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A Stable Finite Element Method for Initial-Boundary Value Problems for First-Order Hyperbolic Systems

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Abstract. A nonstandard finite element method for initial-boundary value problems for first-order hyperbolic systems in one space dimension with general boundary conditions is analyzed. The method can be considered as a generalization of the box scheme. We first establish a stability result for the method and then derive several error estimates.

1. Introduction. The purpose of this paper is to study a finite element method for the first-order system

$$\begin{cases} u_t(x, t) + A(x)u_x(x, t) + B(x)u(x, t) = f(x, t), \\ K_0u(0, t) = g_0(t), \quad K_1u(1, t) = g_1(t), \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

for $(x, t) \in I \times J \equiv [0, 1] \times [0, t^*]$, where $t^* > 0$.

Here $A(x)$ and $B(x)$ are $n \times n$ matrices. We assume that $A(x)$ has n real eigenvalues, $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_n(x)$, and that there exist a constant $\tau_0 > 0$ and an integer m, $0 \leqslant m \leqslant n$, such that

$$-\lambda_{m+1}(x), \lambda_m(x) \geqslant \tau_0 \quad \text{for all } x \in I. \tag{1.2}$$

Furthermore, we assume that for each $x \in I$ there is a nonsingular matrix $Q(x)$ such that

$$A = Q\Lambda Q^{-1},$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and that there exists a constant τ_1 such that

$$|Q(x)|, |Q^{-1}(x)| \leqslant \tau_1 \quad \text{for all } x \in I, \tag{1.3}$$

where $|\cdot|$ denotes the Euclidean matrix norm.

If $y_1(x), y_2(x), \ldots, y_n(x)$ are eigenvectors of $A(x)$, corresponding to the eigenvalues $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$, we let $E^+(x)$ and $E^-(x)$ denote the positive and negative eigenspace, respectively; i.e.,

$$E^+(x) = \operatorname{span}\{y_1(x), y_2(x), \ldots, y_m(x)\}$$

and

$$E^-(x) = \operatorname{span}\{y_{m+1}(x), y_{m+2}(x), \ldots, y_n(x)\}.$$

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The matrices K_0 and K_1 in (1.1) are assumed to be independent of t and of dimension $m \times n$ and $(n - m) \times n$, respectively, and they are assumed to satisfy the condition

$$\ker(K_0) \cap E^+(0) = \{0\} \quad \text{and} \quad \ker(K_1) \cap E^-(1) = \{0\}. \tag{1.4}$$

The data in (1.1) are assumed to be given such that $u_0 \in (L^2(I))^n$, $g_0 \in (L^2(J))^m$, $g_1 \in (L^2(J))^{n-m}$, and $f \in (L^2(I \times J))^n$. Throughout this paper we shall also assume that $B \in (C(I))^{n\times n}$ and $Q, \Lambda, A \in (C^{(1)}(I))^{n\times n}$.

Now let $v(x, t) = Q^{-1}(x)u(x, t)$. It then follows from (1.4) that there exists matrices $S_{0,0}$, $S_{0,1}$, $S_{1,0}$, and $S_{1,1}$ of dimension $m \times (n - m)$, $m \times m$, $(n - m) \times m$, and $(n - m) \times (n - m)$, respectively, such that v satisfies the diagonal system

$$\begin{cases} v_t(x, t) + \Lambda(x)v_x(x, t) + \tilde{B}(x)v(x, t) = Q^{-1}(x)f(x, t), \\ v^+(0, t) = S_{0,0}v^-(0, t) + S_{0,1}g_0(t), \\ v^-(1, t) = S_{1,0}v^+(1, t) + S_{1,1}g_1(t), \\ v(x, 0) = v_0(x), \end{cases} \tag{1.5}$$

where $v_0 = Q^{-1}u_0$, $\tilde{B} = Q^{-1}BQ - \Lambda(Q^{-1})_xQ$, and $v^+ \in \mathbf{C}^m$ and $v^- \in \mathbf{C}^{n-m}$ are such that

$$v = \begin{pmatrix} v^+ \\ v^- \end{pmatrix}.$$

It is also well known (see, for example, Thomée [11]) that the system (1.1) has a unique solution $u \in C(0, t^*; L^2(I)^n)$, under the conditions given above, and that there exists a constant c, independent of u_0, g_0, g_1, and f, such that

$$\sup_{0<t<t^*} \|u(\cdot, t)\|_{L^2(I)} \leqslant c\{\|u_0\|_{L^2(I)} + \|g_0\|_{L^2(J)} + \|g_1\|_{L^2(J)} + \|f\|_{L^2(I\times J)}\}. \tag{1.6}$$

Finite difference methods for general mixed hyperbolic systems of the form (1.1) have been intensively studied (see, for example, Kreiss [8] and Gustafsson, Kreiss, and Sundström [7]), but very little theoretical work has been done in the direction of applying finite element methods to such problems. The reason for this is probably that the standard Galerkin method does not yield optimal error estimates when it is applied to first-order hyperbolic equations (see Dupont [5]) and that this method requires special care in order to treat the boundary conditions in (1.1); (see Gunzburger [6]).

