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On Some Orthogonal Polynomial Integrals

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Abstract. The modified moments of the weight functions $w(x) = x^{\rho}(1-x)^{\alpha}\ln(1/x)$, on [0, 1], with respect to the shifted Jacobi polynomials $P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$, and $w_p(x) = x^{\rho}e^{-x}(\ln x)^p$, $p = 1, 2$, on $[0, \infty)$, with respect to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, are explicitly evaluated.

1. A Jacobi Polynomial Integral. In a recent paper, Gautschi [3], generalizing a result of Blue [2], has considered and explicitly evaluated the modified moments of the weight function

$$w(x) = x^{\mu}\ln(1/x), \qquad \mu > -1,$$

on [0, 1], with respect to the shifted Legendre polynomials $P_n^*(x) = P_n(2x-1)$.

We further generalize these results by considering the weight function

$$w(x) = x^{\rho}(1-x)^{\alpha}\ln(1/x), \qquad \alpha, \rho > -1, \tag{1.1}$$

and evaluating its modified moments on [0, 1] with respect to the shifted Jacobi polynomials $P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$.

It is convenient from now on to replace ρ by $\beta + \mu$; thus, the modified moments we have to examine assume the form

$$\nu_n^{(\alpha,\beta)}(\mu) = \int_0^1 x^{\beta+\mu}(1-x)^{\alpha}\ln(1/x)P_n^{*(\alpha,\beta)}(x)\,dx, \tag{1.2}$$
$$\alpha, \beta, \beta+\mu > -1, n = 0, 1, 2, \ldots .$$

We easily see that

$$\nu_n^{(\alpha,\beta)}(\mu) = -2^{-(\alpha+\beta+\mu+1)}\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}\ln(\tfrac{1}{2}(1+t))P_n^{(\alpha,\beta)}(t)\,dt$$

$$= -2^{-(\alpha+\beta+\mu+1)}\left\{\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}\ln(1+t)P_n^{(\alpha,\beta)}(t)\,dt\right.$$

$$\left.-\ln 2\cdot\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}P_n^{(\alpha,\beta)}(t)\,dt\right\},$$

hence, by putting

$$I_n^{(\alpha,\beta)}(\mu) = \int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}P_n^{(\alpha,\beta)}(t)\,dt, \tag{1.3}$$
$$\alpha, \beta, \beta+\mu > -1, n = 0, 1, 2, \ldots ,$$

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we obtain

$$\nu_n^{(\alpha,\beta)}(\mu) = 2^{-(\alpha+\beta+\mu+1)}\left\{I_n^{(\alpha,\beta)}(\mu)\ln 2 - \frac{d}{d\mu} I_n^{(\alpha,\beta)}(\mu)\right\}. \tag{1.4}$$

The following expression for (1.3),

$$I_n^{(\alpha,\beta)}(\mu) = 2^{\alpha+\beta+\mu+1}\,\frac{\Gamma(\mu+1)}{n!\Gamma(\mu-n+1)}\,\frac{\Gamma(\beta+\mu+1)\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+\mu+2)}, \tag{1.5}$$

is known ([1], [4, p. 256]). Indeed, (1.5) is easily obtained, multiplying on both sides of Rodrigues' formula,

$$(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}\,\frac{d^n}{dt^n}\{(1-t)^{n+\alpha}(1+t)^{n+\beta}\},$$

by $(1+t)^\mu$, integrating from -1 to 1 and carrying out n partial integrations on the right-hand side.

Differentiating (1.5) with respect to μ gives

$$\begin{aligned}\frac{d}{d\mu} I_n^{(\alpha,\beta)}(\mu) = I_n^{(\alpha,\beta)}(\mu)\{&\ln 2 + \psi(\mu+1) + \psi(\beta+\mu+1) \\ &- \psi(\mu-n+1) - \psi(n+\alpha+\beta+\mu+2)\},\end{aligned} \tag{1.6}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function, and, if μ coincides with an integer $m < n$, $m \geqslant 0$, the right-hand member must be replaced by its limit as $\mu \longrightarrow m$.

