MATHEMATICS OF COMPUTATION, VOLUME 35, NUMBER 152
OCTOBER 1980, PAGES 1081–1092

Spectral and Pseudo Spectral Methods for Advection Equations*

By Joseph E. Pasciak

Abstract. Spectral and pseudo spectral methods for advection equations are investigated. A basic framework is given which allows the application of techniques used in finite element analysis to spectral methods with trigonometric polynomials. Error estimates for semidiscrete spectral and pseudo spectral as well as fully discrete explicit pseudo spectral methods are given. The approximation schemes are shown to converge with infinite order.

1. **Introduction.** Spectral and pseudo spectral methods have become popular in approximating solutions of advection equations arising in many sciences. Christensen and Prahm have developed spectral models for dispersion of atmospheric pollutants [4]. Gazdag proposes spectral methods for advection equations and Burger's equation [6]. Gottlieb and Orzag present many spectral applications in [7]. Numerical evidence supporting spectral type approximations is abundant in the literature, see for example [4], [5], [6], [7], [11]. Numerical tests indicate that spectral methods outperform finite difference methods for many hyperbolic problems [7]. The general consensus among users is that spectral methods work well whenever they are stable.

Recently there have been many theoretical advances in the understanding of these methods. Results on stability of semidiscrete spectral type methods have been given in [5], [9], [10]. Investigations into mollifying the method for nonsmooth initial data have appeared in [9], [10]. Error estimates are implied in the above literature, however, the explicit dependence of convergence on smoothness of initial data is not always given. Also, the compatibility requirements on the initial data at the boundary necessary for convergence are not stated.

In this paper a basic framework is given which allows the application of techniques used in finite element theory to spectral and pseudo spectral methods with trigonometric polynomials. The spectral method is just a Galerkin projection and usual finite element analysis leads to error estimates. Unfortunately, the matrices corresponding to the spectral methods are not sparse and any implementation is costly. Pseudo spectral

Received May 11, 1979; revised March 18, 1980.
1980 *Mathematics Subject Classification.* Primary 65M10, 65M15.
*The submitted manuscript has been authored under contract EY-76-C-02-0016 with the U. S. Department of Energy. Accordingly, the U. S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U. S. Government purposes.

0025-5718/80/0000-0152/$04.00

methods involve collocation projections instead of L^2 projections. Using a result given in [9], the finite element analysis carries over for pseudo spectral methods. Furthermore, by use of the fast Fourier transform (FFT) algorithms, implementation of explicit pseudo spectral schemes can be accomplished economically.

For simplicity, the analysis in this paper only deals with advection equations with coefficients which are constant in time. Extensions of the methods and analysis to problems with coefficients that vary with time or to nonlinear problems are possible.

It is illustrated in this paper that spectral techniques can lead to rapidly convergent approximations to evolution equations. A necessary condition for rapid global convergence is that the solution of the equation can be approximated accurately in the finite-dimensional subspace of trigonometric polynomials. The domain under consideration will always be rectangular and periodic boundary conditions will be imposed. In addition, assumptions on smoothness and compatibility of initial data and coefficients of the advection equation shall be made.

The rest of the paper is broken into four parts. In Section 2, the advection equation is defined and "a priori" regularity estimates for its solutions are proven. In Sections 3 and 4 spectral and pseudo spectral approximations are introduced, and error estimates are established for the corresponding semidiscrete schemes. The convergence for the semidiscrete spectral and pseudo spectral approximation is of order s for compatible initial data in the Sobolev space of order $s + 1$. In Section 5, conditions for stability and convergence of fully discrete explicit pseudo spectral schemes are given, and error estimates for the explicit pseudo spectral method are proven.

2. Regularity for the Advection Equation. We shall consider advection equations on rectangles in n-dimensional Euclidean space $\mathbf{R}^n$ with periodic boundary conditions. By changing variables, we may assume, without loss of generality, that the rectangle is the unit rectangle

$$\Omega = [0, 1]^n \equiv [0, 1] \times [0, 1] \times \cdots \times [0, 1].$$

A function f defined on $\mathbf{R}^n$ is periodic if $f(x + z) = f(x)$ for every point x in Ω and every multi-integer z in Z^n. The space $C_p^\infty(\Omega)$ is the set of infinitely differentiable periodic functions defined on $\mathbf{R}^n$. Let $\| \ \|$ denote the L^2 norm on Ω.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-integer with nonnegative entries. Denote $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and define

$$D_\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

For positive integers s, the Sobolev norms on $C_p^\infty(\Omega)$ are given by

$$\|u\|_s = \Bigl(\sum_{|\alpha| \leq s} \|D_\alpha u\|^2\Bigr)^{1/2}.$$

We also denote the Sobolev seminorms

$$|u|_j = \Bigl(\sum_{|\alpha| = j} \|D_\alpha u\|^2\Bigr)^{1/2}.$$

Let $H_p^s(\Omega)$ be the completion of $C_p^\infty(\Omega)$ under the norm $\| \ \|_s$. The space $H_p^s(\Omega)$ is a Hilbert space under the obvious inner product. For $s \geqslant 0$, define the space $H_p^s(\Omega)$ by interpolation; see [8].

