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On Faster Convergence of the Bisection Method for all Triangles

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Abstract. Let ΔABC be a triangle with vertices A, B, and C. It is "bisected" as follows: choose a/the longest side (say AB) of ΔABC, let D be the midpoint of AB, then replace ΔABC by two triangles ΔADC and ΔDBC.

Let Δ_{01} be a given triangle. Bisect Δ_{01} into two triangles Δ_{11} and Δ_{12}. Next bisect each Δ_{1i}, $i = 1, 2$, forming four new triangles Δ_{2i}, $i = 1, 2, 3, 4$. Continue thus, forming an infinite sequence T_j, $j = 0, 1, 2, \ldots$, of sets of triangles, where $T_j = \{\Delta_{ji}: 1 \leqslant i \leqslant 2^j\}$. Let m_j denote the mesh of T_j. It is shown that there exists $N = N(\Delta_{01})$ such that, for $j \geqslant N$, $m_{2j} \leqslant (\sqrt{3}/2)^N(1/2)^{j-N}m_0$, thus greatly improving the previous best known bound of $m_{2j} \leqslant (\sqrt{3}/2)^j m_0$.

It is also shown that only a finite number of distinct shapes occur among the triangles produced, and that, as the method proceeds, Δ_{01} tends to become covered by triangles which are approximately equilateral in a certain sense.

1. Introduction and Summary. Let ΔABC be a triangle with vertices A, B, and C. We define the procedure for "bisecting" ΔABC as follows: choose a/the longest side (say AB) of ΔABC, let D be the midpoint of AB, then divide ΔABC into the two triangles ΔADC and ΔDBC.

Let Δ_{01} be a given triangle. Bisect Δ_{01} into two triangles Δ_{11} and Δ_{12}. Next bisect each Δ_{1i}, $i = 1, 2$, forming four new triangles Δ_{2i}, $i = 1, 2, 3, 4$. Continue thus, forming an infinite sequence T_j, $j = 0, 1, 2, \ldots$, of sets of triangles, where $T_j = \{\Delta_{ji}: 1 \leqslant i \leqslant 2^j\}$. Define m_j, the mesh of T_j, to be the length of the longest side among the sides of the triangles of T_j. Clearly $0 < m_{j+1} \leqslant m_j$, for all $j \geqslant 0$. It is known [2], [1] that $m_j \longrightarrow 0$ as $j \longrightarrow \infty$.

In [1, Theorem 3.1] an explicit bound is obtained for the rate of convergence of m_j: $m_j \leqslant (\sqrt{3}/2)^{[j/2]}m_0$, $j \geqslant 0$, where $[x]$ denotes the integer part of x. This may be written as $m_{2j} \leqslant (\sqrt{3}/2)^j m_0$, $j \geqslant 0$. In [1] it is also mentioned that computer experiments indicate that in many cases this bound is unrealistically high. This prompted the results of [3] where it was shown that if Δ_{01} lay in any one of four sets of similarity classes of triangles, then $m_{2j} \leqslant (\sqrt{3}/2)(1/2)^{j-1}m_0$, for $j \geqslant 1$. Note however that in [3] the inequality was not written in this form; see [3, Corollaries 1 and 2]. In the present paper we show that for any Δ_{01} there exists a positive integer N depending only on Δ_{01} such that

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$$m_{2j} \leqslant (\sqrt{3}/2)^{\min\{j,N\}}(1/2)^{\max\{j-N,0\}}m_0, \quad \text{for } j \geqslant 0.$$

For $j \geqslant N$ this becomes $m_{2j} \leqslant (\sqrt{3}/2)^N(1/2)^{j-N}m_0$, thus generalizing [3, Corollaries 1 and 2]. It is also a great improvement on the result of [1] for j large. We have the interesting result of Corollary 1 below that for any Δ_{01} the bisection method yields only a finite number of distinct shapes among all the triangles in the T_j, $j = 0, 1, 2, \ldots$. Finally Corollary 2 shows that the shapes produced tend to be closer to equilateral than the original Δ_{01}.

2. Results. All notation used is consistent with that of [3]. The results of this paper can also be proven using the methods of [2], [3] (i.e., by consideration of angles in triangles) but the technique below seems simpler.

Given two triangles Δ and Δ', we write $\Delta \sim \Delta'$ to indicate that the triangles are similar.

