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# Exponential Laws for Fractional Differences

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Abstract. In *Math. Comp.*, v. 28, 1974, pp. 185–202, Diaz and Osler gave the following (formal) definition for $\dot{\Delta}^\alpha f(z)$, the $\alpha$th fractional difference of $f(z)$: $\dot{\Delta}^\alpha f(z) = \Sigma_{p=0}^\infty A_p^{-\alpha-1} f(z+\alpha-p)$. They derived formulas and applications involving this difference. They asked whether their differences satisfied an exponent law and what the relation was between their differences and others, such as $\Delta^\alpha f(z) = \Sigma_{p=0}^\infty A_p^{-\alpha-1} f(z+p)$. In this paper an exponent law for their differences is established and a relation found between the two differences mentioned above. Applications of these results are given.

**1. Introduction.** In [2, p. 186] Diaz and Osler give the following definition for $\dot{\Delta}^\alpha f(z)$, the $\alpha$th fractional difference of $f(z)$:

$$(1) \qquad \dot{\Delta}^\alpha f(z) = \sum_{p=0}^\infty A_p^{-\alpha-1} f(z+\alpha-p),$$

where $A_p^{-\alpha-1} = \binom{p-\alpha-1}{p} = (-1)^p \binom{\alpha}{p}$. (Note: in [2] $\dot{\Delta}^\alpha$ is written $\Delta^\alpha$.)

Since $A_p^{-\alpha-1} = O(p^{-\alpha-1})$ as $p \longrightarrow \infty$, the series is convergent for every $z$, if $f(t) = O(t^{\alpha-\epsilon})$ $(\epsilon > 0)$ as $|t| \longrightarrow \infty$. Diaz and Osler show [2, p. 189], that if $z$ and $\alpha$ are fixed and if (in addition to the order condition above) $f(t)$ is analytic in a region $R$ containing the points $t = z + \alpha - p$, $p \geqslant 0$, then $\dot{\Delta}^\alpha f(z)$ may be put in the form of a line integral round a contour in $R$. They ask [2, p. 201] whether there is an exponent law for $\dot{\Delta}^\alpha f(z)$ of the form

$$(2) \qquad \dot{\Delta}^{r+s} f(z) = \dot{\Delta}^r \dot{\Delta}^s f(z).$$

If $s_n = f(n)$, we obtain formally, for the sequence $s_n$,

$$(3) \qquad \dot{\Delta}^\alpha s_n = \sum_{p=0}^\infty A_p^{-\alpha-1} s_{n+\alpha-p}.$$

If $\alpha = 0, 1, 2, \ldots$, the series terminates at $p = \alpha$, and gives successive "backward differences," starting (at $\alpha = 1$) with the difference $\dot{\Delta}^1 s_n = s_{n+1} - s_n$.

**2. An Exponent Law.** In [3] the following definition for the $\alpha$th fractional difference of a sequence $s_n$ was used:

$$(4) \qquad \Delta^\alpha s_n = \sum_{p=0}^\infty A_p^{-\alpha-1} s_{n+p},$$

the series being supposed summable in some Cesàro sense. The definition is due to

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Chapman [1]. For $\alpha = 0, 1, 2, \ldots,$ the series terminates at $p = \alpha$ and we get successive "forward differences" starting (at $\alpha = 1$) with the difference $\Delta^1 s_n = s_n - s_{n+1}$. In fact, as is easily verified,

$$\Delta^\alpha s_n = (-1)^\alpha \dot{\Delta}^\alpha s_n \qquad (\alpha = 0, 1, 2, \ldots). \tag{5}$$

If $\alpha$ is fractional, the formula (3) fails to make sense, since $\dot{\Delta}^\alpha s_n$ takes $s_n$ off its domain; further, (5) is no help since $(-1)^\alpha$ is neither real nor unique.

In [3, Theorem 1] the following exponent formula was obtained for the fractional differences (4):

$$\Delta^{r+s}_{(C,\lambda)} s_n = \Delta^r_{(C,\lambda+s+\epsilon)} \Delta^s s_n, \tag{6}$$

where $\lambda \geqslant -1$, $\lambda + s \geqslant -1$, $r + s \neq 0, 1, 2, \ldots$, $\epsilon = 0$ or $> 0$ according to whether $s$ is or is not an integer, and (unfortunately) $r < 0$ in the case $s \neq 0, 1, 2, \ldots$. Here it is assumed that the left side is summable $(C, \lambda)$. (The series giving $\Delta^s s_n$ is then automatically summable $(C, \mu)$, where $\mu \geqslant \max(\lambda + r, -1)$.)

