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Issues in Nonlinear Hyperperfect Numbers

By Daniel Minoli

Abstract. Hyperperfect numbers (HP) are a generalization of perfect numbers and as such share remarkably similar properties. In this note we show, among other things, that if $m = p_1^{\alpha_1} p_2^{\alpha_2}$ is 2-HP then $\alpha_2 = 1$, with $p_1 = 3$, $p_2 = 3^{\alpha_1+1} - 2$; this is in agreement with the structure of the perfect case (1-HP) stating that such a number is of the form $m = p_1^{\alpha_1} p_2$ with $p_1 = 2$ and $p_2 = 2^{\alpha_1+1} - 1$.

1. **Introduction.** Integers having "some type of perfection" have received considerable attention in the past few years. The most well-known cases are: perfect numbers, [1], [12]–[15]; multiperfect numbers, [1]; quasiperfect numbers, [2]; almost perfect numbers, [3]–[5]; semiperfect numbers, [16], [17]; and unitary perfect numbers, [11]. The related issue of amicable, unitary amicable, quasiamicable and sociable numbers [8], [10], [11], [9], [6], [7] has also been investigated extensively.

The intent of these variations of the classical definition appears to have been the desire to obtain a set of numbers, of nontrivial cardinality, whose elements have properties resembling those of the perfect case. However, none of the existing definitions generates a rich theory and a solution set having structural character emulating the perfect numbers; either such sets are empty, or their euclidean distance from zero is greater than some very large number, or no particularly unique prime decomposition form for the set elements can be shown to exist.

This is in contrast with the abundance (cardinally speaking) and the crystalized form of the n-hyperperfect numbers (n-HP), [18], [19]. These numbers are a natural extension of the perfect case, and as such share remarkably similar properties, as described below.

We begin with a few definitions.

Definition 1. A natural number m is said to be n-hyperperfect if for a positive integer n,

$$m = 1 + n[\sigma(m) - m - 1].$$

For $n = 1$, this reduces to the classical case. Table 1 lists all the n-HP ($n > 1$) up to 1,500,000.

Definition 2. $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}$ is said to be in canonical form if

$$p_1 < p_2 < \cdots < p_j.$$

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In the sequel we assume that numbers are always represented in canonical form; also, p_i refers invariably to a prime number.

Definition 3. If m is n-HP and $m = p_1^{\alpha_1} p_2$, we say that m is a linear n-Hp; otherwise, if $m = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_2 > 1$, we say that m is a nonlinear n-HP.

From observation of Table 1 it appears that the only hyperperfect numbers are the linear n-HP. In this paper we show that, indeed, some nonlinear forms are impossible; a more general theorem which would state that a necessary and sufficient form for a number to be n-HP is that it be a linear n-HP, remains to be established.

We confine our discussion to the product of two distinct primes, since this is the simplest case to analyze; naturally, the multiprime case must eventually be resolved if the conjecture alluded to above is to be established.

The following basic theorem of linear n-HP gives a sufficient form for a hyperperfect number, [19]:

THEOREM 1. *m is a linear n-HP if and only if*

$$p_2 = \frac{n p_1^{\alpha_1+1} - (n-1)p_1 - 1}{p_1^{\alpha_1+1} - (n+1)p_1^{\alpha_1} + n}.$$

2. **Theory**. We wish to show, among other things, that if $m = p_1^{\alpha_1} p_2^{\alpha_2}$ is 2-HP, then $\alpha_2 = 1$.

THEOREM 2. *If $m = p_1^{\alpha_1} p_2^{\alpha_2}$ is* 2-HP, *then* $p_1 = 3$.

Proof. We must have

$$3m = 2\sigma(m) - 1.$$

Substituting $m = p_1^{\alpha_1} p_2^{\alpha_2}$, we obtain

$$p_1^{\alpha_1}(2\Omega - 3\theta) + p_1^{\alpha_1-1} 2\Omega + \cdots + p_1 2\Omega + (2\Omega - 1) = 0,$$

where

$$\theta = p_2^{\alpha_2}, \qquad \Omega = p_2^{\alpha_2} + p_2^{\alpha_2-1} + \cdots + p_2 + 1.$$