The goal of this paper is to prove a stability result similar to (1.6) for a nonstandard finite element method for (1.1) and then use this to derive error estimates for the method. The finite element method, which will be precisely defined in Section 3, consists of using continuous trial functions and discontinuous test functions (both in space and time). The treatment of the boundary conditions in this method is straightforward and can be considered as a generalization of the so-called box scheme. The box scheme was analyzed for problems of the form (1.5) by Thomée [10] and for certain nonlinear problems by Luskin [9]. The method considered here is also closely related to a finite element method analyzed by the author [13] for the nonlinear Korteweg-de Vries equation and it can (except for the treatment of the boundary conditions) be considered as a dual method of the (semidiscrete) method studied for one first-order hyperbolic equation by Baker [2]. The time-stepping part of our method is closely related to discretization in time by

collocation, which was studied for parabolic equations by Douglas and Dupont [4].

Some local properties of the method are derived in Section 4 and the stability estimate is proved in Section 5. L^2-error estimates are given in Section 6 and some superconvergence results are derived in Section 7.

2. Notation. We shall use $\langle \cdot, \cdot \rangle$ to denote the dot product on $\mathbf{C}^n$, and $|\cdot|$ will denote the associated vector and matrix norm. If $w \in \mathbf{C}^n$, then we write

$$w = \begin{pmatrix} w^+ \\ w^- \end{pmatrix},$$

where $w^+ \in \mathbf{C}^m$ and $w^- \in \mathbf{C}^{n-m}$. Occasionally, we shall also use w^+ to denote the vector $\binom{w^+}{0} \in \mathbf{C}^n$, and similarly for w^-. In the same way, we write the diagonal matrix Λ in the block form

$$\Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix}.$$

For an arbitrary Banach space X, we let $\|\cdot\|_X$ denote the norm on X, and, if $j \geqslant 1$ is an integer, then $\|\cdot\|_X$ also denotes the norm on X^j.

For any integer $\nu \geqslant 0$, $H^\nu(I)$ denotes the Sobolev space of functions on I with ν derivatives in $L^2(I)$, and, if $\nu < 0$, then $H^\nu(I)$ is the dual of $H^{-\nu}(I)$ with respect to the inner product on $L^2(I)$. For precise definitions we refer to [1]. We shall frequently use the notation

$$\|\cdot\|_\nu = \|\cdot\|_{H^\nu(I)} \quad \text{and} \quad \|\cdot\| = \|\cdot\|_0.$$

Also, if $\nu_1, \nu_2 \geqslant 0$ are integers and $\varphi \in C^\infty(I \times J)$, define

$$\|\varphi\|^2_{\nu_1,\nu_2} = \sum_{i=0}^{\nu_1} \sum_{j=0}^{\nu_2} \left\| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|^2_{L^2(I\times J)}.$$

We let $H^{\nu_1,\nu_2}(I \times J)$ be the completion of $C^\infty(I \times J)$ in this norm.

Throughout this paper, c will denote a generic constant, not necessarily the same at different occurrences.

3. The Finite Element Method. In this section we shall describe our finite element method for the equation (1.1). For any integer $r \geqslant 1$, let $\mathbf{P}_r$ denote the set of polynomials of degree $< r$. Let Δ be a family of partitions of I; i.e., if $\delta \in \Delta$, then $\delta = \{x_i\}_{i=0}^M$, where

$$0 = x_0 < x_1 < \cdots < x_M = 1.$$

We shall use the notation $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$ and $h = \max_{1\leqslant i\leqslant M} h_i$.

For integers $r \geqslant 1$ and $-1 \leqslant \nu < r - 1$, define

$$S_\delta(r, \nu) = \left\{\chi \in C^{(\nu)}(I) | \chi_{|I_i} \in \mathbf{P}_r,\ 1 \leqslant i \leqslant M\right\},$$

where $C^{(-1)}(I)$ denotes the set of piecewise continuous functions on I. Observe that the spaces $S_\delta(r, \nu)$ have the approximation property that, for any integer j, $\nu + 1 \leqslant j \leqslant r$, there is a constant $c > 0$ such that

$$\inf_{\chi\in S_\delta(r,\nu)} \sum_{i=0}^{\nu+1} h^i \|\varphi - \chi\|_i \leqslant ch^j \|\varphi\|_j, \tag{3.1}$$

for all $\varphi \in H^j(I)$.

