,2)\rangle$

where $4k = g^{2r+1} - 1$ for $r = 1,2,\ \ldots,\frac{1}{2}(p-1)$.

$\Phi_4(2111)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha^p = \beta_2,\ \alpha_i^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(2111)b = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_1^p = \beta_1,\ \alpha^p = \alpha_2^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(2111)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha_2^p = \beta_1,\ \alpha^p = \alpha_1^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_4(1^5) = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha] = \beta_i,\ \alpha^p = \alpha_i^p = \beta_i^p = 1\ (i = 1,2)\rangle$

(5) $\Phi_5(2111) = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2] = [\alpha_3,\alpha_4] = \alpha_1^p = \beta,\ \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1\rangle$

$\Phi_5(1^5) = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2] = [\alpha_3,\alpha_4] = \beta,\ \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1\rangle$

(6) $\Phi_6(221)a = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2] = \beta,\ [\beta,\alpha_i] = \beta_i = \alpha_i^p,\ \beta^p = \beta_i^p = 1\ (i = 1,2)\rangle$

$\Phi_6(221)b_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2] = \beta, [\beta,\alpha_i] = \beta_i, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \beta^p = \beta_i^p = 1\ (i=1,2)\rangle$

where $k = g^r$ for $r = 1,2,\ \ldots,\frac{1}{2}(p-1)$.

$\Phi_6(221)c_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2] = \beta, [\beta,\alpha_i] = \beta_i,\ \alpha_1^p = \beta_2^{-\frac{1}{4}r}, \alpha_2^p = \beta_1^r\beta_2^r, \beta^p = \beta_i^p = 1\ (i=1,2)\rangle$

for $r = 1$ or ν .

$\Phi_6(221)d_0 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i,\ \alpha_1^p = \beta_2,\ \alpha_2^p = \beta_1^\nu,\ \beta^p = \beta_i^p = 1\ (i=1,2) \rangle$

$\Phi_6(221)d_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i,\ \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1\beta_2, \beta^p = \beta_i^p = 1\ (i=1,2) \rangle$

where $4k = g^{2r+1} - 1$ for $r = 1, 2, \ldots, \tfrac{1}{2}(p-1)$.

$\Phi_6(2111)a = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i,\ \alpha_1^p = \beta_1,\ \alpha_2^p = \beta^p = \beta_i^p = 1\ (i=1,2) \rangle$

for $p > 3$

$\Phi_6(2111)b_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i,\ \alpha_2^p = \beta_1^r,\ \alpha_1^p = \beta^p = \beta_i^p = 1\ (i=1,2) \rangle$

for $r = 1$ or ν , and $p > 3$.

$\Phi_6(1^5) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i,\ \alpha_i^p = \beta^p = \beta_i^p = 1\ (i=1,2) \rangle$

(7) $\Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1\ (i=1,2) \rangle$

$\Phi_7(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^p = \beta^p = 1\ (i=1,2) \rangle$

for $r = 1$ or ν .

$\Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^p, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = 1\ (i=1,2) \rangle$

$\Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \cdot (i=1,2) \rangle$

(8) $\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p,\ \beta^{p^2} = \alpha_2^{p^2} = 1 \rangle$

(9) $\Phi_9(2111)a = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1},\ \alpha^p = \alpha_4,\ \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

$\Phi_9(2111)b_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1},\ \alpha_1^{(p)} = \alpha_4^k,\ \alpha^p = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

where $k = g^r$ for $r + 1 = 1, 2, \ldots, (p-1, 3)$

$\Phi_9(1^5) = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1},\ \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

(10) $\Phi_{10}(2111)a_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

where $k = g^r$ for $r + 1 = 1, 2, \ldots, (p-1, 4)$.

$\Phi_{10}(2111)b_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

where $k = g^r$ for $r + 1 = 1, 2, \ldots, (p-1, 3)$, and where $p > 3$.

$\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1\ (i=1,2,3) \rangle$

4.6 The groups of order p^6 .

(1) Abelian: (6), (51), (42), (411), (33), (321), (3111), (222), (2211), (21^4), (1^6).

(2) $\Phi_2(411)a = \Phi_2(41)\times(1)$, $\Phi_2(321)a_i = \Phi_2(32)a_i\times(1)\ (i=1,2)$, $\Phi_2(321)b = \Phi_2(31)\times(2)$,

$\Phi_2(321)e = \Phi_2(21)\times(3)$, $\Phi_2(3111)a = \Phi_2(311)a\times(1)$, $\Phi_2(3111)b = \Phi_2(311)b\times(1)$,

$\Phi_2(3111)c = \Phi_2(311)c\times(1)$, $\Phi_2(3111)d = \Phi_2(111)\times(3)$, $\Phi_2(222)a = \Phi_2(22)\times(2)$,

$\Phi_2(2211)a = \Phi_2(221)a\times(1)$, $\Phi_2(2211)b = \Phi_2(221)b\times(1)$, $\Phi_2(2211)c = \Phi_2(221)c\times(1)$,

$\Phi_2(2211)d = \Phi_2(221)d\times(1)$, $\Phi_2(2211)e = \Phi_2(211)b\times(2)$, $\Phi_2(2211)f = \Phi_2(211)c\times(2)$,

$\Phi_2(21^4)a = \Phi_2(2111)a\times(1)$, $\Phi_2(21^4)b = \Phi_2(2111)b\times(1)$, $\Phi_2(21^4)c = \Phi_2(2111)c\times(1)$,

$\Phi_2(21^4)d = \Phi_2(2111)d\times(1)$, $\Phi_2(1^6) = \Phi_2(1^6)\times(1)$.

$\Phi_2(51) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^4},\ \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(42)a_1 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^3},\ \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$

$\Phi_2(42)a_2 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha_1^p\ ,\ \alpha^{p^4} = \alpha_2^p = 1 \rangle$

$\Phi_2(411)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2 = \gamma^{p^3},\ \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(411)c = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_2,\ \alpha^{p^4} = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(33) = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_2 = \alpha^{p^2},\ \alpha_1^{p^3} = \alpha_2^p = 1\rangle$

$\Phi_2(321)c = \langle \alpha,\alpha_1,\alpha_2,\gamma \mid [\alpha_1,\alpha] = \alpha_2 = \gamma^p,\ \alpha^{p^3} = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(321)d = \langle \alpha,\alpha_1,\alpha_2,\gamma \mid [\alpha_1,\alpha] = \alpha_2 = \gamma^{p^2},\ \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$

$\Phi_2(321)f = \langle \alpha,\alpha_1,\alpha_2 \mid [\alpha_1,\alpha] = \alpha_2,\ \alpha^{p^3} = \alpha_1^{p^2} = \alpha_2^p = 1\rangle$

$\Phi_2(222)b = \langle \alpha,\alpha_1,\alpha_2,\gamma \mid [\alpha_1,\alpha] = \alpha_2 = \gamma^p,\ \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = 1\rangle$

(3) $\Phi_3(3111)a = \Phi_3(311)a\times(1)$, $\Phi_3(3111)b_r = \Phi_3(311)b_r\times(1)$ $(r = 1 \text{ or } \nu)$,

$\Phi_3(2211)a = \Phi_3(221)a\times(1)$, $\Phi_3(2211)b_r = \Phi_3(221)b_r\times(1)$ $(r = 1 \text{ or } \nu)$,

$\Phi_3(2211)c = \Phi_3(211)a\times(2)$, $\Phi_3(2211)e_r = \Phi_3(211)b_r\times(2)$ $(r = 1 \text{ or } \nu)$,

$\Phi_3(21^4)a = \Phi_3(2111)a\times(1)$, $\Phi_3(21^4)b_r = \Phi_3(2111)b_r\times(1)$ $(r = 1 \text{ or } \nu)$,

$\Phi_3(21^4)c = \Phi_3(2111)c\times(1)$, $\Phi_3(21^4)d = \Phi_3(2111)d\times(1)$, $\Phi_3(21^4)e = \Phi_3(2111)e\times(1)$,

$\Phi_3(21^4)f = \Phi_3(1^4)\times(2)$, $\Phi_3(1^6) = \Phi_3(1^5)\times(1)$,

$\Phi_3(411)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha^{p^3}=\alpha_3,\alpha_1^{(p)}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(411)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]^r=\alpha_1^{p^3}=\alpha_3^r,\alpha^p=\alpha_2^p=\alpha_3^p=1\rangle$ $r = 1$ or ν

$\Phi_3(321)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha^{p^2}=\alpha_3,\alpha_1^{p^2}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(321)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]^r=\alpha_1^{(p)}=\alpha_3^r=,\alpha^{p^3}=\alpha_2^p=\alpha_3^p=1\rangle$ $r = 1$ or ν

$\Phi_3(321)c_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]^r=\alpha_1^{p^2}=\alpha_3^r,\alpha^{p^2}=\alpha_2^p=\alpha_3^p=1\rangle$ $r = 1$ or ν