Because of the failure to relate the definitions $\Delta^\alpha s_n$ and $\dot{\Delta}^\alpha s_n$ in the case $\alpha \neq 0, 1, 2, \ldots$, it did not seem likely that (6) could be of help in finding an exponent law of the type (2). However, if we write (2) out formally we obtain

$$\sum_{p=0}^{\infty} A_p^{-r-s-1} f(z + r + s - p) = \sum_{k=0}^{\infty} A_k^{-r-1} \sum_{m=0}^{\infty} A_m^{-s-1} f(z + r + s - k - m), \tag{7}$$

and if we write (6) out, we get

$$\sum_{p=0}^{\infty} A_p^{-r-s-1} s_{n+p} = \sum_{k=0}^{\infty} A_k^{-r-1} \sum_{m=0}^{\infty} A_m^{-s-1} s_{n+k+m}. \tag{8}$$

We see that in (7) the *same* values of $f$ are used on both sides, namely $f(z+r+s-q)$, where $q = 0, 1, 2, \ldots,$ the jump from $f(z)$ to $f(z + s - m)$ occasioned by $\dot{\Delta}^s$ being overlaid by the subsequent jump due to $\dot{\Delta}^r$. Thus, if we put

$$s_q = f(z + r + s - q) \qquad (q = 0, 1, 2, \ldots) \tag{9}$$

in (8), with $n = 0$, we obtain (7). We have thus obtained the following exponent law for Diaz and Osler's differences:

THEOREM 1. *Let* $\lambda \geqslant -1$, $\lambda + s \geqslant -1$, $r + s \neq 0, 1, 2, \ldots$, $\epsilon = 0$ *or* $> 0$ *according as* $s$ *is integral or fractional, and* $r < 0$ *if* $s \neq 0, 1, 2, \ldots$. *Then*

$$\dot{\Delta}^{r+s}_{(C,\lambda)} f(z) = \dot{\Delta}^r_{(C,\lambda+s+\epsilon)} \dot{\Delta}^s f(z), \tag{10}$$

*under the assumption that the left side is summable* $(C, \lambda)$.

**3. A "Converse" Exponent Law.** In [3, Theorem 3] a "converse" result to (6) is given, which in its "convergence" form [3, Theorem 3′] is as follows:

$$\Delta^{r+s}_{(C,0)} s_n = \Delta^r_{(C,0)} \Delta^s_{(C,0)} s_n \tag{11}$$

the two *right* side series being assumed convergent. Here (apart from the trivial cases $r = 0$ or $s = 0$) $r$ and $s$ must be in the first or fourth quadrant or inside the open triangles with vertices $(0, k)$, $(0, k + 1)$, $(-1, k + 1)$, $k = 0, 1, 2, \ldots$. From this we obtain the corresponding formula for Diaz and Osler's differences:

THEOREM 2. *If r and s are in the set S just described,*

$$\dot{\Delta}^{r+s}_{(C,0)}f(z) = \dot{\Delta}^{r}_{(C,0)}\dot{\Delta}^{s}_{(C,0)}f(z), \tag{12}$$

*the two right side series being supposed convergent.*

The last formula is useful in extending known results of Diaz and Osler. In [2, Table 2.1], they give $\dot{\Delta}^{\alpha}f(z)$ for some special functions $f(z)$. In each case it can be seen that the two series on the right side of (12) are convergent for the value of $\alpha$ (= $s$) given, and for $r$ and $s$ in the set $S$; hence, we know that the $r$th difference of the expression $\dot{\Delta}^{\alpha}f(z)$ ($\alpha = s$) given in the table is just the difference $\dot{\Delta}^{r+s}$ of the function $f(z)$. In short, the functions $f(z)$ given in the table all satisfy the exponent law (12) with suitable restrictions on $r$ and $s$.

**4. An Example.** As an example of the above, let

$$f(z) = z^{(p)} = \frac{\Gamma(z+1)}{\Gamma(z+1-p)}.$$

Then by [2, Table 2.1], with $s$ for $\alpha$,

$$\dot{\Delta}^{s}f(z) = \frac{\sin(\pi z)\Gamma(s-p)z^{(p-s)}}{\sin(\pi(z+s))\Gamma(-p)} \tag{13}$$

for $s > p$. (It is assumed that both $z^{(p)}$ and $\dot{\Delta}^{s}f(z)$ are defined by continuity at points of removable singularity, and that $z, p, s$ are chosen so as to avoid points of unremovable singularity in either of them; thus if, in $z^{(p)}$, $z$ is a negative integer, so must $z - p$ be, and if, in $\dot{\Delta}^{s}f(z)$, $z + s$ is an integer, then $p$ is 0 or a positive integer.) Now

$$\dot{\Delta}^{r}\dot{\Delta}^{s}f(z) = \sum_{k=0}^{\infty} A_k^{-r-1}(\dot{\Delta}^{s}f(z+r-k)). \tag{14}$$