We first consider the case where $\mu \neq 0, 1, 2, \ldots, n-1$, whenever $n \geqslant 1$. By inserting (1.5) and (1.6) in (1.4), we obtain

$$\begin{aligned}\nu_n^{(\alpha,\beta)}(\mu) = &\frac{\Gamma(\mu+1)\Gamma(\beta+\mu+1)\Gamma(n+\alpha+1)}{n!\Gamma(\mu-n+1)\Gamma(n+\alpha+\beta+\mu+2)} \\ &\cdot\{\psi(\mu-n+1) + \psi(n+\alpha+\beta+\mu+2) - \psi(\mu+1) - \psi(\beta+\mu+1)\},\end{aligned} \tag{1.7}$$

with $\alpha, \beta, \beta+\mu > -1$, $n = 0, 1, 2, \ldots$ and $\mu \neq 0, 1, 2, \ldots, n-1$ if $n \geqslant 1$.

Taking into account the recurrence relations $\Gamma(x+1) = x\Gamma(x)$ and $\psi(x+1) = \psi(x) + 1/x$, we may derive a useful algorithm for the computation of the modified moments $\nu_n^{(\alpha,\beta)}(\mu)$. Indeed, it is easily seen that, if we put

$$a_0^{(\alpha,\beta)}(\mu) = \frac{\Gamma(\alpha+1)\Gamma(\beta+\mu+1)}{\Gamma(\alpha+\beta+\mu+2)},$$

$$b_0^{(\alpha,\beta)}(\mu) = \psi(\alpha+\beta+\mu+2) - \psi(\beta+\mu+1),$$

and we construct the two sequences $\{a_n^{(\alpha,\beta)}(\mu)\}$ and $\{b_n^{(\alpha,\beta)}(\mu)\}$, defined by the recurrence relationships

$$a_n^{(\alpha,\beta)}(\mu) = a_{n-1}^{(\alpha,\beta)}(\mu)\,\frac{(\alpha+n)(\mu-n+1)}{n(\alpha+\beta+\mu+n+1)},$$

$$b_n^{(\alpha,\beta)}(\mu) = b_{n-1}^{(\alpha,\beta)}(\mu) + \frac{1}{\alpha + \beta + \mu + 1 + n} - \frac{1}{\mu + 1 - n},$$

we have

$$\nu_n^{(\alpha,\beta)}(\mu) = a_n^{(\alpha,\beta)}(\mu) b_n^{(\alpha,\beta)}(\mu).$$

Therefore, this last expression also shows that (1.7) can be written in the following rational form with respect to n

$$\nu_n^{(\alpha,\beta)}(\mu) = \frac{\Gamma(\alpha+1)\Gamma(\beta+\mu+1)}{\Gamma(\alpha+\beta+\mu+2)} \Bigg\{ \psi(\alpha+\beta+\mu+2) - \psi(\beta+\mu+1) + \sum_{k=1}^{n} \left(\frac{1}{\alpha+\beta+\mu+1+k} - \frac{1}{\mu+1-k} \right) \Bigg\} \cdot \prod_{k=1}^{n} \frac{(\alpha+k)(\mu+1-k)}{k(\alpha+\beta+\mu+1+k)}, \tag{1.8}$$

where α, β, and μ satisfy the above-mentioned conditions.

To examine the remaining case $n \geqslant 1$ and $\mu = m = 0, 1, \ldots, n-1$, we recall that for any integer $r \geqslant 0$,

$$\lim_{\epsilon \to 0} \frac{\psi(-r+\epsilon)}{\Gamma(-r+\epsilon)} = (-1)^{r-1} r!.$$

Then, from (1.7), we obtain

$$\nu_n^{(\alpha,\beta)}(m) = \lim_{\mu \to m} \nu_n^{(\alpha,\beta)}(\mu) = \frac{\Gamma(n+\alpha+1)\Gamma(m+1)\Gamma(\beta+m+1)}{n!\,\Gamma(n+\alpha+\beta+m+2)} \lim_{\epsilon \to 0} \frac{\psi(m+\epsilon-n+1)}{\Gamma(m+\epsilon-n+1)},$$

and finally

$$\nu_n^{(\alpha,\beta)}(m) = (-1)^{n-m} \frac{m!\,(n-m-1)!}{n!} \frac{\Gamma(n+\alpha+1)\Gamma(\beta+m+1)}{\Gamma(n+\alpha+\beta+m+2)}, \quad \alpha, \beta > -1,\ m = 0, 1, 2, \ldots, n-1,\ n \geqslant 1. \tag{1.9}$$