Similarly, let Γ be a family of partitions of J. If $\gamma = \{t_j\}_{j=0}^{N} \in \Gamma$, we let $J_j = (t_{j-1}, t_j)$, $k_j = t_j - t_{j-1}$ and $k = \max_{1 \leqslant j \leqslant N} k_j$, and, if $s \geqslant 1$ and $-1 \leqslant \nu < s - 1$, then the spaces $S_\gamma(s, \nu)$ are defined in the same way as above.

We shall be interested in partitions (δ, γ) belonging to a subset Ω of $\Delta \times \Gamma$. We assume that there is a constant $\tau_2 > 1$ such that, for all $(\delta, \gamma) \in \Omega$,

$$\tau_2^{-1} \leqslant k_j / h_i \leqslant \tau_2, \qquad 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N. \tag{3.2}$$

For the rest of this paper, we let r and s be fixed integers $\geqslant 1$ such that $s \leqslant r + 1$. For each $(\delta, \gamma) \in \Omega$, we let $\mathfrak{N}_\delta = S_\delta(r + 1, 0)$, $\mathfrak{N}_\gamma = S_\gamma(s + 1, 0)$, $\mathfrak{N}_\delta = S_\delta(r, -1)$, and $\mathfrak{N}_\gamma = S_\gamma(s, -1)$. Observe that it follows from (3.2) that there is a constant $c > 0$ such that, for all $\chi \in \mathfrak{N}_\delta$ and $0 \leqslant j \leqslant r$,

$$\sum_{i=1}^{M} \|\chi\|_{H^j(I_i)} \leqslant \begin{cases} ch^{-j}\|\chi\|, \\ ck^{-j}\|\chi\|, \end{cases} \tag{3.3}$$

and similar inverse properties holds for the other spaces above.

Adopting a tensor product notation, we define

$$\mathfrak{N}_{\delta,\gamma} = \mathfrak{N}_\delta \otimes \mathfrak{N}_\gamma \quad \text{and} \quad \mathfrak{N}_{\delta,\gamma} = \mathfrak{N}_\delta \otimes \mathfrak{N}_\gamma.$$

Observe that if $U \in \mathfrak{N}_{\delta,\gamma}$, then $U_{xt} \in \mathfrak{N}_{\delta,\gamma}$. We shall let P_δ, P_γ, and $P_{\delta,\gamma}$ be the L^2-projections onto the spaces $\mathfrak{N}_\delta$, $\mathfrak{N}_\gamma$, and $\mathfrak{N}_{\delta,\gamma}$, respectively. Since the spaces $\mathfrak{N}_\delta$ and $\mathfrak{N}_\gamma$ are discontinuous spaces, it follows that all the projections above are completely local. Hence, if $a \in C^{(1)}(I)$ is fixed, then since

$$a\chi - P_\delta(a\chi) = (a - P_\delta a)\chi + (I - P_\delta)((P_\delta a)\chi) - P_\delta((a - P_\delta a)\chi)$$

it follows by inverse properties that there is a constant c such that, for all $\chi \in \mathfrak{N}_\delta$,

$$\|a\chi - P_\delta(a\chi)\| \leqslant ch\|\chi\|. \tag{3.4}$$

A similar property holds for the projection P_γ.

Our finite element method for the equation (1.1) consists of finding $U \in \mathfrak{N}_{\delta,\gamma}^n$ such that

$$\begin{cases} \int_0^{t^*} \int_0^1 \langle U_t + AU_x + BU - f, \chi \rangle \, dx \, dt = 0, \quad \text{for } \chi \in \mathfrak{N}_{\delta,\gamma}^n, \\ K_0 U(0, t) = G_0(t), \quad K_1 U(1, t) = G_1(t), \\ U(x, 0) = U_0(x). \end{cases} \tag{3.5}$$

Here $U_0 \in \mathfrak{N}_\delta^n$, $G_0 \in \mathfrak{N}_\gamma^m$, and $G_1 \in \mathfrak{N}_\gamma^{n-m}$ will be chosen as approximations of u_0, g_0, and g_1, respectively, and we always assume that $K_0 U_0(0) = G_0(0)$ and $K_1 U_0(1) = G_1(0)$. We note that

$$\dim(\mathfrak{N}_{\delta,\gamma}^n) = n(Mr + 1)(Ns + 1), \quad \dim(\mathfrak{N}_{\delta,\gamma}^n) = nMrNs,$$

and that the initial and boundary conditions in (3.5) represent $n(Mr + Ns + 1)$ linear equations. We also observe that the method (3.5) is a time-stepping method in the sense that $U|_{J_j}$ can be computed from $U(\cdot, t_{j-1})$, $G_0|_{J_j}$, $G_1|_{J_j}$, and $f|_{J_j}$. In fact, if we let $r = s = 1$ and if A and B are independent of x, then the method (3.5) is equivalent to the box scheme.

