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On an Accelerated Overrelaxation Iterative Method for Linear Systems With Strictly Diagonally Dominant Matrix

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Abstract. We consider a linear system $Ax = b$ of n simultaneous equations, where A is a strictly diagonally dominant matrix. We get bounds for the spectral radius of the matrix $L_{r,\omega}$, which is accociated with the Accelerated Overrelaxation iterative method (AOR).

Sufficient conditions for the convergence of that method will be given, which improve the results of Theorem 3, Section 4 of [2], applied to this type of matrices.

1. **Introduction.** We want the solution x of a linear system

$$Ax = b, \tag{1.1}$$

where A is an (n, n) real-matrix and x and b are n-real-vectors. We assume A is strictly diagonally dominant. There are several important iterative methods for the approximation of (1.1). We will take the (AOR) of [1]. For that, the matrix is expressed as the matrix sum

$$A = I - E - F, \tag{1.2}$$

where I is the identity and E and F are respectively strictly lower and upper triangular (n, n) matrices.

From the (AOR) method we can write the following equations:

$$x^{(i+1)} = (I - rE)^{-1}[(1-\omega)I + (\omega - r)E + \omega F]x^{(i)} + \omega(I - rE)^{-1}b, \qquad i = 0, 1, 2, \ldots. \tag{1.3}$$

So, $L_{r,\omega}$ will be the point-(AOR)-matrix associated with the matrix A,

$$L_{r,\omega} = (I - rE)^{-1}[(1-\omega)I + (\omega - r)E + \omega F], \tag{1.4}$$

and $\rho(L_{r,\omega})$ the corresponding spectral radius.

2. **Bounds for $\rho(L_{r,\omega})$.** As we assume A strictly diagonally dominant, A is nonsingular and verifies

$$|a_{ii}| > \sum_{j=1;i\neq j}^{n} |a_{ij}|, \qquad i = 1, 2, \ldots, n. \tag{2.1}$$

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Bounds for $\rho(L_{r,\omega})$ are obtained from

THEOREM 1. *If A of* (1.1) *is a strictly diagonally dominant matrix, then* $\rho(L_{r,\omega})$ *satisfies the following*:

$$\min_i \frac{|1-\omega| - |\omega - r|e_i - |\omega|f_i}{1+|r|e_i} \leqslant \rho(L_{r,\omega}) \leqslant \max_i \frac{|\omega - r|e_i + |\omega|f_i + |1-\omega|}{1-|r|e_i}, \tag{2.2}$$

$$i = 1, 2, \ldots, n,$$

where $|r| < 1/e_i$ *and* e_i, f_i *are respectively the i-row sums of the moduli of the entries of E and F, respectively.*

Proof. Since the eigenvalues of $L_{r,\omega}$ are given from

$$\det(L_{r,\omega} - \lambda I) = 0 \tag{2.3}$$

after some manipulations, it is easy to verify that to solve (2.3) is equivalent to solving

$$\det Q = 0, \tag{2.4}$$

where Q is

$$Q = I - \frac{r(\lambda - 1) + \omega}{\lambda - 1 + \omega} E - \frac{\omega}{\lambda - 1 + \omega} F.$$

If we take the parameter r, ω, λ, in order that Q be strictly diagonally dominant, we get

$$|\lambda - 1 + \omega| > |\omega + (\lambda - 1)r|e_i + |\omega|f_i, \quad i = 1, \ldots, n. \tag{2.5}$$

Then, the values of λ satisfying (2.5) will not satisfy (2.4) and cannot be eigenvalues of (2.3). From (2.5), as λ can take any value, we have

$$|\lambda| - |1-\omega| > |\omega - r|e_i + |\lambda r|e_i + |\omega|f_i, \quad i = 1, \ldots, n,$$

or

$$|\lambda|(1 - |r|e_i) > |\omega - r|e_i + |\omega|f_i + |1-\omega|, \quad i = 1, \ldots, n.$$

Assuming $|r| < 1/e_i$, we have

$$|\lambda| > \max_i \frac{|\omega - r|e_i + |\omega|f_i + |1-\omega|}{1-|r|e_i}.$$

If λ satisfies this inequality, it cannot be an eigenvalue of $L_{r,\omega}$, and then

$$\rho(L_{r,\omega}) \leqslant \max_i \frac{|\omega - r|e_i + |\omega|f_i + |1-\omega|}{1-|r|e_i} \quad \text{for } |r| < \frac{1}{e_i}.$$

In order to get the lower bound, from (2.5) we write

$$|1-\omega| - |\lambda| > |\omega - r|e_i + |\lambda r|e_i + |\omega|f_i, \quad i = 1, 2, \ldots, n,$$

and

$$|\lambda| < \min_i \frac{|1-\omega| - |\omega - r|e_i - |\omega|f_i}{1 + |r|e_i}.$$

Since the values of λ satisfying this inequality are not eigenvalues of $L_{r,\omega}$, then

$$\rho(L_{r,\omega}) \geqslant \min_i \frac{|1-\omega| - |\omega - r|e_i - |\omega|f_i}{1 + |r|e_i}, \qquad i = 1, \ldots, n. \quad \text{Q.E.D.}$$

As it is well known, for convenient values of r and ω, the (AOR) method becomes the well-known iterative methods:

$r = 0,\ \omega = 1$	Jacobi Method,
$r = 1,\ \omega = 1$	Gauss-Seidel Method,
$r = 0,\ \omega$	Simultaneous Overrelaxation Method,
$r = \omega$	Successive Overrelaxation Method.