$\Phi_3(321)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha^p=\alpha_3,\alpha_1^{p^3}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(3111)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\gamma \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\gamma^{p^2}=\alpha_3,\alpha^p=\alpha_1^{(p)}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(3111)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha_3,\alpha^{p^3}=\alpha_1^{(p)}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(3111)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha_3,\alpha^p=\alpha_1^{p^3}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(2211)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\gamma \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\gamma^p=\alpha_3,\alpha^p=\alpha_1^{p^2}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(2211)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\gamma \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\gamma^p=\alpha_3,\alpha^{p^2}=\alpha_1^{(p)}=\alpha_2^p=\alpha_3^p=1\rangle$

$\Phi_3(2211)g = \langle \alpha,\alpha_1,\alpha_2,\alpha_3 \mid [\alpha_1,\alpha]=\alpha_2,[\alpha_2,\alpha]=\alpha_3,\alpha^{p^2}=\alpha_1^{p^2}=\alpha_2^p=\alpha_3^p=1\rangle$

(4) $\Phi_4(2211)a = \Phi_4(221)a\times(1)$, $\Phi_4(2211)b = \Phi_4(221)b\times(1)$, $\Phi_4(2211)c = \Phi_4(221)c\times(1)$,

$\Phi_4(2211)d_r = \Phi_4(221)d_r\times(1)$ for $r = 1,2,\ \dots,\tfrac{1}{2}(p-1)$, $\Phi_4(2211)e = \Phi_4(221)e\times(1)$,

$\Phi_4(2211)f_r = \Phi_4(221)f_r\times(1)$ for $r = 0,1,\ \dots,\tfrac{1}{2}(p-1)$, $\Phi_4(21^4)a = \Phi_4(2111)a\times(1)$,

$\Phi_4(21^4)b = \Phi_4(2111)b\times(1)$, $\Phi_4(21^4)c = \Phi_4(2111)c\times(1)$, $\Phi_4(1^6) = \Phi_4(1^5)\times(1)$

$\Phi_4(321)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha^{p^2}=\beta_1,\alpha_2^p=\beta_2,\alpha_1^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(321)b = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha^{p^2}=\beta_1,\alpha_1^p=\beta_2,\alpha_2^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(321)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha^p=\beta_2,\alpha_2^{p^2}=\beta_1,\alpha_1^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(321)d = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha^p=\beta_2,\alpha_1^{p^2}=\beta_1,\alpha_2^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(321)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha_1^{p^2}=\beta_1,\alpha_2^p=\beta_2^r,\alpha^p=\beta_i^p=1\,(i=1,2)\rangle$ for $r=1,2,\dots,p-1$

$\Phi_4(321)f_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha_1^p=\beta_2^r,\alpha_2^{p^2}=\beta_1,\alpha^p=\beta_i^p=1\,(i=1,2)\rangle$ for $r=1$ or ν

$\Phi_4(3111)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha^{p^2}=\beta_1,\alpha_i^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(3111)b = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha_1^{p^2}=\beta_1,\alpha^p=\alpha_2^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(3111)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha_2^{p^2}=\beta_1,\alpha^p=\alpha_1^p=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(222)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i=\alpha_i^p,\alpha^{p^2}=\beta_i^p=1\,(i=1,2)\rangle$

$\Phi_4(222)b_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i,\alpha_1^p=\beta_1^k,\alpha_2^p=\beta_2,\alpha^{p^2}=\beta_i^p=1\,(i=1,2)\rangle$

where $k = g^r$ for $r = 1,2,\ \dots,\tfrac{1}{2}(p-1)$.

$\Phi_4(222)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha^p=\beta_1, \alpha_2^p=\beta_2, \alpha_1^{p^2}=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(222)d_1 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_1^p=\beta_2^{-\frac{1}{4}}, \alpha_2^p=\beta_1\beta_2, \alpha^{p^2}=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(222)d_2 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha^p=\beta_2, \alpha_2^p=\beta_1, \alpha_1^{p^2}=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(222)e_0 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_1^p=\beta_2, \alpha_2^p=\beta_1^{\nu}, \alpha^{p^2}=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(222)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_1^p=\beta_2^k, \alpha_2^p=\beta_1\beta_2, \alpha^{p^2}=\beta_i^p=1 (i=1,2)\rangle$

where $4k = g^{2r+1}-1$ for $r=1,2,\ldots,\frac{1}{2}(p-1)$.

$\Phi_4(2211)g = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i, \gamma^p=\beta_2, \alpha^p=\beta_1, \alpha_i^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)h = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i, \gamma^p=\beta_2, \alpha_1^p=\beta_1, \alpha^p=\alpha_2^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)i = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i, \gamma^p=\beta_2, \alpha_2^p=\beta_1, \alpha^p=\alpha_1^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)j_1 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_1^p=\beta_1, \alpha^{p^2}=\alpha_2^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)j_2 = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha^p=\beta_1, \alpha_1^{p^2}=\alpha_2^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)k = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha^p=\beta_2, \alpha_1^{p^2}=\alpha_2^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)\ell = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_2^p=\beta_1, \alpha^{p^2}=\alpha_1^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)m = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_2^p=\beta_2, \alpha_1^{p^2}=\alpha^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(2211)n = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_2^p=\beta_1, \alpha_1^{p^2}=\alpha^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i, \gamma^p=\beta_1, \alpha^p=\alpha_i^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(21^4)e = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha^{p^2}=\alpha_i^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_4(21^4)f = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\beta_i, \alpha_1^{p^2}=\alpha_2^p=\alpha^p=\beta_i^p=1 (i=1,2)\rangle$

(5) $\Phi_5(21^4)a = \Phi_5(2111)\times(1)$, $\Phi_5(1^6) = \Phi_5(1^5)\times(1)$,

$\Phi_5(311) = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2]=[\alpha_3,\alpha_4]=\alpha_1^{p^2}=\beta, \alpha_2^p=\alpha_3^p=\alpha_4^p=\beta^p=1\rangle$

$\Phi_5(2211)a = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2]=[\alpha_3,\alpha_4]=\alpha_2^p=\beta, \alpha_1^{p^2}=\alpha_3^p=\alpha_4^p=\beta^p=1\rangle$

$\Phi_5(2211)b = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2]=[\alpha_3,\alpha_4]=\alpha_3^p=\beta, \alpha_1^{p^2}=\alpha_2^p=\alpha_4^p=\beta^p=1\rangle$

$\Phi_5(21^4)b = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta,\gamma \mid [\alpha_1,\alpha_2]=[\alpha_3,\alpha_4]=\gamma^p=\beta, \alpha_i^p=\beta^p=1 (i=1,2,3,4)\rangle$

$\Phi_5(21^4)c = \langle \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_1,\alpha_2]=[\alpha_3,\alpha_4]=\beta, \alpha_1^{p^2}=\alpha_2^p=\alpha_3^p=\alpha_4^p=\beta^p=1\rangle$

(6) <u>Note</u>: In this family, $\alpha_1^{(p)} = \alpha_1^p\beta_2^{\binom{p}{3}}$

$\Phi_6(2211)a = \Phi_6(221)a\times(1)$, $\Phi_6(2211)b_r = \Phi_6(221)b_r\times(1)$ $r=1,2,\ldots,\frac{1}{2}(p-1)$,

$\Phi_6(2211)c_r = \Phi_6(221)c_r\times(1)$ $r=1$ or ν, $\Phi_6(2211)d_r = \Phi_6(221)d_r\times(1)$ $r=0,1,\ldots,\frac{1}{2}(p-1)$,

$\Phi_6(21^4)a=\Phi_6(2111)a\times(1)$, $\Phi_6(21^4)b_r=\Phi_6(2111)b_r\times(1)$ $r=1$ or ν, $\Phi_6(1^6)=\Phi_6(1^5)\times(1)$,

$\Phi_6(321)a_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_1^{p^2}=\beta_1, \alpha_2^p=\beta_2^r, \beta^p=\beta_i^p=1 (i=1,2)\rangle$

for $r=1,2,\ldots,p-1$,

$\Phi_6(321)b_{r,s} = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_1^{(p)}=\beta_2^r, \alpha_2^{p^2}=\beta_1^s, \beta^p=\beta_i^p=1 (i=1,2)\rangle$

for $r,s=1$ or ν,

$\Phi_6(3111)a = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_1^{p^2}=\beta_1, \alpha_2^p=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_6(3111)b_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_2^{p^2}=\beta_1^r, \alpha_1^{(p)}=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

for $r=1$ or ν,

$\Phi_6(2211)e = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \gamma^p=\beta_2, \alpha_1^{(p)}=\beta_1, \alpha_2^p=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_6(2211)f_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \gamma^p=\beta_2, \alpha_2^p=\beta_1^r, \alpha_1^{(p)}=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

for $r=1$ or ν,

$\Phi_6(2211)g = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_2^p=\beta_2, \alpha_1^{p^2}=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