Replacing $z$ by $z + r - k$ in (13), we see that $\dot{\Delta}^{s}f(z + r - k)$ is $O(|z| + k)^{p-s}$ as $k \longrightarrow \infty$. Hence, since $A_k^{-r-1}$ is $O(k^{-r-1})$, the series in (14) converges if $r + s > p$. (To avoid unremovable singularities in the terms of the series of (14) we see that if, for any $k$, $z + r - k + s$ is an integer, then we must take $p = 0$ or a positive integer; and it is gratifying to see that this happens if and only if, whenever $z + r + s$ is an integer, then $p = 0, 1, 2, \ldots$, which is the criterion that $\dot{\Delta}^{r+s}f(z)$ has no unremovable singularity.)

Hence the equality in (12) is true for $f(z) = z^{(p)}$ with $s > p$, $r + s > p$, and $r, s$ in the set $S$ (and, of course, $p = 0, 1, 2, \ldots,$ if $z + r + s$ happens to be an integer). In particular, if $p \geqslant 0$ and $z + r + s$ is nonintegral, (12) is true if $s > p$ and $r > 0$, a useful case. The arguments for the other functions $f(z)$ of Table 2.1 are similar.

**5. Relation Between $\Delta^{\alpha}$ and $\dot{\Delta}^{\alpha}$.** Although there is no extension of the Diaz-Osler differences (1) to sequences, for $\alpha$ fractional, there is an immediate extension of the differences (4) to functions $f(z)$:

$$\Delta^{\alpha}f(z) = \sum_{p=0}^{\infty} A_p^{-\alpha-1}f(z+p). \tag{15}$$

Diaz and Osler ask [2, p. 201] whether there is a relation between (1) and other differences. Now for $\alpha = 0, 1, 2, \ldots$, we can replace $\infty$ in (1) by $\alpha$ and then replace $p$ by $\alpha - p$. This shows that by (15),

$$\Delta^\alpha f(z) = (-1)^\alpha \dot{\Delta}^\alpha f(z). \tag{16}$$

But as with $s_n$ in (5), this has no meaning for $\alpha$ fractional.

Let us write for given fixed $z$ and $\alpha$,

$$g(u) = f(2z + \alpha - u). \tag{17}$$

Then it is easy to verify:

THEOREM 3. *If the series for either side converges, then*

$$\Delta^\alpha f(z) = (\dot{\Delta}^\alpha g(u))_{u=z}$$

*where* $g(u)$ *is given by* (17).

This enables us to calculate $\Delta^\alpha f(z)$ from known differences of the $\dot{\Delta}$ type. For example, let

$$f(z) = z^{(p)} = \frac{\Gamma(z+1)}{\Gamma(z-p+1)}.$$

Then

$$g(u) = f(2z + \alpha - u) = (2z + \alpha - u)^{(p)} = \frac{\Gamma(2z + \alpha + 1 - u)}{\Gamma(2z + \alpha + 1 - p - u)} = \frac{\Gamma(A - u)}{\Gamma(B - u)},$$

say. Thus by [2, Table 2.1, #4],

$$\begin{aligned} \Delta^\alpha f(z) = (\dot{\Delta}^\alpha g(u))_{u=z} &= \left(\frac{\Gamma(B - A + \alpha)\Gamma(A - \alpha - u)}{\Gamma(B - A)\Gamma(B - u)}\right)_{u=z} \qquad (B - A > -\alpha) \\ &= \left(\frac{\Gamma(-p + \alpha)\Gamma(2z + 1 - u)}{\Gamma(-p)\Gamma(2z + \alpha + 1 - p - u)}\right)_{u=z} \\ &= \frac{\Gamma(\alpha - p)\Gamma(z + 1)}{\Gamma(-p)\Gamma(z + \alpha + 1 - p)} \\ &= \frac{\Gamma(\alpha - p)}{\Gamma(-p)} z^{(p-\alpha)} \end{aligned}$$

for $\alpha > p$.

By (13) this gives

$$\Delta^\alpha f(z) = \frac{\sin(\pi(z + \alpha))}{\sin(\pi z)} \dot{\Delta}^\alpha f(z) \tag{18}$$

when $\alpha > p$, which is a direct extension of (16) to fractional values of $\alpha$.

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