4. Local Properties. In this section we derive some preliminary results that will be needed in Section 5 in order to prove stability of the method. We first prove a local property for the method when it is applied to a single equation with a constant coefficient.

LEMMA 4.1. *Let r and s be integers such that $r, s \geqslant 1$ and $s \leqslant r + 1$ and let $\bar{a} \neq 0$ be a real number. Assume that $U \in \mathbf{P}_{r+1} \otimes \mathbf{P}_{s+1}$ such that*

$$U(x, 0) \equiv U(x, 1) \equiv 0 \tag{4.1}$$

and

$$\int_0^1 \int_0^1 (U_t + \bar{a}U_x)\chi \, dx \, dt = 0 \quad \text{for } \chi \in \mathbf{P}_r \otimes \mathbf{P}_s. \tag{4.2}$$

Then $U \equiv 0$.

Proof. For $j = 0, 1, \ldots, s - 1$, define $\alpha_j \in \mathbf{P}_{r+1}$ by

$$\alpha_j(x) = \int_0^1 U(x, t)t^j \, dt.$$

Also let $\alpha_{-1} \equiv 0$. It follows from (4.1), (4.2) and integration by parts that, for any $q \in \mathbf{P}_r$,

$$\begin{aligned} \bar{a}\int_0^1 \alpha_j'(x)q(x) \, dx &= \bar{a}\int_0^1 \int_0^1 U_x(x, t)t^j \, dtq(x) \, dx \\ &= -\int_0^1 \int_0^1 U_t(x, t)t^j \, dtq(x) \, dx \\ &= j\int_0^1 \int_0^1 U(x, t)t^{j-1} \, dtq(x) \, dx \end{aligned}$$

or

$$\bar{a}\int_0^1 \alpha_j'(x)q(x) \, dx = j\int_0^1 \alpha_{j-1}(x)q(x) \, dx,$$

for $j = 0, 1, \ldots, s - 1$. Therefore, since $\alpha_{-1} \equiv 0$, we obtain that

$$\bar{a}\alpha_j' = j\alpha_{j-1}, \qquad j = 0, 1, \ldots, s - 1, \tag{4.3}$$

and

$$\alpha_j \in \mathbf{P}_{j+1}, \qquad j = 0, 1, \ldots, s - 1. \tag{4.4}$$

Now let $\{p_i\}_{i=0}^r \subset \mathbf{P}_{s+1}$ be such that

$$U(x, t) = \sum_{i=0}^r p_i(t)x^i.$$

Then

$$\alpha_j(x) = \sum_{i=0}^r x^i \int_0^1 p_i(t)t^j \, dt,$$

and therefore (4.4) implies that

$$\int_0^1 p_i(t)t^j \, dt = 0, \qquad j = 0, 1, \ldots, \min(i, s) - 1. \tag{4.5}$$

For each integer $j \geqslant 0$, let $L_j(t)$ be the Legendre polynomial of degree j on $[0, 1]$ such that $L_j(0) = 1$. Note that this implies that $L_j(1) = (-1)^j$. We now wish to show that $p_i \equiv 0$ for $i = 0, 1, \ldots, r$. We first note that if $i \geqslant s$, then (4.5) implies that $p_i(t) = \mathcal{K}_i L_s(t)$, where $\mathcal{K}_i$ is a constant. Since $p_i(0) = 0$, this implies that $p_i \equiv 0$. Also note that we obtain from (4.5) that

$$p_{s-1}(t) = \mathcal{K}_{s-1}L_{s-1}(t) + \tilde{\mathcal{K}}_{s-1}L_s(t),$$

for suitable constants $\mathcal{K}_{s-1}$ and $\tilde{\mathcal{K}}_{s-1}$. But, since $p_{s-1}(0) = p_{s-1}(1) = 0$ and $L_j(1) = (-1)^j L_j(0)$, this implies that $p_{s-1} \equiv 0$.