This completes the evaluation of the integrals (1.2). Integrals of the form

$$\int_0^1 x^{\beta+\mu}(1-x)^{\alpha}(\ln(1/x))^{p} P_n^{*(\alpha,\beta)}(x)\,dx,$$

may be similarly evaluated by repeatedly differentiating (1.7) with respect to μ.

2. Some Examples. The results derived in the previous section show that if one has to evaluate modified moments of a given weight function of type (1.1) for given values of ρ and α, then one may choose as polynomial basis the Jacobi polynomials

$P_n^{(\alpha,\beta)}(2x - 1)$, with β being a free parameter. For instance, in the case of the weight function

$$w(x) = x^\rho \ln(1/x), \qquad \rho > -1,$$

we can construct the modified moments associated with the basis $\{P_n^{(0,\beta)}(2x - 1)\}$ instead of the particular one, $\{P_n^{(0,0)}(2x - 1)\}$ considered by Gautschi [3].

It may be of some interest to note that the choice $\rho = \beta$ yields very simple expressions for the corresponding modified moments,

$$(2.1)\quad \nu_n^{(0,\beta)}(0) = \int_0^1 x^\beta \ln(1/x) P_n^{(0,\beta)}(2x - 1)\,dx = \begin{cases} 1/(\beta + 1)^2, & n = 0, \\ \dfrac{(-1)^n \Gamma(\beta + 1)(n - 1)!}{\Gamma(n + \beta + 2)}, & n \geqslant 1. \end{cases}$$

Also, in the case of the more general weight functions (1.1), the formulas we obtain are particularly simple when we let $\rho = \beta$,

$$(2.2)\quad \begin{aligned} &\int_0^1 x^\beta (1 - x)^\alpha \ln(1/x) P_n^{(\alpha,\beta)}(2x - 1)\,dx \\ &= \begin{cases} \dfrac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \{\psi(\alpha + \beta + 2) - \psi(\beta + 1)\}, & n = 0, \\ (-1)^n \dfrac{\Gamma(n + \alpha + 1)\Gamma(\beta + 1)}{n\Gamma(n + \alpha + \beta + 2)}, & n \geqslant 1. \end{cases} \end{aligned}$$

An example of (1.1), with $\alpha \neq 0$, could be the weight function

$$w(x) = x^\rho (1 - x)^{-1/2} \ln(1/x), \qquad \rho > -1,$$

for which, recalling that [4, p. 60]

$$P_n^{*(-1/2,-1/2)}(x) = T_n^*(x) \prod_{k=1}^{n} \frac{2k - 1}{2k},$$

where $T_n^*(x) = T_n(2x - 1)$ is the shifted Chebyshev polynomial of degree n. Setting

$$\tau_n(\rho) = \int_0^1 x^\rho (1 - x)^{-1/2} \ln(1/x) T_n^*(x)\,dx, \qquad \rho > -1,$$

and applying (1.9) and (1.8), we have

$$(2.3)\quad \tau_n(\rho) = \begin{cases} (-1)^{n-m} \dfrac{m!\,(n - m - 1)!}{(n + m)!} \pi \displaystyle\prod_{k=1}^{m} \frac{2k - 1}{2}, \\ \qquad\qquad \rho + 1/2 = m < n,\ m \geqslant 0 \text{ an integer}, \\ \dfrac{\sqrt{\pi}\,\Gamma(\rho + 1)}{\Gamma(\rho + 3/2)} \left\{ \psi(\rho + 3/2) - \psi(\rho + 1) + \displaystyle\sum_{k=1}^{n} \left(\frac{1}{\rho + 1/2 + k} - \frac{1}{\rho + 3/2 - k} \right) \right\} \\ \qquad \cdot \displaystyle\prod_{k=1}^{n} \frac{\rho + 3/2 - k}{\rho + 1/2 + k}, \qquad \text{otherwise}. \end{cases}$$

3. Two Laguerre Polynomial Integrals. In this section we consider the problem of evaluating the modified moments of the weight functions

$$w_p(x) = e^{-x}x^{\rho}(\ln x)^p, \qquad \rho > -1, p = 1, 2, \ldots, \tag{3.1}$$

on $[0, \infty)$, with respect to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$.