$\Phi_6(2211)h_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, \alpha_2^p=\beta_1^r, \alpha_1^{p^2}=\beta^p=\beta_i^p=1 (i=1,2)\rangle$

for $r = 1$ or ν

$$\Phi_6(21^4)c = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,\gamma^p=\beta_1,\alpha_1^{(p)}=\alpha_2^p=\beta^p=\beta_i^p=1 (i=1,2)\rangle$$

$$\Phi_6(21^4)d = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,\alpha_1^{p^2}=\alpha_2^p=\beta^p=\beta_i^p=1 (i=1,2)\rangle$$

(7) $\Phi_7(21^4)a = \Phi_7(2111)a\times(1)$, $\Phi_7(21^4)b_r = \Phi_7(2111)b_r\times(1)$ $r = 1$ or ν, $\Phi_7(21^4)c = \Phi_7(2111)c\times(1)$,

$$\Phi_7(1^6) = \Phi_7(1^5)\times(1),$$

$$\Phi_7(3111)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^{p^2}=\alpha_3,\alpha_1^{(p)}=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

$$\Phi_7(3111)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{p^2}=\alpha_3^r,\alpha^p=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_7(3111)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^{p^2}=\alpha_3,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

$$\Phi_7(2211)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^p=\alpha_3,\alpha_1^{p^2}=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

$$\Phi_7(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{(p)}=\alpha_3^r,\alpha^{p^2}=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_7(2211)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^p=\alpha_3,\beta^{p^2}=\alpha_1^{(p)}=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

$$\Phi_7(2211)d_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{(p)}=\alpha_3^r,\beta^{p^2}=\alpha^p=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_7(2211)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^p=\alpha_3,\alpha_1^{p^2}=\alpha^p=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

$$\Phi_7(2211)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^p=\alpha_3,\alpha^{p^2}=\alpha_1^{(p)}=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

$$\Phi_7(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\gamma^p=\alpha_3,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

$$\Phi_7(21^4)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,\alpha^{p^2}=\alpha_1^{(p)}=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

$$\Phi_7(21^4)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,\alpha_1^{p^2}=\alpha^p=\alpha_{i+1}^p=\beta^p=1 (i=1,2)\rangle$$

$$\Phi_7(21^4)g = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,\beta^{p^2}=\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=1 (i=1,2)\rangle$$

(8) $\Phi_8(42) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\alpha_1^p=\beta,[\beta,\alpha_2]=\beta^p,\alpha_2^{p^3}=\beta^{p^2}=1\rangle$

$$\Phi_8(33) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta=\alpha_1^p,[\beta,\alpha_2]=\beta^p,\alpha_2^{p^3}=\beta^{p^2}=1\rangle$$

$$\Phi_8(321)a = \Phi_8(32)\times(1)$$

$$\Phi_8(321)b = \langle \alpha_1,\alpha_2,\beta,\gamma \mid [\alpha_1,\alpha_2]=\beta=\alpha_1^p,[\beta,\alpha_2]=\beta^p=\gamma^p,\alpha_2^{p^2}=\beta^{p^2}=1\rangle$$

$$\Phi_8(321)c_r = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_2]^{r+1}=\beta^{p(r+1)}=\alpha_1^{p^2},\alpha_2^{p^2}=\beta^{p^2}=1\rangle \quad \text{for } r=0,1,\ \ldots,p-2$$

$$\Phi_8(321)c_{p-1} = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_2]=\beta^p=\alpha_2^{p^2},\alpha_1^{p^2}=\beta^{p^2}=1\rangle$$

$$\Phi_8(222) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_2]=\beta^p,\alpha_1^{p^2}=\alpha_2^{p^2}=\beta^{p^2}=1\rangle$$

(9) $\Phi_9(21^4)a = \Phi_9(2111)a\times(1)$, $\Phi_9(21^4)b_r = \Phi_9(2111)b_r\times(1)$ for $r+1 = 1,2,\ \ldots,(p-1,3)$,

$$\Phi_9(1^6) = \Phi_9(1^5)\times(1),$$

$$\Phi_9(3111)a = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha^{p^2}=\alpha_4,\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_9(3111)b_r = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha_1^{p^2}=\alpha_4^k,\alpha^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r+1 = 1,2,\ \ldots,(p-1,3)$

$$\Phi_9(2211)a = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha^p=\alpha_4,\alpha_1^{p^2}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_9(2211)b_r = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha_1^{(p)}=\alpha_4^k,\alpha^{p^2}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r+1 = 1,2,\ \ldots,(p-1,3)$

$$\Phi_9(21^4)c = \langle \alpha,\alpha_1,\ \ldots,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},\gamma^p=\alpha_4,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_9(21^4)d = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha^{p^2}=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_9(21^4)e = \langle \alpha,\alpha_1,\ \ldots,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},\alpha_1^{p^2}=\alpha^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

(10) $\Phi_{10}(21^4)a_r = \Phi_{10}(2111)a_r \times (1)$ for $r+1 = 1,2, \ldots, (p-1,4)$,

$\Phi_{10}(21^4)b_r = \Phi_{10}(2111)b_r \times (1)$ for $r+1 = 1,2, \ldots, (p-1,3)$ and <u>where $p > 3$</u>,

$\Phi_{10}(1^6) = \Phi_{10}(1^5) \times (1)$,

$\Phi_{10}(3111)a_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha^{p^2} = \alpha_4^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

where $k = g^r$ for $r+1 = 1,2, \ldots, (p-1,4)$

$\Phi_{10}(3111)b_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_1^{p^2} = \alpha_4^k, \alpha^p = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

where $k = g^r$ for $r+1 = 1,2, \ldots, (p-1,3)$

$\Phi_{10}(2211)a_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha^p = \alpha_4^k, \alpha_1^{p^2} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

where $k = g^r$ for $r+1 = 1,2, \ldots (p-1,4)$

$\Phi_{10}(2211)b_r = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_1^{(p)} = \alpha_4^k, \alpha^{p^2} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

where $k = g^r$ for $r+1 = 1,2, \ldots, (p-1,3)$

$\Phi_{10}(21^4)c = \langle \alpha, \alpha_1, \ldots, \alpha_4, \gamma \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma^p = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

$\Phi_{10}(21^4)d = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^{p^2} = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

$\Phi_{10}(21^4)e = \langle \alpha, \alpha_1, \ldots, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{p^2} = \alpha^p = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$

(11) $\Phi_{11}(222)a_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

for $r = 1$ or ν

$\Phi_{11}(222)b_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

for $r = 1,2, \ldots, p-2$, where at least two of $-r$, $-(r+1)$, $-(r+1)/r$ are quadratic residues (mod p)

$\Phi_{11}(222)c = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_3^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_2^{-1} \beta_3, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(222)d_0 = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_1^{-p} = \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(222)d_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

for $r = 1,2, \ldots, p-2$, where at most one of $-r$, $-(r+1)$, $-(r+1)/r$ is a quadratic residue (mod p)

$\Phi_{11}(222)d_{p-1} = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_2^p = \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(2211)a = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \alpha_3^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_2^{-1} \beta_3, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(2211)b = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^{-p} = \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(2211)c = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_1^{-p} = \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(2211)d_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1]^r = \alpha_2^p = \beta_2^{\nu^r}, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

for $r = 0,1$

$\Phi_{11}(2211)d_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^{-r^2} \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

for $r = 2,3, \ldots, \frac{1}{2}(p-1)$

$\Phi_{11}(2211)e = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1 \beta_2^{-1}, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(2211)f_r = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^k \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

where $k = -\nu r^2$ for $r = 1,2, \ldots, \frac{1}{2}(p-1)$

$\Phi_{11}(21^4)a = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \alpha_1^p = \beta_2, \alpha_2^p = \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(21^4)b = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_2^p = \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$

$\Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \ldots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1 \ (i=1,2,3) \rangle$

(12) $\Phi_{12}(2211)a = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_1, \beta_1^p = \gamma_2, \alpha_2^p = \beta_2^p = \gamma_i^p = 1 \ (i=1,2) \rangle$

$\Phi_{12}(2211)b = \Phi_2(21) \times \Phi_2(21)$

$\Phi_{12}(2211)c = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_1 \gamma_2, \alpha_2^p = \gamma_2, \beta_i^p = \gamma_i^p = 1 \ (i=1,2) \rangle$