Define the unbounded operator L on $L^2(\Omega)$ by

$$LU = \sum_{i=1}^n V_i \frac{\partial}{\partial x_i} U + \frac{\partial}{\partial x_i}(V_i U),$$

with domain $\mathcal{D}(L) = H_p^1(\Omega)$. We assume for convenience that $V_i(x)$ is in $C_p^\infty(\Omega)$. It follows that multiplication by V_i is a bounded operator from $H_p^k(\Omega)$ into $H_p^k(\Omega)$. Hence L is a bounded operator from $H_p^k(\Omega)$ into $H_p^{k-1}(\Omega)$.

The advection problem is defined by

$$\begin{aligned} &D_t u + Lu = 0, \qquad \Omega \times [0, \infty), \\ &u(x, 0) = u_0(x), \qquad x \in \Omega, \\ &u \text{ periodic on } \partial\Omega. \end{aligned} \tag{2.1}$$

Specifically, the boundary condition u periodic on $\partial\Omega$ means for smooth u that $u(\cdot, t)$ is in $H_p^1(\Omega)$ for every t. The following theorem characterizes the solutions of (2.1).

THEOREM 1. *Let $r \geqslant 1$ and u_0 be in $H_p^r(\Omega)$. There exists a unique function $u(x, t)$ in $C([0, T_0], H_p^r(\Omega))$ satisfying* (2.1) *and a constant C not depending on u_0 or t such that*

$$\|u(\cdot, t)\|_r \leqslant C\|u_0\|_r \quad \text{for } t \text{ in } [0, T_0].$$

Before proving Theorem 1, we shall give an alternative characterization for the spaces $H_p^k(\Omega)$. Let $M = (M_1, \ldots, M_n)$ be a multi-integer and define

$$\varphi_M(X) = \operatorname{Exp}(2\pi i M \cdot X) \quad \text{for } X \text{ in } \mathbf{R}^n.$$

Let

$$\lambda_M = 1 + 4\pi^2 \sum_{j=1}^n M_j^2.$$

φ_M is an eigenfunction with eigenvalue λ_M for the elliptic problem

$$W - \Delta W = f \quad \text{in } \Omega, \qquad W \text{ periodic on } \partial\Omega,$$

where Δ is the Laplace operator $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. For $r \geqslant 0$ consider the sum

$$\sum_{M \in Z^n} \lambda_M^r |\beta_M|^2, \tag{2.2}$$

where $\beta_M = (u, \varphi_M)$ and $(\cdot, \cdot)$ denotes the L_2 inner product on Ω given by

$$(u, v) = \int_\Omega u\bar{v}\, dx.$$

Let $\dot{H}^r$ be the set of functions u in $L^2(\Omega)$ such that the sum in (2.2) is finite. The space $\dot{H}^r$ has the natural norm

$$\|u\|_r = \left(\sum_{M \in Z^n} \lambda_M^r |\beta_M|^2 \right)^{1/2}. \tag{2.3}$$

LEMMA 1. *For $r \geq 0$, the spaces $H_p^r(\Omega)$ and $\dot{H}^r$ coincide and their norms are equivalent.*

Proof. Let k be an even integer and u be in $\dot{H}^k$. Then u is obviously in $H^k(\Omega)$ and the norm given by (2.3) is equivalent to the usual Sobolev norm. The partial sums of the series $u = \Sigma_{M \in Z^n} \beta_M \varphi_M$ exhibit u as an $H^k(\Omega)$ limit of functions in $C_p^\infty(\Omega)$. Hence $\dot{H}^k$ is contained in $H_p^k(\Omega)$.