We now prove that $p_i \equiv 0$ for $0 \leqslant i \leqslant s-1$ by induction. Assume i_0 is an integer such that $0 < i_0 \leqslant s-1$ and $p_{i_0} \equiv 0$. From (4.3) and (4.4) we obtain that, for $0 \leqslant j \leqslant s-1$,

$$j \sum_{i=0}^{j-1} x^i \int_0^1 p_i(t) t^{j-1}\, dt = \bar{a} \sum_{i=0}^{j-1} (i+1) x^i \int_0^1 p_{i+1}(t) t^j\, dt,$$

or

$$j \int_0^1 p_i(t) t^{j-1}\, dt = \bar{a}(i+1) \int_0^1 p_{i+1}(t) t^j\, dt,$$

for $i = 0, 1, \ldots, j-1$. Therefore, since $p_{i_0} \equiv 0$, we have

$$\int_0^1 p_{i_0-1}(t) t^j\, dt = 0, \qquad j = i_0 - 1, i_0, \ldots, s-2.$$

Also note that (4.5) implies that

$$\int_0^1 p_{i_0-1}(t) t^j\, dt = 0, \qquad j = 0, 1, \ldots, i_0 - 2.$$

Hence, there exist constants $\mathcal{K}_{i_0-1}$ and $\tilde{\mathcal{K}}_{i_0-1}$ such that

$$p_{i_0-1}(t) = \mathcal{K}_{i_0-1} L_{s-1}(t) + \tilde{\mathcal{K}}_{i_0-1} L_s(t),$$

and as above this implies that $p_{i_0-1} \equiv 0$. This completes the induction argument and hence $U \equiv 0$. □

Let $0 = \xi_0 < \xi_1 < \cdots < \xi_r = 1$ be a partition of I and let

$$\xi_{ij} = x_{i-1} + \xi_j h_i \quad \text{for } 1 \leqslant i \leqslant M, 0 \leqslant j \leqslant r.$$

For any function $\varphi \in C(I)$, define $R_\delta \varphi \in \mathfrak{N}_\delta$ by interpolating φ at the points $\{\xi_{ij}\}$; i.e.,

$$(R_\delta \varphi)(\xi_{ij}) = \varphi(\xi_{ij}) \quad \text{for } 1 \leqslant i \leqslant M, 0 \leqslant j \leqslant r.$$

We observe that the operator R_δ is defined locally on each subinterval I_i. Hence, since $a\varphi - R_\delta(a\varphi) = (a - R_\delta a)\varphi + (I - R_\delta)((R_\delta a)\varphi)$, it follows, from (3.3) and standard error estimates for polynomial interpolation, that if $a \in C^{(1)}(I)$, then there is a constant c, depending on a, such that, for any $\varphi \in \mathfrak{N}_\delta$,

$$\|a\varphi - R_\delta(a\varphi)\| + h\|a\varphi - R_\delta(a\varphi)\|_1 \leqslant ch\|\varphi\|. \tag{4.6}$$

The operator R_δ will now be used to diagonalize the system (3.5). Assume that $U \in \mathfrak{N}_{\delta,\gamma}^n$ is a solution of (3.5) and define $V \in \mathfrak{N}_{\delta,\gamma}^n$ by

$$V = R_\delta(Q^{-1}U), \tag{4.7}$$

for all $t \in J$. We note that (4.6) implies that there is a constant $c > 1$ such that, for h sufficiently small,

$$c^{-1}\|V(\cdot, t)\| \leqslant \|U(\cdot, t)\| \leqslant c\|V(\cdot, t)\|, \tag{4.8}$$

for all $t \in J$.

Now observe that (3.5) implies that

$$\int_0^{t^*} \int_0^1 \langle Q\{(Q^{-1}U)_t + \Lambda(Q^{-1}U)_x + \tilde{B}(Q^{-1}U)\} - f, \chi\rangle\, dx\, dt = 0,$$

for all $\chi \in \mathfrak{N}_{\delta,\gamma}^n$, where $\tilde{B}$ is defined in Section 1. Hence, it follows from (4.6) that, for all $\chi \in \mathfrak{N}_{\delta,\gamma}^n$,

$$\int_0^{t^*} \int_0^1 \langle Q\{V_t + \Lambda V_x + \tilde{B}V + \omega_0\} - f, \chi\rangle\, dx\, dt = 0, \tag{4.9}$$

where ω_0 satisfies

$$\|\omega_0\|_{L^2(I_i\times J_j)} \leqslant c\|V\|_{L^2(I_i\times J_j)}, \tag{4.10}$$

for $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$, and some constant c independent of V.