$\Phi_{12}(2211)d = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_2, \alpha_2^p = \gamma_1, \beta_i^p = \gamma_i^p = 1 \ (i=1,2) \rangle$

$$\Phi_{12}(2211)e = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\gamma_1\gamma_2,\alpha_2^p=\gamma_1,\beta_i^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(2211)f = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\alpha_2^p=\gamma_1,\beta_1^p=\gamma_2,\beta_2^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(2211)g = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\gamma_1,\alpha_2^p=\beta_1^p=\gamma_2,\beta_2^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(2211)h = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\beta_2^p=\gamma_1,\alpha_2^p=\beta_1^p=\gamma_2,\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(2211)i = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\gamma_1,\alpha_2^p=\gamma_1\gamma_2,\beta_1^p=\gamma_2,\beta_2^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(21^4)a = \Phi_2(21)\times\Phi_2(111)$$

$$\Phi_{12}(21^4)b = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\gamma_1\gamma_2,\alpha_2^p=\beta_i^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(21^4)c = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\gamma_2,\alpha_2^p=\beta_i^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(21^4)d = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\alpha_2^p=\gamma_1,\beta_i^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(21^4)e = \langle \alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2 \mid [\alpha_i,\beta_i]=\gamma_i,\alpha_1^p=\alpha_2^p=\gamma_1\gamma_2,\beta_i^p=\gamma_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{12}(1^6) = \Phi_2(111)\times\Phi_2(111)$$

(13) $$\Phi_{13}(2211)a = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_2^p=\beta_2,\alpha_1^p=\beta_1,\alpha_3^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(2211)b = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_3^p=\beta_2,\alpha_1^p=\beta_1,\alpha_2^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(2211)c_r = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]^r=\alpha_2^p=\beta_2^r,\alpha_3^p=\beta_1,\alpha_1^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{13}(2211)d = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_1^p=\beta_2,\alpha_3^p=\beta_1,\alpha_2^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(2211)e_r = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]^r=\alpha_4^p=\beta_2^r,\alpha_1^p=\beta_1,\alpha_2^p=\alpha_3^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ \ldots,p-1$

$$\Phi_{13}(2211)f = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_4^p=\beta_2,\alpha_3^p=\beta_1,\alpha_1^p=\alpha_2^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(21^4)a = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\beta_2,\alpha_1^p=\beta_1,\alpha_{i+1}^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(21^4)b = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_1^p=\beta_2,\alpha_{i+1}^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(21^4)c = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\alpha_3^p=\beta_2,\alpha_i^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(21^4)d = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\beta_2,\alpha_3^p=\beta_1,\alpha_i^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{13}(1^6) = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_2,\alpha_4]=\beta_2,\alpha_i^p=\alpha_3^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

(14) $$\Phi_{14}(42) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,\alpha_1^{p^2}=\beta,\alpha_2^{p^2}=\beta^{p^2}=1\rangle$$

$$\Phi_{14}(321) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,\alpha_1^{p^2}=\beta^p,\alpha_2^{p^2}=\beta^{p^2}=1\rangle$$

$$\Phi_{14}(222) = \langle \alpha_1,\alpha_2,\beta \mid [\alpha_1,\alpha_2]=\beta,\alpha_1^{p^2}=\alpha_2^{p^2}=\beta^{p^2}=1\rangle$$

(15) $$\Phi_{15}(2211)a = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\alpha_1^p=\beta_1,[\alpha_2,\alpha_4]=\alpha_2^{gp}=\beta_2^g,\alpha_3^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{15}(2211)b_{r,s} = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\beta_1,[\alpha_2,\alpha_4]^k=\alpha_2^{gp}=\beta_2^{gk},\alpha_1^p=\beta_1\beta_2^r,\alpha_3^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

where $k=g^{[\frac{1}{2}n]+s}$ and $g^n=g^2(g-r^2)$ for $r=1,2,\ \ldots,\frac{1}{2}(p-1)$ and $s=0,1,\ \ldots,m$, with $m=\frac{1}{2}(p-3)+n-2[\frac{1}{2}n]$ and $[\frac{1}{2}n]$ = integral part of $\frac{1}{2}n$;

$$\Phi_{15}(2211)c = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\alpha_1^p=\beta_1,[\alpha_2,\alpha_4]=\alpha_4^{-p}=\beta_2^g,\alpha_{i+1}^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{15}(2211)d_r = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\alpha_1^p=\beta_1,[\alpha_2,\alpha_4]=\beta_2^g,\alpha_4^p=\beta_2^k,\alpha_{i+1}^p=\beta_i^p=1\ (i=1,2)\rangle$$

where $k=g^r$ for $r=1,2,\ \ldots,\frac{1}{2}(p-1)$;

$$\Phi_{15}(21^4) = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\alpha_1^p=\beta_1,[\alpha_2,\alpha_4]=\beta_2^g,\alpha_{i+1}^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{15}(1^6) = \langle \alpha_1,\ \ldots,\alpha_4,\beta_1,\beta_2 \mid [\alpha_1,\alpha_{i+1}]=\beta_i,[\alpha_3,\alpha_4]=\beta_1,[\alpha_2,\alpha_4]=\beta_2^g,\alpha_i^p=\alpha_3^p=\alpha_4^p=\beta_i^p=1\ (i=1,2)\rangle$$

(16) $\Phi_{16}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\beta^p=\gamma, \alpha^p=\alpha_3, \alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(2211)b = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\alpha^p=\gamma, \beta^p=\alpha_3, \alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(2211)c_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\beta^p=\gamma, \alpha_1^{(p)}=\alpha_3^r, \alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1,2,\ldots,p-1$

$\Phi_{16}(2211)d_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]^k=\alpha_1^{(p)}=\gamma^k, \beta^p=\alpha_3, \alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

where $k=g^r$ for $r+1=1,2,\ldots,(p-1,3)$

$\Phi_{16}(2211)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\alpha_1^{(p)}=\gamma, \alpha^p=\alpha_3, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(2211)f_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\alpha^p=\gamma, \alpha_1^{(p)}=\alpha_3^r, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν

$\Phi_{16}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\gamma, \alpha^p=\alpha_3, \alpha_1^{(p)}=\beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(21^4)b = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\alpha^p=\gamma, \alpha_1^{(p)}=\beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(21^4)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\beta^p=\gamma, \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\gamma, \beta^p=\alpha_3, \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\gamma, \alpha_1^{(p)}=\alpha_3^r, \alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν

$\Phi_{16}(21^4)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\alpha_1^{(p)}=\gamma, \alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{16}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha]=\gamma, \alpha^p=\alpha_1^{(p)}=\beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

(17) $\Phi_{17}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\beta^p=\gamma, \alpha^p=\alpha_3, \alpha_1^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{17}(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\beta^p=\gamma, \alpha_1^p=\alpha_3^r, \alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν

$\Phi_{17}(2211)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3, \beta^p=\alpha_3\gamma, \alpha_1^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{17}(2211)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\alpha_1^p=\gamma, \beta^p=\alpha_3, \alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{17}(2211)e_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha_1^p=\alpha_3^r, \beta^p=\alpha_3\gamma, \alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν

$\Phi_{17}(2211)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\alpha^p=\gamma, \beta^p=\alpha_3, \alpha_1^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{17}(2211)g_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\beta^p=\gamma, \alpha^p=\alpha_3, \alpha_1^p=\alpha_3^r, \alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1,2,\ldots,p-1$

$\Phi_{17}(2211)h_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\alpha_1^p=\gamma, \alpha^p=\gamma^k, \beta^p=\alpha_3, \alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

where $k=g^r$ for $r+1=1,2,\ldots,(p-1,3)$

$\Phi_{17}(2211)i_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3, \alpha_1^p=\alpha_3^r, \beta^p=\alpha_3\gamma, \alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1,2,\ldots,p-1$

$\Phi_{17}(2211)j = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\alpha_1^p=\gamma, \alpha^p=\alpha_3, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$\Phi_{17}(2211)k_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3^{1-s+rs}\gamma^s, \alpha_1^p=\alpha_3^{r(1-s)}\gamma, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν and $s=0$ or 1

$\Phi_{17}(2211)\ell_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3^{r+s}\gamma, \alpha_1^p=\alpha_3^r\gamma, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν and $s=1,2,\ldots,p-1$

$\Phi_{17}(2211)m_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3^s\gamma, \alpha_1^p=\alpha_3^r, \beta^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

for $r=1$ or ν and $s=0$ or 1

$\Phi_{17}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\beta,\alpha_1]=\gamma, \alpha^p=\alpha_3, \beta^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2) \rangle$

$$\Phi_{17}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha_1^p=\alpha_3^r,\alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{17}(21^4)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\alpha^p=\gamma,\beta^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha^p=\alpha_3\gamma,\beta^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\alpha_1^p=\gamma,\alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)f_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha_1^p=\alpha_3^r\gamma,\alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{17}(21^4)g = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\beta^p=\gamma,\alpha^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)h = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\beta^p=\alpha_3,\alpha^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)i = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\beta^p=\alpha_3\gamma,\alpha^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)j_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha^{rp}=\alpha_1^p=\alpha_3^r,\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{17}(21^4)k = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\alpha^p=\alpha_1^p=\gamma,\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{17}(21^4)\ell_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha^p=\alpha_1^p=\alpha_3^r\gamma,\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{17}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta,\alpha_1]=\gamma,\alpha^p=\beta^p=\alpha_1^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