Let u be in $H_p^k(\Omega)$ and $g = (I - \Delta)^{k/2} u$. Then g is in $L^2(\Omega)$ and the function

$$w = \sum_{M \in Z^n} \lambda_M^{-k/2} (g, \varphi_M) \varphi_M \tag{2.4}$$

is in $\dot{H}^k$ and hence $H_p^k(\Omega)$. Furthermore $(I - \Delta)^{k/2} w = g$. The operator $(I - \Delta)$ is an injective map of $H_p^s(\Omega)$ into $H_p^{s-2}(\Omega)$ so w and u are identical. Hence $H_p^k(\Omega)$ is contained in $\dot{H}^k$ which proves the lemma for even k. Since $\dot{H}^r$ and $H_p^r(\Omega)$ are Hilbert scales, the lemma follows by interpolation [8].

Proof of Theorem 1. The proof of Theorem 1 is essentially a proof given by Taylor in [12]. We shall indicate the changes in Taylor's proof due to our boundary conditions.

Define the mollifier J_ϵ by

$$J_\epsilon U = \sum_{M \in Z^n} e^{-\epsilon |M|^2} (U, \varphi_M) \varphi_M.$$

By Lemma 1, it is evident that J_ϵ is a bounded map of $H_p^k(\Omega)$ into $H_p^{k+r}(\Omega)$ for any real r. Furthermore, for u in $H_p^k(\Omega)$,

$$\|J_\epsilon u\|_k \leq \|u\|_k.$$

The proof of Theorem 1 continues by exhibiting the solution of (2.1) as a limit of solutions of the problems

$$\frac{\partial u_\epsilon}{\partial t} + J_\epsilon L J_\epsilon u_\epsilon = 0, \qquad \Omega \times [0, T_0],$$

$$u_\epsilon(x, 0) = u_0(x), \qquad x \in \Omega,$$

$$u_\epsilon \text{ periodic on } \partial\Omega.$$

From here on the proof proceeds exactly as in Taylor [12] and hence shall be omitted.

3. The Semidiscrete Spectral Approximation. Let M be a multi-integer, $M = (M_1, \ldots, M_n) \in Z^n$, and define $\varphi_M(X) = \operatorname{Exp}(2\pi i M \cdot X)$ for X in $\mathbf{R}^n$. Let

$$\Omega_N = \{(M_1, \ldots, M_n) | -N + 1 \leq M_j \leq N \text{ for } j = 1, \ldots, n\}.$$

The approximation spaces S_N are defined to be the span of φ_M as M varies over Ω_N. We note the approximation properties of S_N given in [3]: Let $m \geqslant j \geqslant 0$. There exists a constant C, not depending on w in $H_p^m(\Omega)$ or N, such that

$$\|w - P_N w\|_j \leqslant CN^{j-m}|w|_m, \tag{3.1}$$

where P_N is the L^2 projection onto S_N. The subspaces S_N also have the inverse properties, for u in S_N,

$$\|u\|_{k+j} \leqslant CN^j\|u\|_k.$$

The spectral approximation L_N is the L^2 projection of L into the subspace S_N, that is

$$L_N f = P_N Lf. \tag{3.2}$$

The operator L_N is obviously skew symmetric on S_N. Since S_N is finite dimensional, L_N generates a unique unitary semigroup. Let U_N be the solution to

$$D_t U_N + L_N U_N = 0, \qquad U_N(0) = P_N u_0.$$

Error analysis for the semidiscrete approximation U_N could proceed by the usual finite element approach. We could first prove convergence estimates for $(I + L_N)^{-1}$ as an approximate to $(I + L)^{-1}$. Then the usual techniques (see [2]) would give rise to semidiscrete error estimates. In general, $(I + L)^{-1}$ is not a smoothing operator and standard techniques in finite element theory only give that $(I + L_N)^{-1}$ is a suboptimal approximation to $(I + L)^{-1}$. Thus, any analysis requiring approximation of $(I + L)^{-1}$ may lead to inferior convergence estimates. In the analysis in the rest of this paper we shall always approximate L.

THEOREM 2. *Let u_0 be in $H_p^{s+1}(\Omega)$ and u be the solution of the advection equation* (2.1). *There is a constant C independent of u_0 and N satisfying*

$$\|u(t) - U_N(t)\|_0 \leqslant CN^{-s}\|u_0\|_{s+1}, \qquad t \in [0, T_0].$$

Proof. Let $X(t) = P_N u(t)$ and $v(t) = u(t) - X(t)$. Then for θ in S_N

$$(D_t X, \theta) + (LX, \theta) = -(Lv, \theta). \tag{3.3}$$

Using the definition of L_N, we have for θ in S_N

$$(D_t U_N, \theta) + (LU_N, \theta) = 0. \tag{3.4}$$

Let $e = U_N - W$. Subtracting (3.3) and (3.4) and setting $\theta = e$ gives $(D_t e, e) + (Le, e) = (Lv, e)$. A similar argument shows $(e, D_t e) + (e, Le) = (e, Lv)$, and since L is skew symmetric,

$$2\|e\|D_t\|e\| = D_t\|e\|^2 = 2\,\mathrm{Re}(e, Lv) \leqslant 2\|e\|\,\|Lv\|.$$