We first examine the case $p = 1$, which is relative to a weight function of mixed sign. By introducing, for notational convenience, a new parameter μ such that $\rho = \alpha + \mu$, we refer to the integrals

$$N_{1,n}^{(\alpha)}(\mu) = \int_0^\infty e^{-x}x^{\alpha+\mu}\ln x\, L_n^{(\alpha)}(x)\,dx, \quad \alpha, \alpha + \mu > -1, n = 0, 1, 2, \ldots, \tag{3.2}$$

for which, if we set

$$I_n^{(\alpha)}(\mu) = \int_0^\infty e^{-x}x^{\alpha+\mu}L_n^{(\alpha)}(x)\,dx, \tag{3.3}$$

we have

$$N_{1,n}^{(\alpha)}(\mu) = \frac{d}{d\mu} I_n^{(\alpha)}(\mu). \tag{3.4}$$

The evaluation of $I_n^{(\alpha)}(\mu)$, hence of $N_{1,n}^{(\alpha)}(\mu)$, can be carried out in a way much similar to that concerning the Jacobi polynomials described in Section 1. To this end, we need to recall the Rodrigues formula for Laguerre polynomials

$$e^{-x}x^{\alpha}L_n^{(\alpha)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

Indeed, inserting this last formula in (3.3) and integrating by parts n times, we obtain

$$I_n^{(\alpha)}(\mu) = \frac{(-1)^n}{n!}\frac{\Gamma(\mu + 1)\Gamma(\mu + \alpha + 1)}{\Gamma(\mu - n + 1)}.$$

At this point it is not difficult to derive from (3.4) the following two expressions

$$\begin{aligned} N_{1,n}^{(\alpha)}(\mu) = {} & \frac{(-1)^n}{n!}\frac{\Gamma(\mu + 1)\Gamma(\mu + \alpha + 1)}{\Gamma(\mu - n + 1)} \\ & \cdot \{\psi(\mu + 1) + \psi(\mu + \alpha + 1) - \psi(\mu - n + 1)\}, \end{aligned} \tag{3.5}$$

with $\alpha, \alpha + \mu > -1, n = 0, 1, 2, \ldots$ and $\mu \neq 0, 1, \ldots, n - 1$ if $n \geqslant 1$, and

$$N_{1,n}^{(\alpha)}(m) = (-1)^{m-1}\frac{m!\,(n - m - 1)!}{n!}\Gamma(m + \alpha + 1), \qquad m = 0, 1, \ldots, n - 1, n \geqslant 1; \tag{3.6}$$

the second being obtained from the first by taking its limit as $\mu \longrightarrow m$.

Finally, we remark that (3.5) may be put in the following form

$$N_{1,n}^{(\alpha)}(\mu) = \Gamma(\mu + \alpha + 1)\left\{\psi(\mu + \alpha + 1) - \sum_{k=1}^{n} \frac{1}{k - \mu - 1}\right\} \prod_{k=1}^{n} \frac{k - \mu - 1}{k}. \tag{3.7}$$

The evaluation of the modified moments

$$N_{p,n}^{(\alpha)}(\mu) = \int_0^\infty e^{-x} x^{\alpha+\mu} (\ln x)^p L_n^{(\alpha)}(x)\, dx, \qquad p \geqslant 2,$$

associated to the weight functions (3.1), can be obtained by repeatedly differentiating (3.5) with respect to μ. We shall only examine, with some details, the case $p = 2$.