Notice that

$$2\Omega - 3\theta = -p_2^{\alpha_2} + 2\left(\frac{p_2^{\alpha_2} - 1}{p_2 - 1}\right) < -p_2^{\alpha_2} + \frac{2}{p_2 - 1} p_2^{\alpha_2} = p_2^{\alpha_2}\left\{\frac{2}{p_2 - 1} - 1\right\} \leqslant 0$$

if $p_2 \geqslant 3$; the case $p_2 = 2$ is impossible. [20] shows that if

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

then any root z satisfies

$$\left|z + \frac{a_1}{a_0}\right| \leqslant 1, \quad \text{or} \quad |z| \leqslant 1 + \max_{2 \leqslant j \leqslant n-1} \left|\frac{a_j}{a_0}\right|, \quad \text{or} \quad |z| \leqslant \left|\frac{a_n}{a_0}\right|.$$

TABLE 1

n-HP up to 1,500,000, $n \geqslant 2$

n	m	Prime decomp of m
2	21	3×7
6	301	7×43
3	325	$5^2\times 13$
12	697	17×41
18	1,333	31×43
18	1,909	23×83
12	2,041	13×157
2	2,133	$3^3\times 79$
30	3,901	47×83
11	10,693	$17^2\times 37$
6	16,513	$7^2\times 337$
2	19,521	$3^4\times 241$
60	24,601	73×337
48	26,977	53×509
19	51,301	$29^2\times 61$
132	96,361	173×557
132	130,153	157×829
10	159,841	$11^2\times 1321$
192	163,201	293×557
2	176,661	$3^5\times 727$
31	214,273	$47^2\times 97$
168	250,321	193×1297
108	275,833	133×2441
66	296,341	67×4423
35	306,181	$53^2\times 109$
252	389,593	317×1229
18	486,877	79×6163
132	495,529	137×3617
342	542,413	499×1087
366	808,861	463×1747
390	1,005,421	479×2099
168	1,005,649	173×5813
348	1,055,833	401×2633
282	1,063,141	307×3463
498	1,232,053	691×1783
540	1,284,121	829×1549
546	1,403,221	787×1783
59	1,433,701	$89^2\times 181$

This translates into

$$\left|p_1 + \frac{2\Omega}{2\Omega - 3\theta}\right| \leqslant 1, \quad \text{or} \quad |p_1| \leqslant 1 + \left|\frac{2\Omega}{2\Omega - 3\theta}\right|, \quad \text{or} \quad |p_1| \leqslant \left|\frac{2\Omega - 1}{2\Omega - 3\theta}\right| < \left|\frac{2\Omega}{2\Omega - 3\theta}\right|.$$

Because $2\Omega - 3\theta < 0$, these three bounds may be shown to be equivalent to the bound

$$p_1 \leqslant 1 + \frac{2\Omega}{3\theta - 2\Omega}, \quad \text{or} \quad p_1 \leqslant 1 + \frac{2p_2^{\alpha_2} + 2p_2^{\alpha_2 - 1} + \cdots + 2}{p_2^{\alpha_2} - 2p_2^{\alpha_2 - 1} - \cdots - 2},$$

which, by long division, yields

$$p_1 \leqslant 3 + \frac{6p_2^{\alpha_2 - 1} + 6p_2^{\alpha_2 - 3} + \cdots + 6}{p_2^{\alpha_2} - 2p_2^{\alpha_2 - 1} - \cdots - 2},$$

or, using the sum formula,

$$p_1 \leqslant 3 + \frac{6p_2^{\alpha_2} - 6}{(p_2 - 3)p_2^{\alpha_2} + 2} = Q(p_2, \alpha_2).$$

Now, $Q(p_2, \alpha_2)$ decreases monotonically to 3 as $p_2 \longrightarrow \infty$, α_2 fixed; for $p_2 = 3$, $p_1 = 2$ is the only choice; for $p_2 = 5$, $p_1 = 2$ or 3; thus, the maximum value for $Q(p_2, \alpha_2)$ must be obtained for $p_2 = 7$; but here $p_1 \leqslant 4.5$, on the other hand, $Q(p_2, \alpha_2)$ increases monotonically to $3 + 6/(p_2 - 3)$ from below as $\alpha_2 \longrightarrow \infty$, p_2 fixed. Thus $p_1 \leqslant 3$. Since $p_1 \geqslant 3$, [19], we obtain the desired result. Q.E.D.

THEOREM 3. *There are no n-*HP *of the form* $m = p_1^{\alpha_1}p_2^2$ *with* $p_1 = n + 1$; *i.e., if an n-*HP *of the form* $p_1^{\alpha_1}p_2^2$ *exists, then* $p_1 > n + 1$.