Integrating the above equation gives

$$\|e\| \leqslant C \sup_{t \in [0, T_0]} \|Lv(t)\|. \tag{3.5}$$

By the approximation properties of S_N and Theorem 1

$$\|Lv\| \leqslant C\|u - P_N u\|_1 \leqslant CN^{-s}\|u_0\|_{s+1}. \tag{3.6}$$

Combining (3.5) and (3.6) and again using (3.1),

$$\|u(t) - U_N(t)\| \leqslant \|e(t)\| + \|v(t)\| \leqslant CN^{-s}\|u_0\|_{s+1},$$

which completes the proof of the theorem.

Remark. I suspect that Theorem 2 is not sharp. Indeed, for V_j constant, one can prove the stronger result that N^{-s} convergence is achieved with initial data in $H_p^s(\Omega)$. The proof does not generalize to nonconstant V_j due to the lack of commutativity between P_h and multiplication by V_j. Numerical tests on a few variable coefficient problems suggest that the stronger result holds. The numerical results, however, could be misleading since the cases computed were by no means extensive.

4. The Semidiscrete Pseudo Spectral Approximation. The spectral approximation L_N, defined by (3.2), is given by the alternative formula

$$L_N f = \sum_{j=1}^{n} P_N\left(V_j \frac{\partial}{\partial x_j} f\right) + \frac{\partial}{\partial x_j} P_N(V_j f). \tag{4.1}$$

That $\partial/\partial x_j$ and P_N commute is readily seen by expansion in the basis of trigonometric polynomials. The pseudo spectral approximation to L will be defined by replacing the L^2 projections in (4.1) by collocation projections.

Let $h = 1/2N$ and set $x_j = jh$. Let Ω_x be the collection of points

$$\Omega_x = \{(x_{j_1}, \ldots, x_{j_n}) | 0 \leqslant j_k < 2N\}.$$

For any continuous function f on $\bar{\Omega}$, define $P_c f$ to be the function in S_N which interpolates f on the grid points of Ω_x. To see that P_c is well defined, we shall introduce discrete Fourier transforms. Let C_x be the space of complex-valued functions on Ω_x and define the inner product

$$(f, g)_x = h^n \sum_{y \in \Omega_x} f(y)\bar{g}(y) \quad \text{for } f, g \in C_x.$$

Let C_N be the space of complex-valued functions on Ω_N and define the inner product

$$(f, g)_N = \sum_{I \in \Omega_N} f(I)\bar{g}(I) \quad \text{for } f, g \in C_N.$$

The discrete Fourier transform $\digamma_N$ maps C_x onto C_N and is defined by

$$\digamma_N f(I) = h^n \sum_{y \in \Omega_x} f(y) \operatorname{Exp}(-2\pi i y \cdot I).$$

$\digamma_N$ is a unitary transformation from C_x onto C_N. The inverse of $\digamma_N$ is given by

$$\digamma_N^{-1} f(y) = \sum_{I \in \Omega_N} f(I) \operatorname{Exp}(2\pi i y \cdot I).$$

We shall see that $P_c f$ is the function g given by

$$g(y) = \sum_{I \in \Omega_N} \mathcal{F}_N f(I) \operatorname{Exp}(2\pi i y \cdot I). \tag{4.2}$$

Indeed, the inversion formula $\mathcal{F}_N^{-1} \mathcal{F}_N = I$ implies that g is a function in S_N which equals f at each point of Ω_x. An easy exercise in linear algebra shows that g is the unique function in S_N assuming the values of f on Ω_x. Thus P_c is well defined. We note that (4.2) gives an algorithm for finding the coefficients of $P_c f$ in the basis of trigonometric polynomials given the nodal values of f on Ω_x. Also note that the inner product $(\cdot, \cdot)_x$ is defined so that P_c is a unitary transformation of C_x onto S_N with L^2 inner product. Thus for functions in S_N, the $(\cdot, \cdot)_x$ inner product and L^2 inner product and hence their respective norms are interchangeable.

The next theorem is essentially an n-dimensional version of Theorem 3.3 of [9] and demonstrates that P_c has approximation properties similar to those given by (3.1) for P_N. We shall give a new proof of Theorem 3 using arguments which are similar to those used to derive approximation properties for finite element interpolation.