Differentiating (3.5) once, with respect to μ, gives

$$\begin{aligned} N_{2,n}^{(\alpha)}(\mu) = \frac{(-1)^n}{n!} \frac{\Gamma(\mu + 1)\Gamma(\mu + \alpha + 1)}{\Gamma(\mu - n + 1)} & \\ \cdot \{(\psi(\mu + 1) + \psi(\mu + \alpha + 1) - \psi(\mu - n + 1))^2 & \\ + \psi'(\mu + 1) + \psi'(\mu + \alpha + 1) - \psi'(\mu - n + 1)\}, & \end{aligned} \tag{3.8}$$

which holds for all $n \geqslant 0$ with $\mu \neq 0, 1, 2, \ldots, n - 1$, when $n \geqslant 1$.

A more convenient form of (3.8), obtained by using the previously recalled properties of the functions $\Gamma(x)$ and $\psi(x)$, together with the recurrence relation $\psi'(x + 1) = \psi'(x) - 1/x^2$, is

$$\begin{aligned} N_{2,n}^{(\alpha)}(\mu) = \Gamma(\mu + \alpha + 1)\Bigg\{ \left(\psi(\mu + \alpha + 1) - \sum_{k=1}^{n} \frac{1}{k - \mu - 1}\right)^2 & \\ + \psi'(\mu + \alpha + 1) - \sum_{k=1}^{n} \frac{1}{(k - \mu - 1)^2} \Bigg\} \prod_{k=1}^{n} \frac{k - \mu - 1}{k}. & \end{aligned} \tag{3.9}$$

If $\mu = m = 0, 1, 2, \ldots, n - 1, n \geqslant 1$, from (3.8) we have

$$N_{2,n}^{(\alpha)}(m) = \lim_{\mu \to m} N_{2,n}^{(\alpha)}(\mu) = \frac{(-1)^n}{n!} \Gamma(m + 1)\Gamma(m + \alpha + 1)\{A_n(m) - 2B_n(m)\}, \tag{3.10}$$

where

$$A_n(m) = \lim_{\epsilon \to 0} \frac{\psi^2(m + \epsilon - n + 1) - \psi'(m + \epsilon - n + 1)}{\Gamma(m + \epsilon - n + 1)},$$

$$B_n(m) = \lim_{\epsilon \to 0} \{\psi(m + \epsilon + 1) + \psi(m + \epsilon + \alpha + 1)\} \frac{\psi(m + \epsilon - n + 1)}{\Gamma(m + \epsilon - n + 1)}.$$

By means of the two series expansions

$$\begin{aligned} \Gamma(x) &= \frac{(-1)^r}{r!} \frac{1}{x + r} + \sum_{k=0}^{\infty} a_k (x + r)^k, \\ & \qquad\qquad\qquad\qquad r = 0, 1, 2, \ldots, \\ \psi(x) &= \frac{-1}{x + r} + \psi(1 + r) + \sum_{k=0}^{\infty} b_k (x + r)^k, \end{aligned}$$

which are valid for $|x + r| < 1$, it is easily seen that

$$A_n(m) = (-1)^{n-m} 2(n - m - 1)!\, \psi(n - m),$$
$$B_n(m) = (-1)^{n-m}(n - m - 1)!\{\psi(m + 1) + \psi(m + \alpha + 1)\}.$$

Hence, substituting these last two expressions into (3.10), we obtain the final result

$$N_{2,n}^{(\alpha)}(m) = (-1)^m 2\, \frac{m!\,(n - m - 1)!}{n!}\, \Gamma(m + \alpha + 1) \cdot \{\psi(n - m) - \psi(m + 1) - \psi(m + \alpha + 1)\}, \quad m = 0, 1, \ldots, n - 1,\ n \geqslant 1. \tag{3.11}$$

4. Some Particular Cases. The results derived in Section 3 assume a very simple form when $\mu = 0$, that is in the cases where the weight functions

$$e^{-x}x^\alpha \ln x \quad \text{and} \quad e^{-x}x^\alpha(\ln x)^2, \qquad \alpha > -1,$$

and the polynomials $L_n^{(\alpha)}(x)$ have the same parameter α.