Definitions. Given a triangle Δ, define $d(\Delta)$, the diameter of Δ, to be length of the longest side of Δ. Define $t(\Delta)$ to be (area $\Delta)/d^2(\Delta)$. Note that if $\Delta \sim \Delta'$, then $t(\Delta) = t(\Delta')$. (This similarity invariant t is closely related to J. H. C. Whitehead's "relative thickness of simplexes" [5, pp. 811, 812].)

Note that when a triangle is bisected, its area is bisected. This fact is used implicitly many times in what follows.

Definition. We say that Δ_{01} is good if it has an associated positive integer N such that

$$m_{2j} \leqslant (\sqrt{3}/2)^{\min\{j,N\}}(1/2)^{\max\{j-N,0\}}m_0, \quad \text{for } j \geqslant 0.$$

For the rest of this paper we always assume that we are working with the smallest such N associated with a given Δ_{01}.

Remarks. Clearly all triangles generated from a good Δ_{01}, by the bisection method, are also good, using the result of [1] that $m_{2j} \leqslant (\sqrt{3}/2)^j m_0$, $j \geqslant 0$, for any Δ_{01}. In [3, Corollary 1 (i)] it was shown that for $\Delta_{01} = \Delta ABC$, as in Figure 1 satisfying (length of $AC =$) $AC \leqslant BC \leqslant AB$, $AC \geqslant \max\{CD, AD\}$, and $CD \geqslant CF$, where D, F are the midpoints of AB, BC respectively, we have $m_{2j} \leqslant (\sqrt{3}/2)^{\min\{j,1\}}(1/2)^{\max\{j-1,0\}}m_0$, for $j \geqslant 0$, so such triangles are good. We note that these inequalities were expressed differently in [3]. In [3] three other classes of triangles satisfying conditions on the length of their sides and medians were considered and similar inequalities for m_{2j} were proven; however, these three classes can be shown to be subclasses of the one specified above (this passed unnoticed in [3]).

LEMMA 1. *If $t(\Delta_{01}) \geqslant \sqrt{7}/8$, then Δ_{01} is good.*

Proof. We use the result just quoted from [3]. Take $\Delta_{01} = \Delta ABC$ in Figure 1. Assuming $AC \leqslant BC \leqslant AB$, we show that $AC \geqslant \max\{CD, AD\}$ and $CD \geqslant CF$.

Since $t(\Delta)$ is invariant under similarity, we may take $AB = 1$. In Figure 1, CL is perpendicular to AB. Let $CL = h$, $AL = x$, and $LB = 1 - x$. Now $7/64 \leqslant t^2(\Delta ABC) = h^2/4$ by definition of $t(\Delta)$, so $h^2 \geqslant 7/16$. Hence $1 = AB^2 \geqslant BC^2 = h^2 + (1-x)^2 \geqslant 7/16 + (1-x)^2$, which gives $x \geqslant 1/4$.

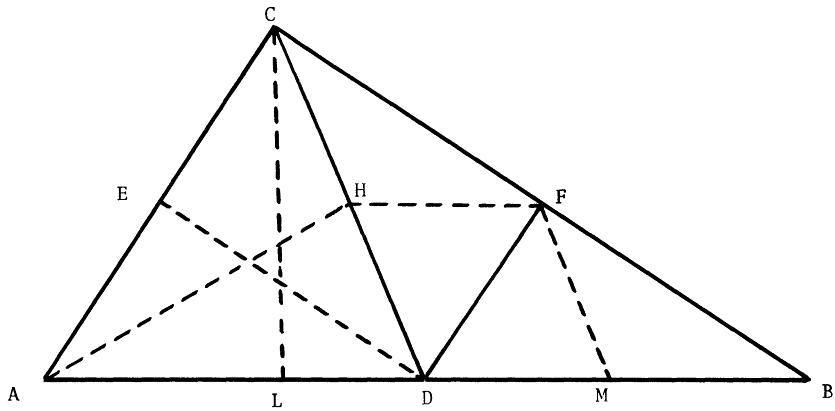

FIGURE 1

Now

$$\max\{CD^2, AD^2\} = \max\{h^2 + (1/2 - x)^2, 1/4\}$$

$$\leqslant h^2 + x^2 \quad \text{since } h^2 \geqslant 7/16 \text{ and } 1/2 \geqslant x \geqslant 1/4 = AC^2.$$

Also

$$CF^2 = BC^2/4 = (h^2 + (1-x)^2)/4$$

$$\leqslant h^2 \quad \text{since } x \geqslant 1/4 \text{ and } h^2 \geqslant 7/16 \leqslant CD^2.$$

By the remarks above, $\Delta_{01} = \Delta ABC$ is good.