(18)
$$\Phi_{18}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^p=\alpha_3,[\alpha,\beta]=\beta^p=\gamma,\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{18}(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^p=\alpha_3,[\alpha,\beta]^r=\alpha^p=\gamma^r,\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{18}(2211)c_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{(p)}=\alpha_3^r,[\alpha,\beta]=\beta^p=\gamma,\alpha^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ \ldots,p-1$

$$\Phi_{18}(2211)d_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^k=\beta^p=\alpha_3^k,[\alpha,\beta]=\alpha_1^{(p)}=\gamma,\alpha^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

where $k=g^r$ for $r+1=1,2,\ \ldots,(p-1,3)$

$$\Phi_{18}(2211)e_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^k=\alpha^p=\alpha_3^k,[\alpha,\beta]=\alpha_1^{(p)}=\gamma,\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

where $k=g^r$ for $r+1=1,2,\ \ldots,(p-1,4)$

$$\Phi_{18}(2211)f_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{(p)}=\alpha_3^r,[\alpha,\beta]^s=\alpha^p=\gamma^s,\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν and $s=1,2,\ \ldots,p-1$

$$\Phi_{18}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^p=\alpha_3,[\alpha,\beta]=\gamma,\beta^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{18}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,[\alpha,\beta]^r=\alpha^p=\gamma^r,\beta^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{18}(21^4)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,[\alpha,\beta]=\beta^p=\gamma,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{18}(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^p=\alpha_3,[\alpha,\beta]=\gamma,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{18}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=\alpha_1^{(p)}=\alpha_3^r,[\alpha,\beta]=\gamma,\alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{18}(21^4)f = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,[\alpha,\beta]=\alpha_1^{(p)}=\gamma,\alpha^p=\beta^p=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{18}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_3,[\alpha,\beta]=\gamma,\alpha^p=\beta^p=\alpha_1^{(p)}=\alpha_{i+1}^p=\gamma^p=1\ (i=1,2)\rangle$$

(19)
$$\Phi_{19}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\alpha_i^p=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{19}(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha_1^p=\beta_1,\alpha_2^p=\beta_2^k,\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

where $k=g^r$ for $r=1,2,\ \ldots,\tfrac{1}{2}(p-1)$

$$\Phi_{19}(2211)c_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^r,\alpha_2^p=\beta_2^s,\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν and $s=2,3,\ldots,p-1$

$$\Phi_{19}(2211)d_{0,0,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{-\nu}=\alpha_2^p=\beta_1^{-\nu},\alpha_1^p=\beta_2^{\nu},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

$$\Phi_{19}(2211)d_{r,0,t} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{t}=\alpha_2^p=\beta_1^{t},\alpha_1^p=\beta_1\beta_2^{r},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1$ or ν, $t=1,2,\ldots,p-1$, and $1+4rt$ is a quadratic residue (mod p)

$$\Phi_{19}(2211)d_{r,s,t} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^{r},\alpha_2^p=\beta_1^{s}\beta_2^{k},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,\nu$ and $s=1,2,\ldots,p-1$, where $k=g^t$ $(t=0,1,\ldots,\frac{1}{2}(p-1))$, $k\neq rs$ and $(1-k)^2+4rs$ is a quadratic residue (mod p), and where $-s$ is a non-quadratic residue (mod p) whenever $r=\nu$ and $k=\pm 1$

$$\Phi_{19}(2211)e_{r} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^{r},\alpha_2^p=\beta_2,\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(2211)f_{r,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{t}=\alpha_2^p=\beta_1^{t},\alpha_1^p=\beta_1\beta_2^{r},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

where $rt=-\frac{1}{4}$ for $r=1$ or ν

$$\Phi_{19}(2211)f_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^{r},\alpha_2^p=\beta_1^{t}\beta_2^{k},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,\nu$ where $rt=-\frac{1}{4}(1-k)^2$ and $k=g^s$ $(s=1,2,\ldots,\frac{1}{2}(p-3))$

$$\Phi_{19}(2211)g_{r,0,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{r}=\alpha_2^p=\beta_1^{r},\alpha_1^p=\beta_2,\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(2211)g_{r,0,t} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{t}=\alpha_2^p=\beta_1^{t},\alpha_1^p=\beta_1\beta_2^{r},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1$ or ν, $t=1,2,\ldots,p-1$, and $1+4rt$ is a non-quadratic residue (mod p)

$$\Phi_{19}(2211)g_{r,s,t} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^{r},\alpha_2^p=\beta_1^{s}\beta_2^{k},\alpha^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,\nu$ and $s=1,2,\ldots,p-1$, where k, r and s satisfy the same conditions as for $d_{r,s,t}$ except that $(1-k)^2+4rs$ is a <u>non</u>-quadratic residue (mod p)

$$\Phi_{19}(2211)h_{r} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_1,\alpha_1^p=\beta_2^{r},\alpha_2^p=\beta_2,\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$

$$\Phi_{19}(2211)i = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_1,\alpha_2^p=\beta_2,\alpha_1^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

$$\Phi_{19}(2211)j_{r} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]^{r}=\alpha_i^p=\beta_i^{r},[\alpha,\alpha_1]=\beta_1,\alpha^p=\beta_1\beta_2,\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,2,\ldots,\frac{1}{2}(p-1)$

$$\Phi_{19}(2211)k_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{r}=\alpha_1^p=\beta_1^{r},\alpha^p=\beta_1\beta_2,\alpha_2^p=\beta_2^{s-r},\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$ and $s=0,1,\ldots,\frac{1}{2}(p-1)$ where $s-r$ and $2r-s$ are not divisible by p

$$\Phi_{19}(2211)\ell_{r} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^{r}=\alpha_1^p=\beta_1^{r},\alpha^p=\beta_1\beta_2,\alpha_2^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$

$$\Phi_{19}(2211)m_{r} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_1,\alpha_1^p=\beta_2^{r},\alpha_2^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha_1^p=\beta_1,\alpha^p=\alpha_2^p=\beta^p=\beta_i^p=1 \quad (i=1,2)\rangle$$

$$\Phi_{19}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^r,\alpha^p=\alpha_2^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(21^4)c_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha_1^p=\beta_1,\alpha_2^p=\beta_1^r,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(21^4)d_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^r,\alpha_2^p=\beta_1^{k/r}\beta_2^k,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν, where $k=g^s$ $(s=0,1,\ \ldots,\frac{1}{2}(p-3))$ and $d_{\nu,0}$ only exists for $p\equiv 1 \pmod 4$

$$\Phi_{19}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_2^r,\alpha^p=\alpha_2^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(21^4)f_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha_1^p=\beta_1\beta_2^r,\alpha_2^p=\beta_1^{-1/r}\beta_2^{-1},\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{19}(21^4)g = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_1,\alpha_i^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

$$\Phi_{19}(21^4)h = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha^p=\beta_1\beta_2,\alpha_i^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

$$\Phi_{19}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_1,\alpha^p=\alpha_i^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

(20) <u>Note</u>: In this family, $\alpha_2^{(p)}=\alpha_2^p\beta_1^{-\binom{p}{3}}$.

$$\Phi_{20}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha_2^{(p)}=\beta_2,\alpha_1^p=\beta_1,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

$$\Phi_{20}(2211)b_{r-1} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_2^{(p)}=\beta_2^r,\alpha_1^p=\beta_1,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=2,3,\ \ldots,p-1$, <u>and $p>3$</u>

$$\Phi_{20}(2211)c_{r,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_1^p=\beta_2^r,\alpha_2^{(p)}=\beta_1^r,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(2211)c_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^s=\alpha_1^p=\beta_2^s,\alpha_2^{(p)}=\beta_1^r\beta_2,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν and $s=1,2,\ \ldots,p-1$, where $1+4rs$ is a quadratic residue (mod p)

$$\Phi_{20}(2211)d_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha_2^{(p)}=\beta_2,\alpha_1^p=\beta_1\beta_2^r,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(2211)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^s=\alpha_1^p=\beta_2^s,\alpha_2^{(p)}=\beta_1^r\beta_2,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν, where $1+4rs=0$

$$\Phi_{20}(2211)f_{r,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_1^p=\beta_2^r,\alpha_2^{(p)}=\beta_1^{\nu/r},\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(2211)f_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^s=\alpha_1^p=\beta_2^s,\alpha_2^{(p)}=\beta_1^r\beta_2,\alpha^p=\beta^p=\beta_i^p=1 \ (i=1,2)$$

for $r=1$ or ν and $s=1,2,\ \ldots,p-1$, where $1+4rs$ is a non-quadratic residue (mod p)

$$\Phi_{20}(2211)g = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_2,\alpha_1^p=\beta_1,\alpha_2^{(p)}=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

$$\Phi_{20}(2211)h_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^k=\alpha_2^{(p)}=\beta_2^k,\alpha^p=\beta_1,\alpha_1^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

where $k=g^r$ for $r+1=1,2,\ \ldots,(p-1,3)$

$$\Phi_{20}(2211)i_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_1^p=\alpha_2^{(p)}=\beta_2^r,\alpha^p=\beta_1,\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1,2,\ \ldots,p-1$

$$\Phi_{20}(2211)j_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\alpha^p=\beta_2,\alpha_2^{(p)}=\beta_1^r,\alpha_1^p=\beta^p=\beta_i^p=1 \ (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$

$$\Phi_{20}(2211)k_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]^{\ell}=\alpha_1^p=\beta_2^{\ell}, \alpha^p=\beta_1, \alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

where $\ell = g^r$ for $r+1=1,2,\ldots,(p-1,4)$

$$\Phi_{20}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, \alpha_1^p=\beta_1, \alpha^p=\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