THEOREM 3. *Let $0 \leq j \leq m$ and $m > n/2$. There exists a constant C, independent of w in $H_p^m(\Omega)$ and N, such that*

$$\|w - P_c w\|_j \leq C N^{-m+j} |w|_m .$$

Proof. First we introduce the notation for the proof. Let $\hat{\Omega}$ be the rectangle $[0, 2N]^n$. For a function f defined on Ω, let $\hat{f}$ be the function defined on $\hat{\Omega}$ by $\hat{f}(x) = f(x/2N)$. Let $\hat{S}_N$ be the image of S_N under the above dilation. We also denote the seminorms on $\hat{\Omega}$,

$$|f|_{j,\hat{\Omega}} = \left(\sum_{|\alpha|=j} \|D_\alpha f\|^2_{L^2(\hat{\Omega})} \right)^{1/2},$$

and the corresponding Sobolev norms

$$\|f\|_{j,\hat{\Omega}} = \left(\sum_{k=0}^{j} |f|^2_{k,\hat{\Omega}} \right)^{1/2}.$$

We make the following observations:

(i) $\hat{S}_N$ is a space of trigonometric polynomials and the interpolation projection $\hat{P}_c$ satisfying $\hat{P}_c \hat{f}(2Nx) = \hat{f}(2Nx)$, for all x in Ω_x, is well defined.

(ii) $\widehat{P_c f} = \hat{P}_c \hat{f}$ for all f in C_x.

(iii) $(\hat{P}_c - I) f = 0$ for f in $\hat{S}_N$.

(iv) For w in $H_p^m(\Omega)$,

$$\|\hat{w} - \widehat{P_N w}\|_{m,\hat{\Omega}} \leq C |\hat{w}|_{m,\hat{\Omega}} .$$

Observations (i), (ii), and (iii) follow easily from the definitions of S_N, P_c, $\hat{P}_c$ and earlier arguments. For (iv) we first note that a change of variables implies

$$|f|_k = (2N)^{k-n/2}|\hat{f}|_{k,\hat{\Omega}}. \tag{4.3}$$

Using (3.1) and (4.3) gives

$$\|\hat{w} - \widehat{P_N w}\|^2_{m,\hat{\Omega}} = \sum_{k=0}^{j} (2N)^{n-2k}|w - P_N w|_k^2$$
$$\leqslant C(2N)^{-2m+n}|w|_m^2 = C|\hat{w}|^2_{m,\hat{\Omega}},$$

which proves (iv).

Using (ii) and (iii), we can compute

$$\|w - P_c w\|_j \leqslant (2N)^{j-n/2}\|\hat{w} - \hat{P}_c \hat{w}\|_{j,\hat{\Omega}}$$
$$\leqslant (2N)^{j-n/2}\|(I - \hat{P}_c)(\hat{w} - \widehat{P_N w})\|_{j,\hat{\Omega}}.$$

Using (iv) and (4.3) gives

$$\|w - P_c w\|_j \leqslant CN^{j-n/2}\|I - \hat{P}_c\|_{L(H^m(\hat{\Omega}),H^j(\hat{\Omega}))}|\hat{w}|_{m,\hat{\Omega}}$$
$$\leqslant CN^{j-m}\|I - \hat{P}_c\|_{L(H^m(\hat{\Omega}),H^j(\hat{\Omega}))}|w|_m.$$

Thus, *the theorem will follow if we can bound the operator norm of* $\hat{P}_c$ *in* $L(H^m(\hat{\Omega}), H^j(\hat{\Omega}))$ *independently of N.*

Let $\hat{f}$ be in $H^m(\hat{\Omega})$, then Sobolev inequalities on the domain Ω imply

$$\sum_{x\in\Omega_x} |\hat{f}(2Nx)|^2 \leqslant C \sum_{x\in\Omega_x} \|\hat{f}(\cdot + 2Nx)\|_m^2 \leqslant C\|\hat{f}\|^2_{m,\hat{\Omega}}. \tag{4.4}$$

By (4.2)

$$\hat{P}_c\hat{f}(X) = \sum_{I\in\Omega_N} \mathcal{F}_N f(I)\, \operatorname{Exp}\left(\frac{\pi i}{N} X \cdot I\right). \tag{4.5}$$

Using the fact that $\mathcal{F}_N$ is an isometry from C_x onto C_N and (4.4) gives

$$\|\hat{P}_c\hat{f}\|_{0,\hat{\Omega}} = \left(\sum_{x\in\Omega_x} |\hat{f}(2Nx)|^2\right)^{1/2} \leqslant C\|\hat{f}\|_{m,\hat{\Omega}}.$$

Finally (4.5) implies

$$\|\hat{P}_c\hat{f}\|_{j,\hat{\Omega}} \leqslant C\|\hat{P}_c\hat{f}\|_{0,\hat{\Omega}} \leqslant C\|\hat{f}\|_{m,\hat{\Omega}},$$

which completes the proof of the theorem.