Taking these values, we get from (2.2) the known results

(2.6a) $$\rho(L_{0,1}) \leqslant \max_i (e_i + f_i),$$

(2.6b) $$\rho(L_{1,1}) \leqslant \max_i \frac{f_i}{1 - e_i},$$

(2.6c) $$\rho(L_{0,\omega}) \leqslant \max_i |\omega|(e_i + f_i) + |1 - \omega|,$$

(2.6d) $$\rho(L_{\omega,\omega}) \leqslant \max_i \frac{|\omega|f_i + |1-\omega|}{1 - |\omega|e_i}.$$

3. Convergence of the (AOR) Method.

THEOREM 2. *If A of* (1.1) *is a strictly diagonally dominant matrix and $\omega \geqslant r \geqslant 0$, then a sufficient condition for the convergence of the* (*AOR*) *method is*

$$0 < \omega < \frac{2}{1 + \max_i (e_i + f_i)}.$$

Proof. From [1, Section 3], with A strictly diagonally dominant, the (AOR) method is convergent if $0 < \omega \leqslant 1$, $0 \leqslant r \leqslant 1$. From (2.2) we see that $\rho(L_{r,\omega})$ will be less than 1 if

(3.1) $$|\omega - r|e_i + |\omega|f_i + |1 - \omega| + |r|e_i < 1, \qquad i = 1, \ldots, n.$$

With $\omega > r \geqslant 0$ these conditions will be fulfilled.

Taking $f(\delta) = (\delta - r)e_i + \delta f_i + (1 - \delta) + re_i$, we see that $f(\delta)$ is a nonincreasing function of δ if $0 \leqslant \delta \leqslant 1$ and $f(0) = 1$, $f(1) = e_i + f_i < 1$. With $\delta > 1$, $f(\delta)$ is an increasing function with $f(\delta) = 1$ for $\delta = 2/(1 + e_i + f_i)$. Then $\rho(L_{r.\omega}) < 1$ if

$$0 < \omega < \frac{2}{1 + \max_i (e_i + f_i)},$$

and the (AOR) method will be convergent. Q.E.D.

Let us consider a first-degree linear stationary iterative method

$$x^{(i+1)} = Gx^{(i)} + d|_{i=0,1,2,\ldots}, \tag{3.2}$$

with G a known $n \times n$ matrix, d a known n-vector, and $x^{(0)}$ an arbitrary initial approximation for the solution x.

The following method

$$x^{(i+1)} = [(1-\omega)I + \omega G]x^{(i)} + \omega d|_{i=0,1,2,\ldots}, \tag{3.3}$$

will be called the extrapolated method of (3.2).

We recall now, the Theorem of Extrapolation [2, p. 2], as we need it in the sequel.

THEOREM 3 (THEOREM OF EXTRAPOLATION). *The sufficient conditions for the convergence of* (3.3) *are*:

(1) *The original* (3.2) *is convergent,*

(2) $0 < \omega < 2/(1 + \rho(G))$.

Setting $r = 0$ in (1.3), we obtain

$$x^{(i+1)} = [(1-\omega)I + \omega(E+F)]x^{(i)} + \omega b|_{i=0,1,2,\ldots}.$$

This is the extrapolated Jacobi method, where ω is the extrapolation parameter.

After some computation, it is easy to verify that (1.3) is the extrapolated SOR method, when $r \neq 0$ and its extrapolation parameter is ω/r.

THEOREM 4. *If A from* (1.1) *is strictly diagonally dominant, then* $\rho(L_{0,\omega}) < 1$ *provided* $0 < \omega < 2/(1 + \rho(L_{0,1}))$.

Proof. Bearing in mind that A strictly diagonally dominant implies $\rho(L_{0,1}) < 1$ [4, p. 73], it is an immediate consequence of Theorem 3.

THEOREM 5. *If A of* (1.1) *is strictly diagonally dominant, then* $\rho(L_r, \omega) < 1$ *with* $0 < r \leqslant 1$ *provided* $0 < \omega < 2r/(1 + \rho(L_{r,r}))$.

Proof. From (2.6d) we easily deduce that the SOR method will converge with $0 < r \leqslant 1$ (A is strictly diagonally dominant).

Then, by the Theorem of Extrapolation, the AOR method will converge for $0 < r \leqslant 1$ and $0 < \omega < 2r/(1 + \rho(L_{r,r}))$.

THEOREM 6. *The AOR method is convergent, i.e.* $\rho(L_{r,\omega}) < 1$, *for*:

(i) $0 \leqslant r \leqslant \omega$ *and* $0 < \omega < 2/(1 + \max_i(e_i + f_i))$ *if*

$$\frac{2r}{1 + \rho(L_{r,r})} \leqslant \frac{2}{1 + \max\limits_i (e_i + f_i)};$$

(ii) $0 < r \leqslant 1$ *and* $0 < \omega < 2r/(1 + \rho(L_{r,r}))$ *or* $1 < r < \omega$ *and* $0 < \omega < 2/(1 + \max_i(e_i + f_i))$ *if*

$$\frac{2r}{1 + \rho(L_{r,r})} > \frac{2}{1 + \max\limits_i (e_i + f_i)}.$$

Proof. These results come from Theorems 2 and 5. They improve the results of Theorem 3, Section 4 of [2], when the matrix A of (1.1) is strictly diagonally dominant:

$$\rho(L_{r,\omega}) < 1 \quad \text{if } 0 \leqslant r < 1 \quad \text{and} \quad 0 < \omega \leqslant \max\left\{1, \frac{2r}{1+\rho(L_{r,r})}\right\}.$$

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