<u>for $p>3$</u>

$$\Phi_{20}(21^4)b = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\alpha_2^{(p)}=\beta_2, \alpha^p=\alpha_1^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

<u>for $p>3$</u>

$$\Phi_{20}(21^4)c_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, \alpha_2^{(p)}=\beta_1^r\beta_2, \alpha^p=\alpha_1^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(21^4)d_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]^r=\alpha_1^p=\beta_2^r, \alpha^p=\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, \alpha_2^{(p)}=\beta_1^r, \alpha^p=\alpha_1^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{20}(21^4)f = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\alpha^p=\beta_2, \alpha_1^p=\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{20}(21^4)g = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, \alpha^p=\beta_1, \alpha_1^p=\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{20}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, \alpha^p=\alpha_1^p=\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

(21) <u>Note</u>: In this family, $\alpha_1^{(p)}=\alpha_1^p\beta_2^{-\binom{p}{3}}$, $\alpha_2^{(p)}=\alpha_2^p\beta_1^{\binom{p}{3}}$ and $t = 1-\tfrac{1}{2}(p-1,4)$.

$$\Phi_{21}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\alpha_2^{(p)}=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{21}(2211)b_{r,s,1} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^r\beta_2^{(s-1)/\nu}, \alpha_2^{(p)}=\beta_1^{s+1}\beta_2^r, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,\tfrac{1}{2}(p-1)$ and $s=2,3,\ldots,p-2$ or 0, where $\nu r^2 \not\equiv s^2-1 \pmod p$ and s^2-1 is a non-quadratic residue (mod p)

$$\Phi_{21}(2211)b_{r,s,2} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^{r+1}\beta_2^{(s-t)/\nu}, \alpha_2^{(p)}=\beta_1^{s+t}\beta_2^{r-1}, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,p-1$ and $s=0,1,\ldots,\tfrac{1}{2}(p-1)$, where $\nu r^2 \not\equiv s^2-t^2+\nu \pmod p$ and $s^2-t^2+\nu$ is a non-quadratic residue (mod p)

$$\Phi_{21}(2211)c_{r,1} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^r, \alpha_2^{(p)}=\beta_1^2\beta_2^r, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,\tfrac{1}{2}(p-1)$

$$\Phi_{21}(2211)c_{r,2} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]^r=\alpha_2^{(p)}=\beta_2^r, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^r\beta_2^{-2/\nu}, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,\tfrac{1}{2}(p-1)$

$$\Phi_{21}(2211)c_{r,3} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^{r+1}\beta_2^{-1/(s+t)}, \alpha_2^{(p)}=\beta_1^{s+t}\beta_2^{r-1}, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$, where s is the smallest positive solution to $s^2\equiv t^2-\nu \pmod p$

$$\Phi_{21}(2211)d_{r,0,0} = \langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta, [\beta,\alpha_i]=\beta_i, [\alpha,\alpha_1]=\beta_2, [\alpha,\alpha_2]=\beta_1^{\nu}, \alpha_1^{(p)}=\beta_1^r\beta_2^{1/\nu}, \alpha_2^{(p)}=\beta_1\beta_2^r, \alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,\frac{1}{2}(p-1)$

$$\Phi_{21}(2211)d_{r,s,1}=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_1^{r}\beta_2^{(s-1)/\nu},\alpha_2^{(p)}=\beta_1^{s+1}\beta_2^{r},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,\frac{1}{2}(p-1)$ and $s=2,3,\ldots,p-2$ or 0, where s^2-1 is a quadratic residue (mod p)

$$\Phi_{21}(2211)d_{r,s,2}=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_1^{r+1}\beta_2^{(s-t)/\nu},\alpha_2^{(p)}=\beta_1^{s+1}\beta_2^{r-1},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,p-1$ and $s=0,1,\ldots,\frac{1}{2}(p-1)$, where $s^2-t^2+\nu$ is a quadratic residue (mod p)

$$\Phi_{21}(2211)e_r=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_2^{(p)}=\beta_2^{r},[\alpha,\alpha_2]=\alpha^{\nu p}=\beta_1^{\nu},\alpha_1^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,\frac{1}{2}(p-1)$

$$\Phi_{21}(2211)f_{r,s}=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_2^{(p)}=\beta_2^{r},[\alpha,\alpha_2]=\alpha^{\nu p}=\beta_1^{\nu},\alpha_1^{(p)}=\beta_2^{s},\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,\frac{1}{2}(p-1)$ and $s=1,2,\ldots,p-1$

$$\Phi_{21}(2211)g_r=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]^r=\alpha_1^{(p)}=\beta_2^{r},[\alpha,\alpha_2]=\alpha^{\nu p}=\beta_1^{\nu},\alpha_2^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=1,2,\ldots,p-1$

$$\Phi_{21}(21^4)a_{r,1}=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_1^{s}\beta_2^{(r-1)/\nu},\alpha_2^{(p)}=\beta_1^{r+1}\beta_2^{s},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=2,3,\ldots,p-2$ or 0, where r^2-1 is a non-quadratic residue (mod p) and s is the smallest positive solution to $\nu s^2\equiv r^2-1$ (mod p)

$$\Phi_{21}(21^4)a_{r,2}=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_1^{r+1}\beta_2^{(s-t)/\nu},\alpha_2^{(p)}=\beta_1^{s+t}\beta_2^{r-1},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0,1,\ldots,p-1$, where $\nu(r^2-1)+t^2$ is a quadratic residue (mod p), and s is the smallest positive solution to $s^2-t^2\equiv\nu(r^2-1)$ (mod p)

$$\Phi_{21}(21^4)b_r=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_2^{-2r/\nu},\alpha_2^{(p)}=\beta_1^{2(1-r)},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=0$ or 1

$$\Phi_{21}(21^4)b_r=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha_1^{(p)}=\beta_1\beta_2^{(s-t)/\nu},\alpha_2^{(p)}=\beta_1^{s+t}\beta_2^{-1},\alpha^p=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

for $r=2$ or 3, corresponding respectively to the two smallest positive solutions to $s^2-t^2\equiv -\nu$ (mod p)

$$\Phi_{21}(21^4)c=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\alpha^{\nu p}=\beta_1^{\nu},\alpha_i^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

$$\Phi_{21}(1^6)=\langle \alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha,\alpha_1]=\beta_2,[\alpha,\alpha_2]=\beta_1^{\nu},\alpha^p=\alpha_i^{(p)}=\beta^p=\beta_i^p=1\ (i=1,2)\rangle$$

(22) $$\Phi_{22}(21^4)a=\langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta_1,\beta_2]=\alpha^p=\alpha_3,\alpha_1^{(p)}=\beta_i^p=\alpha_{i+1}^p=1\ (i=1,2)\rangle$$

$$\Phi_{22}(21^4)b_r=\langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta_1,\beta_2]^r=\alpha_1^{(p)}=\alpha_3^{r},\alpha^p=\beta_i^p=\alpha_{i+1}^p=1\ (i=1,2)\rangle$$

for $r=1$ or ν

$$\Phi_{22}(21^4)c=\langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta_1,\beta_2]=\beta_1^p=\alpha_3,\alpha^p=\alpha_1^{(p)}=\beta_2^p=\alpha_{i+1}^p=1\ (i=1,2)\rangle$$

$$\Phi_{22}(21^4)d_r=\langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta_1,\beta_2]=\beta_1^p=\alpha_3,\alpha_1^{(p)}=\alpha_3^{r},\alpha^p=\beta_2^p=\alpha_{i+1}^p=1\ (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_{22}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\beta_1,\beta_2]=\alpha_3,\alpha^p=\alpha_1^{(p)}=\beta_i^p=\alpha_{i+1}^p=1 \ (i=1,2)\rangle$$