We can now define the pseudo spectral approximation to L by replacing P_N in (4.1) by P_c:

$$L_c f = \sum_{j=1}^{n} P_c\left(V_j \frac{\partial}{\partial x_j} f\right) + \frac{\partial}{\partial x_j} P_c(V_j f).$$

The operator L_c is skew symmetric on S_N, indeed, for f, g in S_N

$$(L_c f, g) = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} f, V_j g\right)_x - \left(V_j f, \frac{\partial}{\partial x_j} g\right)_x$$

$$= \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} f, P_c V_j g\right) - \left(f, V_j \frac{\partial}{\partial x_j} g\right)_x = -(f, L_c g).$$

As in Section 3, L_c generates a unitary semigroup on S_N. Let U be the solution to the problem

$$D_t U + L_c U = 0, \qquad U(0) = P_c u_0. \tag{4.6}$$

THEOREM 4. *Let u_0 be in $H_p^{s+1}(\Omega)$ for $s > n/2$ and let u be the solution to the advection equation* (2.1). *There exists a constant C independent of u_0 satisfying*

$$\|u(t) - U(t)\| \leqslant CN^{-s}\|u_0\|_{s+1}, \qquad t \in [0, T_0].$$

Theorem 4 can also be viewed as an extension to a theorem given by B. Fornberg. In [5], Fornberg derives estimates for semidiscrete pseudo spectral methods which bound the errors by norms of certain remainder terms. Fornberg gives heuristic arguments to show that these bounding norms are small. The norms in Fornberg's theorem can be estimated using Theorem 1 and Theorem 3. We give a proof which is similar to familiar finite element proofs.

Proof of Theorem 4. Let X and v be defined as in the proof of Theorem 2 and set $e = U - X$. Using the definition of U, we have

$$(D_t U, \theta) + (L_c U, \theta) = 0,$$

for all θ in S_N. Subtracting (3.3) and setting $\theta = e$ gives

$$(D_t e, e) + (L_c e, e) = (Lv, e) + ((L - L_c)X, e).$$

As in the proof of Theorem 2, it follows that

$$\|e\| \leqslant \|e(0)\| + T_0 \sup_{t \in [0, T_0]} \{\|(L - L_c)X(t)\| + \|Lv(t)\|\}.$$

From the proof of Theorem 2,

$$\|Lv(t)\| \leqslant CN^{-s}\|u_0\|_{s+1}.$$

Also the triangle inequality and Theorem 3 imply

$$\|(L - L_c)X\| \leqslant \sum_{j=1}^{n} \left\|\frac{\partial}{\partial x_j}(I - P_c)V_j X\right\| + \left\|(I - P_c)V_j \frac{\partial}{\partial x_j} X\right\| \tag{4.7}$$

$$\leqslant CN^{-s}\|X\|_{s+1} \leqslant CN^{-s}\|u_0\|_{s+1}.$$

The last inequality made use of (3.1) and Theorem 1. Theorem 3 implies

$$\|e(0)\| \leqslant CN^{-s}\|u_0\|_s.$$

Combining the above estimates proves Theorem 4.

5. Fully Discrete Pseudo Spectral Approximation. In this section we shall describe fully discrete explicit pseudo spectral approximations to (2.1). By taking advantage of the fast Fourier transform package, the pseudo spectral operator L_c can be evaluated economically. Thus fully discrete explicit timestepping procedures will run efficiently. To get error estimates, arguments proceed along the lines given by Baker, Bramble, and Thomée in [1]. We prove "smooth" data estimates (Lemma 3) and then expand the discretization error and apply the smooth estimates. In [1], Baker et al. expand the discretization error in terms of differences of T and T_h. For advection equations, expanding the errors in terms of $L_N - L_c$ gives rise to better error estimates because the operator $(I + L)^{-1}$, corresponding to T in [1], is not a smoothing operator.