Proof. If $m = p_1^{\alpha_1}p_2^2$ is n-HP, then p_2 must satisfy the polynomial

$$(n + 1)p_1^{\alpha_1}p_2^2 = n\frac{p_1^{\alpha_1 + 1} - 1}{p_1 - 1}(p_2^2 + p_2 + 1) - n + 1$$

or, solving in p_2,

$$p_2^2[p_1^{\alpha_1}(n + 1 - p_1) - n] + p_2[np_1^{\alpha_1 + 1} - n] + [np_1^{\alpha_1 + 1} + p_1 - np_2 - 1] = 0.$$

The discriminant of this quadratic can be shown to be

$$\Delta = p_1^{2(\alpha_1 + 1)}\left[n(n + 4) - \frac{4n(n + 1)}{p_1}\right] + p_1^{\alpha_1 + 1}\left[6n^2 - 8 + \frac{4(n + 1)}{p_1} - 4p_1(n - 1)\right]$$
$$+ [n(n - 4) - p_1(4n)(n - 1)].$$

Now if we assume that $p_1 = n + 1$, we get

$$\Delta = p_1^{2(\alpha_1 + 1)}[n^2] + p_1^{\alpha_1 + 1}[2n^2] + [n^2(1 - 4n)]$$
$$= n^2\{(p_1^{\alpha_1 + 1} + 1)^2 - 4n\} = n^2\{[(n + 1)^{\alpha_1 + 1} + 1]^2 - 4n\}.$$

However, $[(n+1)^{\alpha_1+1}+1]^2-4n$ cannot be a perfect square. $[(n+1)^{\alpha_1+1}+1]^2$ is a perfect square; the next (smaller) perfect square is $[(n+1)^{\alpha_1+1}]^2$; the distance between these two squares is $2(n+1)^{\alpha_1+1}+1$; but this is greater than $4n$ for all α and n. Therefore, $p_1 > n+1$. Q.E.D.

THEOREM 4. *There are no* 2-HP *of the form* $m = p_1^{\alpha_1}p_2^2$.

Proof. From Theorem 3, if $m = p_1^{\alpha_1}p_2^2$ is n-HP, $p_1 > n+1$; however, from Theorem 2 a 2-HP number must have $p_1 = n+1 = 3$. This is a contradiction. Thus, no 2-HP of the form $p_1^{\alpha_1}p_2^2$ exists. Q.E.D.

More importantly,

THEOREM 5. *There are no* 2-HP *of the form* $m = p_1^{\alpha_1}p_2^{\alpha_2}$, $\alpha_2 > 1$.

Proof. From Theorem 2, $p_1 = 3$; let $j = \alpha_1 + 1$. $(n+1)m = n\sigma(m) - n + 1$ leads to

$$3^j p_2^{\alpha_2} = (3^j - 1)(p_2^{\alpha_2} + p_2^{\alpha_2-1} + \cdots + 1) - n + 1$$

for this case.

We thus obtain

$$P(p_2) = p_2^{\alpha_2} - (3^j-1)p_2^{\alpha_2-1} - (3^j-1)p_2^{\alpha_2-2} - \cdots - 3^j + 2 = 0.$$

Using the same bound used earlier, we obtain $p_2 \leqslant 3^j$; because of the primality condition, $p_2 \leqslant 3^j - 1$. For $\alpha_2 > 1$ the following facts hold:

Fact 1. $P(3^j) = 2$ since we get a finite telescoping sequence.

Fact 2. $P(3^j - 1) < 0$ since

$$P(3^j-1) = -(3^j-1)^{\alpha_1-1} - (3^j-1)^{\alpha_1-2} - \cdots - (3^j-1) + 1 < 0.$$

Fact 3. $P(x) < 0$ for $1 < x < 3^j - 1$ since we have

$$P(x) = x^{\alpha_2} - (3^j-1)x^{\alpha_2-1} \underbrace{- (3^j-1)x^{\alpha_2-2} - \cdots - (3^j-1) + 1}_{\text{negative since } \alpha_2 > 1}.$$

Therefore,

$$\begin{aligned} P(x) &= x^{\alpha_2} - (3^j-1)x^{\alpha_2-1} - K < x^{\alpha_2} - (3^j-1)x^{\alpha_2-1} \\ &= x^{\alpha_2-1}(x - (3^j-1)). \end{aligned}$$

But $x < (3^j - 1)$, therefore, $x - (3^j-1) < 0$. This implies $P(x) < 0$. This shows that any positive root r of this polynomial satisfies $3^j - 1 < r < 3^j$, thus no integral values of p_2 exist.