(23) $$\Phi_{23}(2211)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\alpha_1^p=\gamma,\alpha^p=\alpha_4,\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2,3)\rangle$$

<u>for $p > 3$</u>

$$\Phi_{23}(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\alpha_4,\alpha_1^p=\alpha_4^r\gamma,\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2,3)\rangle$$

for $r = 1,2, \ldots,p-1$ <u>and $p > 3$</u>

$$\Delta_{23}(2211)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^3=\alpha_4\gamma^r,\alpha_1^3\alpha_2^3\alpha_3=\alpha_4,\alpha_{i+1}^{(3)}=\gamma^3=1 \ (i=1,2,3)\rangle$$

for $r = 1$ or 2

$$\Phi_{23}(2211)c_{r,s} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\gamma^r,\alpha_1^{(p)}=\alpha_4^k,\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

for $r = 1$ or ν, where $k = g^s$ for $s + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{23}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\alpha_4,\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

$$\Phi_{23}(21^4)b_{r,0} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\gamma^r,\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

for $r = 1$ or ν

$$\Phi_{23}(21^4)b_{r,1} = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\alpha_4\gamma^k,\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$

$$\Phi_{23}(21^4)c_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha_1^{(p)}=\alpha_4^k,\alpha^p=\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{23}(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\alpha_1^p=\gamma,\alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2,3)\rangle$$

<u>for $p > 3$</u>

$$\Phi_{23}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha_1^p=\alpha_4^k\gamma,\alpha^p=\alpha_{i+1}^p=\gamma^p=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots(p-1,3)$ <u>and $p > 3$</u>

$$\Delta_{23}(21^4)e = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^3=\alpha_1^3\alpha_2^3\alpha_3=\alpha_4,\alpha_{i+1}^{(3)}=\gamma^3=1 \ (i=1,2,3)\rangle$$

$$\Phi_{23}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\gamma \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\gamma,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=\gamma^p=1 \ (i=1,2,3)\rangle$$

(24) $$\Phi_{24}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha^p=\alpha_4,\alpha_1^{(p)}=\beta^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_{24}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^k=\alpha_1^{(p)}=\alpha_4^k,\alpha^p=\beta^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{24}(21^4)c = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\beta^p=\alpha_4,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

$$\Phi_{24}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=\alpha_4,\alpha^p=\alpha_1^{(p)}=\beta^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

(25) and (26) $$\Phi_{25+x}(321) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_3,\alpha]=\alpha_4,\alpha^{p^2}=\alpha_4^y,\alpha_i^{(p)}=\alpha_{i+2}^y,\alpha_{i+2}^p=1 \ (i=1,2)\rangle$$

$$\Phi_{25+x}(222) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_3,\alpha]=\alpha_4,\alpha_i^{(p)}=\alpha_{i+2}^y,\alpha^{p^2}=\alpha_{i+2}^p=1 \ (i=1,2)\rangle$$

where $y = \nu^x$ and $x = 0$ (for Φ_{25}) or $x = 1$ (for Φ_{26})

(27) $$\Phi_{27}(21^4)a_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^k=[\alpha_1,\alpha_2]^k=\alpha^p=\alpha_4^k,\alpha_1^{(p)}=\beta^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$

$$\Phi_{27}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^k=[\alpha_1,\alpha_2]^k=\alpha_1^{(p)}=\alpha_4^k,\alpha^p=\beta^p=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{27}(21^4)c_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]^r=[\alpha_1,\alpha_2]^r=\beta^p=\alpha_4^r,\alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1 \ (i=1,2,3)\rangle$$

for $r = 1$ or ν

$$\Phi_{27}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\beta]=[\alpha_1,\alpha_2]=\alpha_4,\alpha^p=\alpha_1^{(p)}=\beta^p=\alpha_{i+1}^{(p)}=1\ (i=1,2,3)\rangle$$

(28) and (29) $\Phi_{28+x}(321)a_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_3,\alpha]=[\alpha_1,\alpha_2]=\alpha_4,\alpha^{p^2}=\alpha_4^r,\alpha_i^{(p)}=\alpha_{i+2}^y,\alpha_{i+2}^p=1\ (i=1,2)\rangle$

for $r = 1,2, \ldots,p-1$

$$\Phi_{28+x}(222) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_3,\alpha]=[\alpha_1,\alpha_2]=\alpha_4,\alpha_i^{(p)}=\alpha_{i+2}^y,\alpha^{p^2}=\alpha_{i+2}^p=1\ (i=1,2)\rangle$$

where $y = \nu^x$ and $x = 0$ (for Φ_{28}) or $x = 1$ (for Φ_{29})

(30) $\Phi_{30}(21^4)a = \langle \alpha,\alpha_1, \ldots,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_i,\beta]=\alpha_{i+2},[\alpha_3,\alpha]=\alpha^p=\alpha_4,\alpha_i^p=\beta^p=\alpha_{i+2}^p=1\ (i=1,2)\rangle$

<u>for $p > 3$</u>

$$\Phi_{30}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_i,\beta]=\alpha_{i+2},[\alpha_3,\alpha]^k=\alpha_1^{(p)}=\alpha_4^k,\alpha^p=\beta^p=\alpha_2^{(p)}=\alpha_{i+2}^p=1\ (i=1,2)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{30}(21^4)c = \langle \alpha,\alpha_1, \ldots,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_i,\beta]=\alpha_{i+2},[\alpha_3,\alpha]=\beta^p=\alpha_4,\alpha^p=\alpha_i^{(p)}=\alpha_{i+2}^p=1\ (1,2)\rangle$$

$$\Phi_{30}(21^4)d_r = \langle \alpha,\alpha_1, \ldots,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_i,\beta]=\alpha_{i+2},[\alpha_3,\alpha]=\beta^p=\alpha_4,\alpha_1^{(p)}=\alpha_4^k,\alpha^p=\alpha_2^{(p)}=\alpha_{i+2}^p=1\ (i=1,2)\rangle$$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,3)$

$$\Phi_{30}(1^6) = \langle \alpha,\alpha_1, \ldots,\alpha_4,\beta \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_i,\beta]=\alpha_{i+2},[\alpha_3,\alpha]=\alpha_4,\alpha^p=\alpha_i^{(p)}=\beta^p=\alpha_{i+2}^p=1\ (i=1,2)\rangle$$

(31) and (32) $\Phi_{31+x}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\alpha^p=\gamma,[\alpha_2,\beta_2]=\gamma^y,\alpha_i^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$

$$\Phi_{31+x}(21^4)b = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\alpha_2^p=\gamma,[\alpha_2,\beta_2]=\gamma^y\alpha^p=\alpha_1^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{31+x}(21^4)c = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\alpha_1^p=\gamma,[\alpha_2,\beta_2]=\gamma^y,\alpha_2^p=\gamma^j,\alpha^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

where j is the smallest positive solution to $j^2 + y \equiv 0 \pmod p$

$$\Phi_{31+x}(21^4)d = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\alpha^p=\alpha_1^p=\gamma,[\alpha_2,\beta_2]=\gamma^y,\alpha_2^p=\gamma^j,\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

where j is as in the previous group

$$\Phi_{31+x}(21^4)e_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\alpha_2^p=\gamma,[\alpha_2,\beta_2]=\gamma^y,\alpha^p=\gamma^r,\alpha_1^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r = 1,\nu$

$$\Phi_{31+x}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\alpha_1,\beta_1]=\gamma,[\alpha_2,\beta_2]=\gamma^y,\alpha^p=\alpha_i^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

where $y = \nu^x$ and $x = 0$ (for Φ_{31}) or $x = 1$ (for Φ_{32})

(33) <u>Note</u>: In this family, $\alpha_2^{(p)} = \alpha_2^p\gamma^{\binom{p}{3}}$

$$\Phi_{33}(21^4)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\beta_2,\alpha]=[\alpha_1,\beta_1]=\alpha_1^p=\gamma,\alpha^p=\alpha_2^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

$$\Phi_{33}(21^4)b_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\beta_2,\alpha]^r=[\alpha_1,\beta_1]^r=\alpha^p=\gamma^r,\alpha_1^p=\alpha_2^{(p)}=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_{33}(21^4)c_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\beta_2,\alpha]^r=[\alpha_1,\beta_1]^r=\alpha_2^{(p)}=\gamma^r,\alpha^p=\alpha_1^p=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

for $r = 1$ or ν

$$\Phi_{33}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i,[\beta_2,\alpha]=[\alpha_1,\beta_1]=\gamma,\alpha^p=\alpha_1^p=\alpha_2^{(p)}=\beta_i^p=\gamma^p=1\ (i=1,2)\rangle$$