Let $P_J(\tau)$ be the truncated Taylor series

$$P_J(\tau) = \sum_{j=0}^{J} \frac{\tau^j}{j!}.$$

We shall only consider J such that there exists $\delta > 0$ satisfying

$$|P_J(i\tau)| \leqslant 1 \quad \text{for real } \tau \text{ with } |\tau| \leqslant \delta. \tag{5.1}$$

By expanding $|P_J(i\tau)|$ as a function of τ, it is easily checked that the values $J = 3, 4, 7, 8$ satisfy the above assumption. Inequality (5.1), however, fails to hold for $J = 1, 2, 5$ or 6 for all choices of δ.

We approximate the solution of (2.1) by the sequence

$$W^0 = P_c u_0, \qquad W^{j+1} = P_j(-L_c k) W^j \quad \text{for } j = 0, 1, 2, \ldots. \tag{5.2}$$

Then W^j approximates $u(t_j)$ for $t_j = kj$. Note that the evaluation of (5.2) requires only J evaluations of L_c per timestep. Evaluation of L_c only requires multiplications and discrete Fourier transforms. Each timestep of (5.2) involves work of order $m \log(m)$ where m is the number of points in Ω_x.

Let the maximum norm over the grid points of Ω_x be denoted

$$\|V\|_{x,\infty} = \max_{y \in \Omega_x} |V(y)|.$$

For stability of (5.2) we use the following lemma:

LEMMA 2. *Let θ be an element of S_N. The following estimate holds:*

$$\|L_c \theta\| \leqslant C_1 N \|\theta\|, \quad \textit{where } C_1 = 4\pi n \max_{j=1,\ldots,n} \|V_j\|_{x,\infty}.$$

Proof. We clearly have

$$\|V\theta\|_x \leqslant \|V\|_{x,\infty} \|\theta\|_x.$$

Also, for θ in S_N,

$$\left\| \frac{\partial \theta}{\partial x_i} \right\|_x = \left\| \frac{\partial \theta}{\partial x_i} \right\| \leqslant 2\pi N \|\theta\| = 2\pi N \|\theta\|_x.$$

The lemma follows immediately from the above estimates and the definition of L_c.

As a consequence of (5.1) and Lemma 2, an obvious eigenfunction analysis gives that (5.2) is stable in L^2 whenever $kC_1N \leqslant \delta$. The following theorem is the main result of this section.

THEOREM 5. *Let u be the solution of* (2.1) *and W be the solution of* (5.2) *with $kC_1N \leqslant \delta$. Let u_0 be in $H_p^\tau(\Omega)$ where $\tau = \max(s + 1, J + 1)$. For $\tau > n/2 + 1$ there exists a constant C independent of u_0 satisfying*

$$\|W^n - u(t_n)\| \leqslant C\{N^{-s} + k^J\}\|u_0\|_\tau \quad \textit{for } t_n \leqslant T_0.$$

As a result of Theorem 4, to prove Theorem 5, it is sufficient to analyze the error between (5.2) and (4.6). To accomplish this we shall break the error into pieces and analyze the pieces. Introduce the error function $E_j(g)$ for g in S_N defined by $E_j(g) = W^j - U(t_j)$ where W^j satisfies (5.2) with initial data $W^0 = g$ and U satisfies (4.6) with data $U(0) = g$. Note that $E_j(g)$ is a linear function of g and that to prove Theorem 5 we need to bound $E_j(P_c u_0)$.

Since L_c is skew symmetric on a finite-dimensional space, the operator $T_c = (I + L_c)^{-1}P_N$ is well defined. We prove the following approximation result:

LEMMA 3. *For $kj \leqslant T_0$ and $kC_1N \leqslant \delta$ the following estimates hold*:
(i) $\|E_j(g)\| \leqslant C\|g\|$,
(ii) $\|E_j(T_c^m g)\| \leqslant Ck^{m-1}\|g\|$ *for* $2 \leqslant m \leqslant J + 1$.

Proof. (i) is just stability in L^2 of (5.2) and (4.6). For (ii) we note the following recursion

$$E_{j+1}(f) = P_J(-L_c k)E_j(f) + [P_J(-L_c k) - \mathrm{Exp}(-L_c k)]\, U(t_j), \tag{5.3}$$

where U is defined by (4.6) with initial data f. By (5.1)

$$\|P_J(-L_c k)E_j(f)\| \leqslant \|E_j(f)\|.$$

Now for $f = T_c^m g$, $U(t) = T_c^m \widetilde{U}(t)$ where $\widetilde{U}$ satisfies (4.6) with initial data g. Expanding the second term in (5.3) in the eigenfunctions of L_c implies that

$$\|[P_J(-L_c k) - \mathrm{Exp}(-L_c k)]\, T_c^m \widetilde{U}(t)\| \leqslant Ck^m\|\widetilde{U}(t)\|.$$

Using the stability of (4.6) and combining the above results gives

$$\|E_{j+1}(T_c^m g)\| \leqslant \|E_j(T_c^m g)\| + Ck^m\|g\|,$$

and summing proves Lemma 3.