Define now $\omega_1 \in \mathfrak{V}^n_{\delta,\gamma}$ by

$$\begin{aligned}\int_0^{t^*}\int_0^1 \langle \omega_1, P_\delta Q^*\chi\rangle\, dx\, dt \\ = -\int_0^{t^*}\int_0^1 \langle V_t + \Lambda V_x + \tilde{B}V + \omega_0, (P_\delta - I)Q^*\chi\rangle\, dx\, dt,\end{aligned} \tag{4.11}$$

for all $\chi \in \mathfrak{V}^n_{\delta,\gamma}$. By (3.3), (3.4), and (4.10), we obtain that ω_1 is uniquely defined for h sufficiently small and there is a constant c such that

$$\|\omega_1\|_{L^2(I_i\times J_j)} \leqslant c\|V\|_{L^2(I_i\times J_j)}, \qquad 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N. \tag{4.12}$$

Also note that it follows from (3.4) that $P_\delta Q$ maps $\mathfrak{V}^n_{\delta,\gamma}$ onto $\mathfrak{V}^n_{\delta,\gamma}$ for h sufficiently small. Let $(P_\delta Q)^{-1}$ denote $(P_\delta Q|_{\mathfrak{V}^n_{\delta,\gamma}})^{-1}$ and define $\tilde{f} \in \mathfrak{V}^n_{\delta,\gamma}$ by $\tilde{f} = (P_\delta Q)^{-1}P_\delta f$. Then there is a constant c such that, for h sufficiently small,

$$\|\tilde{f}\|_{L^2(I_i\times J_j)} \leqslant c\|f\|_{L^2(I_i\times J_j)}, \qquad 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N. \tag{4.13}$$

If we let $\omega = \omega_0 + \omega_1$, it follows from (4.9) and (4.11) that V satisfies the equation

$$\int_0^{t^*}\int_0^1 \langle V_t + \Lambda V_x + \tilde{B}V + \omega - \tilde{f}, P_\delta Q^*\chi\rangle\, dx\, dt = 0,$$

for all $\chi \in \mathfrak{V}^n_{\delta,\gamma}$, and by (4.10) and (4.12)

$$\|\omega\|_{L^2(I_i\times J_j)} \leqslant c\|V\|_{L^2(I_i\times J_j)}, \qquad 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N, \tag{4.14}$$

where the constant c is independent of V. Finally, we note as above that, for h sufficiently small, $P_\delta Q^*$ maps $\mathfrak{V}^n_{\delta,\gamma}$ onto $\mathfrak{V}^n_{\delta,\gamma}$ and hence in this case V solves the following discrete analog of (1.5).

$$\begin{cases} \displaystyle\int_0^{t^*}\int_0^1 \langle V_t + \Lambda V_x + \tilde{B}V + \omega - \tilde{f}, \chi\rangle\, dx\, dt = 0 \quad \text{for } \chi \in \mathfrak{V}^n_{\delta,\gamma}, \\ V^+(0,t) = S_{0,0}V^-(0,t) + S_{0,1}G_0(t), \\ V^-(1,t) = S_{1,0}V^+(1,t) + S_{1,1}G_1(t), \\ V(x,0) = V_0(x), \qquad \text{where } V_0 = R_\delta(Q^{-1}U_0). \end{cases} \tag{4.15}$$

The following result is now a consequence of Lemma 4.1.

LEMMA 4.2. *Assume that $U \in \mathfrak{V}^n_{\delta,\gamma}$ is a solution of* (3.5) *and let V be defined by* (4.7). *Then there is a constant c such that, for h sufficiently small and $1 \leqslant j \leqslant N$,*

$$\|V\|^2_{L^2(I\times J_j)} \leqslant ck_j\{\|V(\cdot, t_{j-1})\|^2 + \|V(\cdot, t_j)\|^2 + k_j\|f\|^2_{L^2(I\times J_j)}\}.$$

Proof. Let i and j be fixed integers such that $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. Define $\hat{V}$ and $\hat{f}$ in $(\mathbf{P}_{r+1} \otimes \mathbf{P}_{s+1})^n$ by

$$\hat{V}(x,t) = V(x_{i-1} + xh_i, t_{j-1} + tk_j)$$

and

$$\hat{f}(x,t) = \tilde{f}(x_{i-1} + xh_i, t_{j-1} + tk_j),$$

for $(x, t) \in \mathbf{R}^2$, and let $\bar{\Lambda} = \Lambda(x_i)$. Then $\hat{V}$ satisfies the equation

$$\int_0^1 \int_0^1 \left\langle \hat{V}_t + \frac{k_j}{h_i} \bar{\Lambda} \hat{V}_x + k_j \mathcal{E}(\hat{V}) - k_j \hat{f}, \chi \right\rangle dx\, dt = 0$$

for all $\chi \in (\mathbf{P}_r \otimes \mathbf{P}_s)^n$, where $\mathcal{E}(\hat{V})$ depends linearly on $\hat{V}$ and by (3.3) and (4.14)

$$\|\mathcal{E}(\hat{V})\|_{L^2(I\times I)} \leqslant c \|\hat{V}\|_{L^2(I\times I)}.$$