(34) $\Phi_{34}(321)a = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i\ (i=1,2), [\beta_2,\alpha]=[\alpha_1,\beta_1]=\beta_1^p=\gamma, \alpha^p=\beta_1, \alpha_1^p=\beta_2, \alpha_2^p=\beta_2^p=\gamma^p=1\rangle$

$\Phi_{34}(321)b_r = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha]=\beta_i\ (i=1,2), [\beta_2,\alpha]=[\alpha_1,\beta_1]=\beta_1^p=\gamma, \alpha^p=\beta_1, \alpha_1^p=\beta_2, \alpha_2^p=\gamma^r, \beta_2^p=\gamma^p=1\rangle$

for $r = 1$ or ν

(35) $\Phi_{35}(21^4)a = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, \alpha^p=\alpha_5, \alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

$\Phi_{35}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, \alpha_1^{(p)}=\alpha_5^k, \alpha^p=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$

$\Phi_{35}(1^6) = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

(36) $\Phi_{36}(21^4)a_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]^k=\alpha^p=\alpha_5^k, \alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,6)$

$\Phi_{36}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_5, \alpha_1^{(p)}=\alpha_5^k, \alpha^p=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$ and $\underline{p > 3}$

$\Phi_{36}(1^6) = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_5, \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^{(p)}=1\ (i=1,2,3,4)\rangle$

(37) Note this family does not exist for $p = 3$

$\Phi_{37}(21^4)a_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_2,\alpha_3]^r=[\alpha_3,\alpha_1]^r=[\alpha_4,\alpha_1]^r=\alpha^p=\alpha_5^r, \alpha_1^p=\alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

for $r = 1$ or ν

$\Phi_{37}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_2,\alpha_3]=[\alpha_3,\alpha_1]=[\alpha_4,\alpha_1]=\alpha_1^p=\alpha_5, \alpha^p=\alpha_5^k, \alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$

$\Phi_{37}(21^4)b_p = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_2,\alpha_3]=[\alpha_3,\alpha_1]=[\alpha_4,\alpha_1]=\alpha_1^p=\alpha_5, \alpha^p=\alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

$\Phi_{37}(1^6) = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_2,\alpha_3]=[\alpha_3,\alpha_1]=[\alpha_4,\alpha_1]=\alpha_5, \alpha^p=\alpha_1^p=\alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

(38) Note this family does not exist for $p = 3$

$\Phi_{38}(21^4)a_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4\alpha_5^{-1}, [\alpha_1,\alpha_3]^k=\alpha^p=\alpha_5^k, \alpha_1^{(p)}=\alpha_{i+1}^p=1\ (i=1,2,3,4)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,5)$

$\Phi_{38}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4\alpha_5^{-1}, [\alpha_1,\alpha_3]^r=\alpha^p=\alpha_1^{(p)}=\alpha_5^r, \alpha_{i+1}^p=1\ (i=1,2,3,4)\rangle$

for $r = 1,2, \ldots,p-1$

$\Phi_{38}(21^4)b_{p+r} = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4\alpha_5^{-1}, [\alpha_1,\alpha_3]^k=\alpha_1^{(p)}=\alpha_5^k, \alpha^p=\alpha_{i+1}^p=1\ (i=1,2,3,4)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,4)$

$\Phi_{38}(1^6) = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4\alpha_5^{-1}, [\alpha_1,\alpha_3]=\alpha_5, \alpha^p=\alpha_1^{(p)}=\alpha_{i+1}^p=1\ (i=1,2,3,4)\rangle$

(39) Note this family does not exist for $p = 3$

$\Phi_{39}(21^4)a_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4, [\alpha_2,\alpha_3]^k=[\alpha_3,\alpha_1]^k=[\alpha_4,\alpha_1]^k=\alpha^p=\alpha_5^k, \alpha_1^p=\alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

where $k = g^r$ for $r + 1 = 1,2, \ldots,(p-1,6)$

$\Phi_{39}(21^4)b_r = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4, [\alpha_2,\alpha_3]^r=[\alpha_3,\alpha_1]^r=[\alpha_4,\alpha_1]^r=\alpha^p=\alpha_1^p=\alpha_5^r, \alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

for $r = 1,2, \ldots,p-1$

$\Phi_{39}(21^4)b_{p+r} = \langle \alpha,\alpha_1, \ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1}, [\alpha_1,\alpha_2]=\alpha_4, [\alpha_2,\alpha_3]^k=[\alpha_3,\alpha_1]^k=[\alpha_4,\alpha_1]^k=\alpha_1^p=\alpha_5^k, \alpha^p=\alpha_{i+1}^p=\alpha_5^p=1\ (i=1,2,3)\rangle$

where $k = g^r$ for $r + 1 = 1,2,\ldots,(p-1,5)$

$$\Phi_{39}(1^6) = \langle \alpha,\alpha_1,\ldots,\alpha_5 \mid [\alpha_i,\alpha]=\alpha_{i+1},[\alpha_1,\alpha_2]=\alpha_4,[\alpha_2,\alpha_3]=[\alpha_3,\alpha_1]=[\alpha_4,\alpha_1]=\alpha_5,\alpha^p=\alpha_1^p=\alpha_{i+1}^p=\alpha_5^p=1 \ (i=1,2,3)\rangle$$

(40) $$\Phi_{40}(21^4)a_r = \langle\alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\beta_1,\alpha_2]=[\beta_2,\alpha_1]=\alpha_1^p=\gamma,\alpha_2^p=\gamma^k,\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

where $k = g^r$ for $r + 1 = 1,2,\ldots,(p-1,3)$

$$\Phi_{40}(21^4)a_p = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\beta_1,\alpha_2]=[\beta_2,\alpha_1]=\alpha_1^p=\gamma,\alpha_2^p=\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

$$\Phi_{40}(1^6) = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\beta_1,\alpha_2]=[\beta_2,\alpha_1]=\gamma,\alpha_i^p=\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

(41) $$\Phi_{41}(21^4)a_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_1,\beta_1]^k=\alpha_1^p=\gamma^k,[\alpha_2,\beta_2]=\gamma^{-\nu},\alpha_2^p=\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

where $k = g^r$ for $r + 1 = 1,2,\ldots,(p-1,3)$

$$\Delta_{41}(21^4)a_1 = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_i,\beta_i]=\alpha_i^3=\gamma,\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

<u>for p = 3</u>

$$\Phi_{41}(1^6) = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_1,\beta_1]^{-\nu}=[\alpha_2,\beta_2]=\gamma^{-\nu},\alpha_i^p=\beta^p=\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

(42) $$\Phi_{42}(222)a_0 = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_1,\beta_2]=[\alpha_2,\beta_1]=\beta^p=\gamma,\alpha_1^p=\beta_1^{-1}\gamma^{-\frac{1}{2}},\alpha_2^p=\beta_2\gamma^{\frac{1}{2}},\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

$$\Phi_{42}(222)a_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\gamma \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_1,\beta_2]=[\alpha_2,\beta_1]=\beta^p=\gamma,\alpha_1^p=\beta_1^{-1}\gamma^{\frac{1}{2}},\alpha_2^p=\beta_2\gamma^{r+\frac{1}{2}},\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

for $r = 1,2,\ldots,p$

(43) $$\Phi_{43}(222)a_r = \langle \alpha_1,\alpha_2,\beta,\beta_1,\beta_2 \mid [\alpha_1,\alpha_2]=\beta,[\beta,\alpha_i]=\beta_i,[\alpha_1,\beta_1]^{-\nu}=[\alpha_2,\beta_2]=\gamma^{-\nu},\alpha_1^p=\beta_2\gamma^k \ \alpha_2^p=\beta_1^\nu\gamma^\ell,\beta^p=\gamma^n \ ;\beta_i^p=\gamma^p=1 \ (i=1,2)\rangle$$

where $n = \nu + \binom{p}{3}$, and k,ℓ are the smallest positive integers satisfying $(k-\nu)^2 - \nu(\ell+\nu)^2 \equiv r \pmod p$, for $r = 0,1,\ldots,p-1$.

Added in Proof. In her M.Sc. thesis (Australian National University, 1979), Miss A. M. Küpper has pointed out an error in the second line of the above list 4.6 (25) and (26). This should read:

"$$\Phi_{25+x}(222)a_r = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \mid [\alpha_i,\alpha] = \alpha_{i+1},[\alpha_3,\alpha]^y = \alpha_2^{(p)} = \alpha_4^y,\alpha_1^{(p)} = \alpha_3^y\alpha_4^{r-y},\alpha^{p^2} = \alpha_{i+2}^p = 1 \ (i = 1,2)\rangle$$

for $r = 0, 1, \ldots, \frac{1}{2}(p-1)$" yielding another $p-1$ groups of order p^6 (and so another 2 groups of order 3^5). I am indebted to her for this correction.

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