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# Five-Diagonal Sixth Order Methods for Two-Point Boundary Value Problems Involving Fourth Order Differential Equations

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**Abstract.** We present a sixth order finite difference method for the two-point boundary value problem $y^{(4)} + f(x, y) = 0$, $y(a) = A_0$, $y(b) = B_0$, $y'(a) = A_1$, $y'(b) = B_1$. In the case of linear differential equations, our difference scheme leads to five-diagonal linear systems.

Consider the two-point boundary value problem

$$(1)\quad y^{(4)} + f(x, y) = 0, \quad y(a) = A_0, \quad y(b) = B_0, \quad y'(a) = A_1, \quad y'(b) = B_1.$$

In a recent paper Usmani [1] has given finite difference methods of orders two, four and six for the boundary value problem (1) in the case when $f(x, y)$ is linear. While his methods of orders two and four can be easily adapted for nonlinear $f(x, y)$ and lead to five-diagonal linear systems when $f(x, y)$ is linear, the sixth order method given by Usmani leads to a nine-diagonal linear system. In the following we present a sixth order method for the nonlinear boundary value problem (1) which, in the case of linear $f(x, y)$, leads to five-diagonal linear systems.

At the grid points $x_k$, $k = 2(1)N-1$, where $x_k = a + kh$, $k = 0(1)N + 1$, $N \geqslant 5$, the differential equation in (1) can be discretized by

$$(2)\quad \delta^4 y_k + h^4[2a_0 f_k + a_1(f_{k+1} + f_{k-1}) + a_2(f_{k+2} + f_{k-2})] + T_k(h) = 0,$$

where we have set $y_k = y(x_k)$ and $f_k = f(x_k, y_k)$.

In order that $T_k(h) = O(h^{10})$, we find that

$$(a_0, a_1, a_2) = (1/720)\,(237, 124, -1).$$

Note that $y_0 = A_0$, $y_{N+1} = B_0$. Let $y'_k = y'(x_k)$, $k = 0, N + 1$. The discretizations for the boundary conditions $y'_0 = A_1$, $y'_{N+1} = B_1$ can be obtained following Chawla and Katti [2]. Now, for the boundary conditions $y'_0 = A_1$ and $y'_{N+1} = B_1$, consider the discretizations

$$(3a)\quad \sum_{k=0}^{3} b_k y_k + chy'_0 + h^4\left(\sum_{k=0}^{3} d_k f_k + \sum_{k=0}^{1} d_k^* \bar{f}_{k+1/2}\right) + T_1(h) = 0,$$

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and

$$\sum_{k=0}^{3} b_k y_{N+1-k} - chy'_{N+1} + h^4\left(\sum_{k=0}^{3} d_k f_{N+1-k} + \sum_{k=0}^{1} d_k^* \bar{f}_{N-k+1/2}\right) + T_N(h) = 0, \tag{3b}$$

where we have set $\bar{f}_{k+1/2} = f(x_{k+1/2}, \bar{y}_{k+1/2})$, $x_{k+1/2} = x_k + h/2$, $k = 0(1)N$, and where

$$\bar{y}_{k+1/2} = \sum_{m=0}^{3} u_{k,m} y_m + h^4 \sum_{m=0}^{3} w_{k,m} f_m, \quad k = 0, 1, \tag{4a}$$

and

$$\bar{y}_{N-k+1/2} = \sum_{m=0}^{3} u_{k,m} y_{N+1-m} + h^4 \sum_{m=0}^{3} w_{k,m} f_{N+1-m}, \quad k = 0, 1. \tag{4b}$$

In order that $T_1(h)$ and $T_N(h) = O(h^{10})$, we find the following values for the parameters in (3) and (4):

$$(b_0, b_1, b_2, b_3) = \left(-\frac{11}{2}, 9, -\frac{9}{2}, 1\right), \quad c = -3,$$

$$(d_0, d_1, d_2, d_3) = \left(\frac{1}{4200}\right)(20, 1335, 460, 7),$$

$$(d_0^*, d_1^*) = \left(\frac{1}{4200}\right)(488, 840),$$

$$(u_{0,0}, u_{0,1}, u_{0,2}, u_{0,3}) = \left(\frac{1}{16}\right)(5, 15, -5, 1),$$

$$(w_{0,0}, w_{0,1}, w_{0,2}, w_{0,3}) = \left(\frac{1}{256}\right)(-3, 13, 0, 0),$$

$$(u_{1,0}, u_{1,1}, u_{1,2}, u_{1,3}) = \left(\frac{1}{16}\right)(-1, 9, 9, -1),$$

$$(w_{1,0}, w_{1,1}, w_{1,2}, w_{1,3}) = \left(\frac{3}{256}\right)(0, -1, -1, 0),$$

While determining these parameters, we have set the free parameters $w_{0,2}, w_{0,3}, w_{1,0}, w_{1,3} = 0$ for simplicity, and we have fixed $b_3 = 1$ for the reason that when the discretization is written in a matrix form the inverse of the coefficient matrix $(D)$ would be available in [1]. We also note that then

$$T_k(h) = -\frac{h^{10}}{3024} y_k^{(10)} + O(h^{12}), \quad k = 2(1)N-1,$$

and

$$T_1(h) = h^{10}\left[\frac{1}{38400} y_0^{(10)} - \left\{\frac{(61F_{1/2} + 9F_{3/2})}{46080}\right\} y_0^{(6)}\right] + O(h^{11}),$$

$$T_N(h) = h^{10}\left[\frac{1}{38400}\, y_{N+1}^{(10)} - \left\{\frac{(61F_{N+1/2} + 9F_{N-1/2})}{46080}\right\} y_{N+1}^{(6)}\right] + O(h^{11}),$$

where $F = \partial f/\partial y$.

Now, a method based on the discretizations (3a), (2) and (3b) can be expressed in the matrix form as

$$D\widetilde{Y} + G(\widetilde{Y}) = \mathbf{0}. \tag{5}$$

For the derivation of the above difference scheme, we have assumed that $y \in C^{10}[a, b]$, and for $x \in [a, b]$, $-\infty < y < \infty$, $f$ is six times continuously differentiable and that $\partial f/\partial y$ exists and is continuous.

Following arguments given in Usmani [1], we can show that if $E = \widetilde{Y} - Y$, then in the uniform norm, for sufficiently small $h$,

$$\|E\| = O(h^6),$$

provided $U^* < 2592/(7K(b - a)^4)$, where

$$K = \frac{181}{180} \quad \text{and} \quad U^* = \max\left|\frac{\partial f}{\partial y}\right|.$$

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