Proof of Theorem 5. We note the following identity: For θ in S_N,

$$\theta = \sum_{j=1}^{m} T_c^j(L_c - L_N)(I + L_N)^{j-1}\theta + T_c^m(I + L_N)^m\theta. \tag{5.4}$$

The identity is obvious for $m = 1$, and validity for all m follows from an easy induction argument.

Using (5.4) and the linearity of E_m gives

$$E_m(P_c u_0) = \sum_{j=1}^{J+1} E_m(T_c^j(L_c - L_N)(I + L_N)^{j-1} P_N u_0)$$
$$+ E_m(T_c^{J+1}(I + L_N)^{J+1} P_N u_0) + E_m(P_c u_0 - P_N u_0).$$

As a consequence of (3.1), for f in $H_p^m(\Omega)$,

$$\|(I + L_N)^j f\|_{m-j} \leqslant C\|f\|_m \tag{5.5}$$

Lemma 3, the triangle inequality, and (3.1) give

$$\begin{aligned}\|E_m(P_c u_0)\| \leqslant \sum_{j=2}^{J+1} Ck^{j-1}\|(L_c - L_N)(I + L_N)^{j-1} P_N u_0\| \\ + C\|(L_c - L_N)P_N u_0\| + Ck^J\|u_0\|_{J+1} + CN^{-s}\|u_0\|_s.\end{aligned} \tag{5.6}$$

By estimates similar to those given on (4.7), the inverse properties of S_N, and (5.5)

$$\|(L_c - L_N)(I + L_N)^{j-1} P_N u_0\| \leqslant CN^{-s}\|P_N u_0\|_{s+j} \leqslant CN^{-s+j-1}\|u_0\|_{s+1}. \tag{5.7}$$

Combining (5.6) and (5.7) with the inequality $kNC_1 \leqslant \delta$ proves the theorem.

Acknowledgment. The author would like to thank R. Scott and the referee for their helpful comments and suggestions.

Applied Mathematics Department
Brookhaven National Laboratory
Upton, New York 11973

1. G. A. BAKER, J. H. BRAMBLE & V. THOMÉE, "Single step Galerkin approximation for parabolic problems," *Math. Comp.*, v. 31, 1977, pp. 818–847.

2. J. H. BRAMBLE, A. H. SCHATZ, V. THOMÉE & L. B. WAHLBIN, "Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations," *SIAM J. Numer. Anal.*, v. 14, 1977, pp. 218–241.

3. P. L. BUTZER & R. J. NESSEL, *Fourier Analysis and Approximation*, Vol. 1, Academic Press, New York, 1971, p. 97.

4. O. CHRISTENSEN & L. PRAHM, "A pseudo spectral model for dispersion of atmospheric pollutants," *J. Appl. Meteor.*, v. 15, 1976, pp. 1284–1294.

5. B. FORNBERG, "On a Fourier method for the integration of hyperbolic equations," *SIAM J. Numer. Anal.*, v. 12, 1975, pp. 509–528.

6. J. GAZDAG, "Numerical convective schemes based on accurate computation of space derivatives," *J. Comput. Phys.*, v. 13, 1973, pp. 100–113.

7. D. GOTTLIEB & S. ORSZAG, *Numerical Analysis of Spectral Methods*, SIAM-CBMS-NSF Conference Series, 1977, pp. 121–138.

8. S. G. KREIN & Y. I. PETUNIN, "Scales of Banach spaces", *Russian Math. Surveys*, v. 21, 1966, pp. 85–160.

9. H. KREISS & J. OLIGER, "Stability of the Fourier method," *SIAM J. Numer. Anal.*, v. 16, 1979, pp. 421–433.

10. A. MAJDA, J. McDONOUGH & S. OSHER, "The Fourier method for nonsmooth initial data," *Math. Comp.*, v. 32, 1978, pp. 1041–1081.

11. S. ORZAG, "Comparison of pseudospectral and spectral approximations," *Stud. Appl. Math.*, v. 51, 1972, pp. 253–259.

12. M. TAYLOR, *Pseudo Differential Operators*, Springer-Verlag, Berlin and New York, 1974, pp. 62–65.