Here the constant c is independent of $\hat{V}$ and k_j. Hence, it follows from Lemma 4.1 and (3.2) that there is a constant c such that, for k_j sufficiently small,

$$\|\hat{V}\|^2_{L^2(I\times I)} \leqslant c\{\|\hat{V}(\cdot, 0)\|^2 + \|\hat{V}(\cdot, 1)\|^2 + k_j^2 \|\hat{f}\|^2_{L^2(I\times I)}\}$$

or

$$\|V\|^2_{L^2(I_i\times J_j)} \leqslant c k_j \{\|V(\cdot, t_{j-1})\|^2 + \|V(\cdot, t_j)\|^2 + k_j \|\tilde{f}\|^2_{L^2(I_i\times J_j)}\}.$$

The desired result now follows by summation from $i = 1$ to $i = M$ and by (4.13). □

5. L^2-Stability. The purpose of this section is to prove an estimate similar to (1.6) for the method (3.5). The result below will be obtained by combining the local results from the previous section with a generalization of an argument used by Thomée [10] in order to prove stability of the box scheme.

THEOREM 5.1. *Assume that $U \in \mathcal{M}^n_{\delta,\gamma}$ is a solution of* (3.5). *Then there is a constant c independent of U, such that, for h sufficiently small,*

$$\max_{\bar{t}\in\gamma} \|U(\cdot, \bar{t})\| \leqslant c\{\|U_0\| + \|G_0\|_{L^2(J)} + \|G_1\|_{L^2(J)} + \|f\|_{0,0}\}.$$

Before we prove the result above we note that, since (3.5) is a system of linear equations with as many equations as unknowns, the following corollary follows directly from Theorem 5.1.

COROLLARY 5.2. *The system* (3.5) *has a unique solution $U \in \mathcal{M}^n_{\delta,\gamma}$ for h sufficiently small.*

Proof of Theorem 5.1. First observe that because of (4.8) it is enough to show that

$$\max_{\bar{t}\in\gamma} \|V(\cdot, \bar{t})\| \leqslant c\{\|V_0\| + \|G_0\|_{L^2(J)} + \|G_1\|_{L^2(J)} + \|f\|_{0,0}\}, \tag{5.1}$$

for some constant c independent of V, where V is defined by (4.7).

Now let $\varepsilon \in (0, 1)$ be fixed, where the value of ε will be chosen later. Define two piecewise constant functions d^+ and d^- on I by

$$d^+(x) = (1 - \varepsilon)x_{i-1} + \varepsilon \quad \text{for } x \in I_i,$$

and

$$d^-(x) = 1 - (1 - \varepsilon)x_{i-1} \quad \text{for } x \in I_i.$$

Observe that

$$d^+(x_i+) - d^+(x_i-) = d^-(x_i-) - d^-(x_i+) = (1 - \varepsilon)h_i. \tag{5.2}$$

Let D be the $n \times n$ matrix given by

$$D = \begin{bmatrix} D^+ & 0 \\ 0 & D^- \end{bmatrix},$$

where $D^+ = \operatorname{diag}(d^+, d^+, \ldots, d^+)$ and $D^- = \operatorname{diag}(d^-, d^-, \ldots, d^-)$ are matrices of dimension $m \times m$ and $(n-m) \times (n-m)$, respectively. We note that

$$I \geqslant D \geqslant \varepsilon I. \tag{5.3}$$

We now prove the estimate (5.1) by energy arguments. For an integer j, $1 \leqslant j \leqslant N$, let $\chi \in \mathcal{U}^n_{\delta,\gamma}$ be given by

$$\chi(x, t) = \begin{cases} D(P_{\delta,\gamma}V)(x, t) & \text{if } t \in J_j, \\ 0 & \text{otherwise.} \end{cases}$$

By using χ as a test function in (4.15), we obtain

$$\int_{t_{j-1}}^{t_j} \int_0^1 \langle V_t + \Lambda V_x + \tilde{B}V + \omega - \tilde{f}, D(P_{\delta,\gamma}V)\rangle \, dx \, dt = 0. \tag{5.4}$$