COROLLARY 1. *If* $\alpha_2 = 1$, $p_2 = 3^j - 2$ *is the only allowable second prime.*

Proof. From Theorem 5, if $\alpha_2 = 1$, $P(p_2) = p_2 - 3^j + 2 = 0$ or $p_2 = 3^j - 2$. Q.E.D.

For the general n-HP case we can obtain a few (weaker) results as follows.

THEOREM 6. *If* $m = p_1^{\alpha_1} p_2^{\alpha_2}$ *is* n-HP, *then*

$$\alpha_2 \leqslant \frac{\operatorname{Log}[n(p_1^{\alpha_1} + p_1^{\alpha_1 - 1} + \cdots + 1) + (p_2 - 1)(n - 1)]}{\operatorname{Log} p_2}.$$

Proof. $(n + 1)m = m\sigma(m) - n + 1$ leads to

$$p_2^{\alpha_2}\left[n(p_1^{\alpha_1} + \cdots + 1)\frac{p_2}{p_2 - 1} - (n + 1)p_1^{\alpha_1}\right] = n(p_1^{\alpha_1} + \cdots + 1)\frac{1}{p_2 - 1} + n - 1,$$

so that

$$\alpha_2 = \operatorname{Log}\left[\frac{n(p_1^{\alpha_1} + \cdots + 1) + (p_2 - 1)(n - 1)}{n(p_1^{\alpha_1} + \cdots + 1)p_2 - (n + 1)(p_2 - 1)p_1^{\alpha_1}}\right]\Big/ \operatorname{Log} p_2$$

from which the result follows. This bound is actually attained in many circumstances. Q.E.D.

THEOREM 7. *If* $m = p_1^{\alpha_1} p_2^2$ *is* n-HP, *then* $n + 2 \leqslant p_1 \leqslant (n + 1)^2$.

Proof. As in Theorem 3,

$$(n + 1)p_1^{\alpha_1} p_2^2 = n\frac{p_1^{\alpha_1 + 1} - 1}{p_1 - 1}(p_2^2 + p_2 + 1) - n + 1;$$

or, solving in p_1,

$$p_1^{\alpha_1}[-p_2^2 + np_2 + n] + p_1^{\alpha_1 - 1}[n(p_2^2 + p_2 + 1)] + p_1^{\alpha_1 - 2}[n(p_2^2 + p_2 + 1)]$$
$$+ \cdots + p_1[n(p_2^2 + p_2 + 1)] + [np_2^2 + np_2 + 1] = 0.$$

Note that $-p_2^2 + np_2 + n = -p_2(p_2 - n) + n < 0$ since $p_2 > n$. Using the bounds of Theorem 2, from [20], we obtain

$$p_1 \leqslant 1 + \frac{np_2^2 + np_2 + n}{p_2^2 - np_2 - n} = n + 1 + \frac{(n + n^2)(p_2 + 1)}{p_2^2 - np_2 - n}$$

$$= n + 1 + \frac{(n + n^2)(p_2^2 - 1)}{[p_2 - (n + 1)]p_2^2 + n}.$$

This bound decreases monotonically to $n + 1$ as $p_2 \longrightarrow \infty$; thus the maximum is obtained at $p_2 = n + 2$ $(p_2 > p_1 \geqslant n + 1)$, so that

$$p_1 \leqslant (n + 1) + \frac{(n + n^2)(n + 3)}{(n + 2)^2 - n(n + 2) - n} = (n + 1) + \frac{n(n + 1)(n + 3)}{n + 4} \leqslant (n + 1)^2.$$

Note $p_1 \geqslant n + 2$ by Theorem 3 above. Q.E.D.

3. Conclusion. It appears empirically that the only n-HP are the linear n-HP. In this paper we have shown that there are no nonlinear 2-HP. A bound on α_2 for nonlinear n-HP, $n \neq 2$, has also been derived. A more general theorem stating that there are not nonlinear n-HP (for any n) is currently being sought.

Computer time (PDP 11/70) for Table 1 was over 10 hours.

Bell Telephone Laboratories
Holmdel, New Jersey 07733

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