Definitions. Using the notation of the introduction, an iteration of the bisection method is defined to be the progression from T_j to T_{j+1}, for any $j \geqslant 0$. A cycle of the bisection method is defined to be two successive iterations, i.e., the progression from T_j to T_{j+2}, for any $j \geqslant 0$.

LEMMA 2. *After one cycle of the bisection procedure, applied to Δ_{01}, we have four triangles Δ_{2i}, $i = 1, 2, 3, 4$. Suppose that for each i either Δ_{2i} is good or $\Delta_{2i} \sim \Delta_{01}$ with $d(\Delta_{2i}) = d(\Delta_{01})/2$. Then Δ_{01} is good.*

Proof. Set $N = 1 + \max\{N_i\colon \Delta_{2i}$ good with associated positive integer $N_i\}$. We show that

$$m_{2j} \leqslant (\sqrt{3}/2)^{\min\{j,N\}}(1/2)^{\max\{j-N,0\}}m_0, \quad \text{for } j \geqslant 0.$$

To do this we use induction on j.

For $j \leqslant N$, this is the $n = 2$ case of [1, Theorem 3.1]. Fix $k \geqslant N$ and assume that the result holds for $j = k$. We now prove it for $j = k + 1$.

Each triangle $\Delta_{2(k+1),i'}$, $i' = 1, 2, \ldots, 2^{2(k+1)}$, is obtained by applying k bisection cycles to one of the Δ_{2i}, $i = 1, 2, 3, 4$. We consider the possibilities for Δ_{2i} separately. If Δ_{2i} is good with associated positive integer N_i, then, after k bisection cycles applied to it, any resulting triangle $\Delta_{2(k+1),i'}$ satisfies

$$\begin{aligned} d(\Delta_{2(k+1),i'}) &\leqslant (\sqrt{3}/2)^{N_i}(1/2)^{k-N_i}d(\Delta_{2i}) \\ &\leqslant (\sqrt{3}/2)^{N_i+1}(1/2)^{k-N_i}m_0 \quad \text{by [1, Theorem 3.1]} \\ &\leqslant (\sqrt{3}/2)^{N}(1/2)^{k+1-N}m_0 \quad \text{since } N_i + 1 \leqslant N. \end{aligned}$$

If $\Delta_{2i} \sim \Delta_{01}$ with $d(\Delta_{2i}) = d(\Delta_{01})/2$, then, after k bisection cycles applied to it, any resulting triangle $\Delta_{2(k+1),i'}$ satisfies

$$d(\Delta_{2(k+1),i'}) \leqslant (\sqrt{3}/2)^{N}(1/2)^{k-N}d(\Delta_{2i}),$$

by the inductive hypothesis applied to Δ_{2i} since $\Delta_{2i} \sim \Delta_{01}$. Now use $d(\Delta_{2i}) = m_0/2$ to complete the proof for $j = k + 1$.

THEOREM 1. *Any triangle Δ_{01} is good.*

Proof. Let S be the set of all triangles Δ_{01} which are not good. Assuming S is not empty, set $\hat{t} = \sup\{t(\Delta)\colon \Delta \in S\}$. By Lemma 1, $\hat{t} \leqslant \sqrt{7}/8$. Now choose $\Delta_{01} \in S$ such that $t^* \equiv t(\Delta_{01}) \geqslant \max\{3\hat{t}/4, \hat{t}(1 - 9\hat{t}^2/2)\}$. We show that in fact Δ_{01} is good, contradicting the assumption that S is nonempty.

Take $\Delta_{01} = \Delta ABC$ in Figure 1, with $AC \leqslant BC \leqslant AB$. Here D, E, F, H, and M are the midpoints of AB, AC, BC, CD, and BD respectively. When the bisection method is applied to ΔABC, AB and BC must be bisected as shown. Any of the three sides in ΔCAD may be bisected next, while in ΔCDF either CD or CF will be bisected since $DF = AC/2 \leqslant BC/2 = CF$ (if $DF = CF$, we can assume that CF is bisected since either choice has the same effect). This gives six possible combinations of bisections of sides. We analyze these cases separately.

Case (i): *AC in ΔCAD, CD in ΔCDF.* In this case we are in the situation of [3, Corollary 1(i)]; by the Remarks above ΔABC is good.