For the first weight function, applying (3.7) and (3.6), we find

$$\int_0^\infty e^{-x}x^\alpha \ln x\, L_n^{(\alpha)}(x)\,dx = \begin{cases} \Gamma(\alpha + 1)\psi(\alpha + 1), & n = 0, \\ -\Gamma(\alpha + 1)/n, & n \geqslant 1, \end{cases} \tag{4.1}$$

which may be regarded as a generalization of the well-known integral representation

$$\gamma = -\int_0^\infty e^{-x} \ln x\, dx,$$

of the Euler-Mascheroni constant $\gamma = -\psi(1) = .57721\ 56649\ldots.$

For the second weight function, by using (3.9) and (3.11), we obtain

$$\int_0^\infty e^{-x}x^\alpha(\ln x)^2 L_n^{(\alpha)}(x)\,dx = \begin{cases} \Gamma(\alpha + 1)\{\psi^2(\alpha + 1) + \psi'(\alpha + 1)\}, & n = 0, \\ \dfrac{2}{n}\Gamma(\alpha + 1)\left\{\displaystyle\sum_{k=1}^{n-1} \frac{1}{k} - \psi(\alpha + 1)\right\}, & n \geqslant 1. \end{cases} \tag{4.2}$$

Two other cases of interest, which may be used, for instance, in constructing the modified moments of the weight functions

$$\exp(-x^2)\ln|x| \quad \text{and} \quad \exp(-x^2)(\ln|x|)^2,$$

on $(-\infty, \infty)$, with respect to the Hermite polynomials $H_n(x)$, are obtained by means of Szegö's relationships [4, p. 106] between Hermite and Laguerre polynomials,

$$H_{2n}(x) = (-1)^n 2^{2n} n!\, L_n^{(-1/2)}(x^2), \qquad H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\, x L_n^{(1/2)}(x^2). \tag{4.3}$$

Indeed, setting $x = t^2$ in the integral (3.2), with $\alpha = -1/2$ and $\mu = 0$, from (3.7) and (3.6) we obtain

$$4\int_0^\infty \exp(-t^2)\ln t\, L_n^{(-1/2)}(t^2)\,dt = \begin{cases} \sqrt{\pi}\psi(1/2) = -\sqrt{\pi}(\gamma + 2\ln 2), & n = 0, \\ -\sqrt{\pi}/n, & n \geqslant 1, \end{cases}$$

hence, by applying the first relation in (4.3),

$$\int_0^\infty \exp(-x^2)\ln x\, H_{2n}(x)\,dx = \begin{cases} \dfrac{-\sqrt{\pi}}{4}(\gamma + 2\ln 2), & n = 0, \\ (-1)^{n-1}(n-1)!\,2^{2(n-1)}\sqrt{\pi}, & n \geqslant 1. \end{cases} \tag{4.4}$$

Similarly, if we assume $\alpha = ½$ and $\mu = -½$ in (3.2), then (3.7), together with the second relation in (4.3), gives

$$\begin{aligned} &\int_0^\infty \exp(-x^2)\ln x\, H_{2n+1}(x)\,dx \\ &\quad = (-1)^{n-1}2^{n-1}\left(\gamma + 2\sum_{k=1}^{n}\frac{1}{2k-1}\right)\prod_{k=1}^{n}(2k-1), \\ &\qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.5}$$

The integrals involving the weight function $\exp(-x^2)(\ln x)^2$ can be dealt with in the same way. Recalling that $\psi'(½) = \pi^2/2$ and $\psi'(1) = \pi^2/6$, by use of (3.9) and (3.11), this leads to the following results,

$$\begin{aligned} &\int_0^\infty \exp(-x^2)(\ln x)^2 H_{2n}(x) \\ &\quad = \begin{cases} \dfrac{\sqrt{\pi}}{8}\{\psi^2(½) + \psi'(½)\} = 1.94752\,21803\ldots, & n = 0, \\ (-1)^n 2^{2(n-1)}(n-1)!\sqrt{\pi}\left(\gamma + 2\ln 2 + \displaystyle\sum_{k=1}^{n-1}\frac{1}{k}\right), & n \geqslant 1, \end{cases} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\int_0^\infty \exp(-x^2)(\ln x)^2 H_{2n+1}(x)\,dx \\ &\quad = (-1)^n 2^{n-2}\left\{\left(\gamma + \sum_{k=1}^{n}\frac{2}{2k-1}\right)^2 + \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{4}{(2k-1)^2}\right\}\prod_{k=1}^{n}(2k-1), \\ &\qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.7}$$

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