Case (ii): *CD in ΔCAD, CF in ΔCDF.* This is impossible unless $CD = CF$, since $CD \geqslant AD = AB/2 \geqslant BC/2 = CF$ and if $CD > CF$, then CF cannot be bisected in ΔCDF. If $CD = CF$, we proceed to Case (vi) below.

For the remaining cases we shall use the well-known formula for the length of the median CD:

$$(*) \qquad CD^2 = AC^2/2 + BC^2/2 - AB^2/4.$$

Case (iii): *AC in ΔCAD, CF in ΔCDF.* Now $BC^2/4 = CF^2 \geqslant CD^2 \geqslant AD^2/2 + BC^2/2 - AB^2/4$, hence $AB^2/2 \geqslant BC^2$. Thus $t(\Delta CDF) = (\text{area } \Delta CDF)/CF^2 = (\text{area } \Delta ABC)/BC^2 \geqslant 2t^*$. By choice of t^* ΔCDF is good.

When ΔCAD is bisected, we get $\Delta ADE \sim \Delta ABC$ with $d(\Delta ADE) = d(ABC)/2$ and $\Delta CED \sim \Delta CDF$, so ΔCED is good. Also $\Delta DBF \sim \Delta ABC$ with $d(\Delta DBF) = d(\Delta ABC)/2$. By Lemma 2 ΔABC is good.

Case (iv): *AD in ΔCAD, CF in ΔCDF.* Here $t(\Delta CAD) = (\text{area } \Delta CAD)/AD^2 =$

2(area $\Delta ABC)/AB^2 = 2t^*$, so ΔCAD is good, and consequently the two triangles obtained from it by bisection are good.

Now $t^* = (AB \cdot AC \cdot \sin C\hat{A}B)/(2AB^2) \leqslant AC/(2AB)$ so $AC \geqslant 2 \cdot AB \cdot t^*$. Hence $BC^2/4 = CF^2 \geqslant CD^2 \geqslant 2 \cdot AB^2 \cdot (t^*)^2 + BC^2/2 - AB^2/4$ using $(*) \Rightarrow (1 - 8(t^*)^2)AB^2 \geqslant BC^2$. Thus $t(\Delta CDF) = (\text{area } \Delta CDF)/CF^2 = (\text{area } \Delta ABC)/BC^2 \geqslant t^*/(1 - 8(t^*)^2)$. Now $t^* \geqslant 3\hat{t}/4$, so $t(\Delta CDF) \geqslant t^*/(1 - 9\hat{t}^2/2) > \hat{t}$, since $t^* > \hat{t}(1 - 9\hat{t}^2/2)$ and $1 - 9\hat{t}^2/2 > 0$ by Lemma 1 and the definition of $\hat{t}$. Therefore ΔCDF is good. Also $\Delta DBF \sim \Delta ABC$ with $d(\Delta DBF) = d(\Delta ABC)/2$.

We can now apply Lemma 2 to obtain ΔABC good.

Case (v): *AD in ΔCAD, CD in ΔCDF.* As in Case (iv), ΔCAD is good. In ΔCDF bisection yields $\Delta CHF \sim \Delta BCD$ and $\Delta DFH \sim \Delta CAD$ (consequently good). In ΔFDB ($\sim \Delta ABC$) bisection yields $\Delta BFM \sim \Delta BCD$ and $\Delta MFD \sim \Delta CAD$ (consequently good). Applying Lemma 2 to ΔBCD now shows that ΔBCD is good. But ΔFDB, obtained from ΔBCD by bisection, must then be good, so $\Delta ABC \sim \Delta FDB$ is good.

Case (vi): *CD in ΔCAD, CD in ΔCDF.* If we can show that ΔCAD is good, then repeating the argument of Case (v) shows that ΔABC is good. We therefore analyze ΔCAD.

If $CD^2 \leqslant 3AB^2/8$, then $t(\Delta CAD) = (\text{area } \Delta CAD)/CD^2 \geqslant 4t^*/3 > \hat{t}$, so ΔCAD is good. Therefore assume that $CD^2 \geqslant 3AB^2/8$. We claim that this implies that ΔCAD satisfies the hypotheses of [3, Corollary 1(i)] and is consequently good (see the remarks above).

Using $BC \leqslant AB$ in $(*)$ gives

$$AC^2/2 + AB^2/2 - AB^2/4 \geqslant CD^2 \geqslant 3AB^2/8 \Rightarrow AC^2 \geqslant AB^2/4, \quad \text{i.e.,} \quad AC \geqslant AD.$$

Thus in ΔCAD, $CD \geqslant AC \geqslant AD$. When ΔCAD is bisected, the median formed is AH. To prove our claim, we must show that (a) $AD \geqslant AH$, (b) $AD \geqslant DH$, and (c) $AH \geqslant AE$ all hold.

Note that $AH^2 = AC^2/2 + AD^2/2 - CD^2/4$. Now

$$\text{(a)} \quad AD^2 - AH^2 = AD^2/2 - AC^2/2 + CD^2/4$$
$$= AB^2/16 - 3AC^2/8 + BC^2/8 \quad \text{using } (*)$$
$$\geqslant 3BC^2/16 - 3AC^2/8 \quad \text{since } AB \geqslant BC.$$

But $AC^2 \leqslant CD^2 \leqslant AC^2/2 + BC^2/2 - BC^2/4$ from $(*) \Rightarrow AC^2 \leqslant BC^2/2$. Hence $AD^2 - AH^2 \geqslant 3BC^2/16 - 3BC^2/16 = 0$ as required.

$$\text{(b)} \quad AD = AB/2 \geqslant BC/2 \geqslant CD/2 = DH.$$

$$\text{(c)} \quad AH^2 - AE^2 = AC^2/4 + AD^2/2 - CD^2/4$$
$$= AC^2/8 + 3AB^2/16 - BC^2/8 \quad \text{using } (*)$$
$$> 0 \quad \text{since } AB \geqslant BC.$$

This completes the proof that in all cases ΔABC is good, and the theorem is proven.

Remark. The above proof can also be used to derive other properties of triangles under repeated bisection. If we redefine the term "good" in such a way that

(a) all triangles generated from a good Δ_{01} are also good,

(b) Lemma 1 still holds, and

(c) Lemma 2 still holds,

then Theorem 1 is still true using exactly the same proof. In this way we now give two corollaries. It is possible to derive more quickly the inequality in the original definition of "good", using [1, Theorem 3.1] and [2, Theorem], but this method of proof does not seem to extend to proving results of the type below.

Definition. We now say that Δ_{01} is good if, identifying similar triangles, only a finite number of distinct triangles are produced from Δ_{01} by the bisection method.

It is easy to verify (a), (b) (use [2, Lemma 4]), and (c) above under this definition of "good". Consequently by the above Remark we have:

COROLLARY 1. *Every triangle yields only a finite number of similarity-distinct triangles under repeated bisections.*

Definition. Let $p_k(\Delta_{01})$, $k = 0, 1, 2, \ldots$, by that fraction of the total area of Δ_{01} that after k cycles of the bisection method applied to Δ_{01} is covered by triangles which satisfy the conditions of [2, Lemma 4] (these are the same as the conditions of [3, Corollary 1(i)]).

On examining [2, Lemma 4], we see that p_k is a monotonically increasing sequence. Thus $p(\Delta_{01}) \equiv \lim_{k\to\infty} p_k(\Delta_{01})$ always exists.

Definition. We now say that Δ_{01} is good if $p(\Delta_{01}) = 1$.

COROLLARY 2. *For every triangle* Δ_{01}, $P(\Delta_{01}) = 1$.

Proof. We must check (a), (b), and (c) of the Remark above. Clearly (a) holds as, otherwise, we have a contradiction. For (b) use [2, Lemma 4].

To prove (c) (i.e. to prove Lemma 2), we may assume that of the four triangles produced, after one cycle of the bisection method applied to Δ_{01}, at least two are not similar to Δ_{01}, as, otherwise, elementary calculations show that Δ_{01} satisfies the conditions of [2, Lemma 4] and so by (b) is good.

By hypothesis in Lemma 2, triangles which are not similar to Δ_{01} are good. If there are exactly two such triangles, we have

$$p(\Delta_{01}) = 1/4 + 1/4 + p(\Delta')/4 + p(\Delta'')/4,$$

where Δ', Δ'' are the other two triangles. But $\Delta_{01} \sim \Delta' \sim \Delta''$, so $p(\Delta_{01}) = p(\Delta') = p(\Delta'')$. Hence $p(\Delta_{01}) = 1$. Similarly, if there are three triangles which are not similar to Δ_{01}, we get $p(\Delta_{01}) = 1$. This completes the proof of Lemma 2 and also the proof of Corollary 2.

Corollary 2 is of use in explaining an apparent anomaly elsewhere [4].

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