First observe that, since for each $x \in I$, $V_t(x, \cdot) \in \mathcal{U}^n_\gamma$, we have that

$$\int_{t_{j-1}}^{t_j} \int_0^1 \langle V_t, D(P_{\delta,\gamma}V)\rangle \, dx \, dt = \frac{1}{2} \int_{t_{j-1}}^{t_j} \frac{d}{dt} \|D^{1/2}(P_\delta V)\|^2 \, dt$$

$$= \frac{1}{2}\{\|D^{1/2}(P_\delta V)(\cdot, t_j)\|^2 - \|D^{1/2}(P_\delta V)(\cdot, t_{j-1})\|^2\}.$$

Also note that, by (4.13), (4.14), and (5.3), we obtain

$$\int_{t_{j-1}}^{t_j} \int_0^1 \langle \tilde{B}V + \omega - \tilde{f}, D(P_{\delta,\gamma}V)\rangle \, dx \, dt \leqslant c\{\|V\|^2_{L^2(I\times J_j)} + \|f\|^2_{L^2(I\times J_j)}\},$$

where the constant c is independent of V. Let $\bar{\Lambda}$ be the piecewise constant matrix on I given by $\bar{\Lambda}(x) = \Lambda(x_i)$ for $x \in I_i$. Then there is a constant c such that

$$\int_{t_{j-1}}^{t_j} \int_0^1 \langle (\Lambda - \bar{\Lambda})V_x, D(P_{\delta,\gamma}V)\rangle \, dx \, dt$$

$$\leqslant ch\|V_x\|_{L^2(I\times J_j)}\|V\|_{L^2(I\times J_j)} \leqslant c\|V\|^2_{L^2(I\times J_j)},$$

where we have used the inverse property (3.3). Finally, we obtain from (5.2) that, for any $t \in J_j$,

$$\int_{t_{j-1}}^{t_j} \int_0^1 \langle \bar{\Lambda} V_x, D(P_{\delta,\gamma}V)\rangle \, dx \, dt = \frac{1}{2} \int_{t_{j-1}}^{t_j} \int_0^1 \langle D\bar{\Lambda}(P_\gamma V), P_\gamma V\rangle \, dx \, dt$$

$$= \frac{1}{2} \int_{t_{j-1}}^{t_j} \sum_{i=1}^{M} \{\langle D\bar{\Lambda}(P_\gamma V), P_\gamma V\rangle(x_i-, t) - \langle D\bar{\Lambda}(P_\gamma V), P_\gamma V\rangle(x_{i-1}+, t)\} \, dt$$

$$\geqslant \frac{1}{2} \int_{t_{j-1}}^{t_j} \{(1-\varepsilon)|(\Lambda^+(1))^{1/2}(P_\gamma V)^+(1, t)|^2 - \varepsilon|(\Lambda^+(x_1))^{1/2}(P_\gamma V)^+(0, t)|^2$$

$$+ (1-\varepsilon)|(-\Lambda^-(x_1))^{1/2}(P_\gamma V)^-(0, t)|^2 - \varepsilon|(-\Lambda^-(1))^{1/2}(P_\gamma V)^-(1, t)|^2\} \, dt$$

$$- c \int_{t_{j-1}}^{t_j} \sum_{i=1}^{M-1} h_i |P_\gamma V(x_i, t)|^2 \, dt.$$

Since there is a constant c such that

$$\int_{t_{j-1}}^{t_j} \sum_{i=1}^{M-1} h_i |P_\gamma V(x_i, t)|^2 \, dt \leqslant \sum_{i=1}^{M-1} h_i \int_{t_{j-1}}^{t_j} |V(x_i, t)|^2 \, dt$$

$$\leqslant c\|V\|^2_{L^2(I\times J_j)},$$

it therefore follows from (5.4) and the estimates above that there is a constant c such that

$$
\begin{aligned}
&\|D^{1/2}(P_\delta V)(\cdot, t_j)\|^2 - \|D^{1/2}(P_\delta V)(\cdot, t_{j-1})\|^2 \\
&\qquad + (1-\varepsilon)\big\{\|(\Lambda^+(1))^{1/2}(P_\gamma V)^+(1,\cdot)\|^2_{L^2(J_j)} \\
&\qquad\qquad + \|(-\Lambda^-(0))^{1/2}(P_\gamma V)^-(0,\cdot)\|^2_{L^2(J_j)}\big\} \\
&\qquad - \varepsilon\big\{\|(\Lambda^+(0))^{1/2}(P_\gamma V)^+(0,\cdot)\|^2_{L^2(J_j)} \\
&\qquad\qquad + \|(-\Lambda^-(1))^{1/2}(P_\gamma V)^-(1,\cdot)\|^2_{L^2(J_j)}\big\} \\
&\qquad \leqslant c\big\{\|V\|^2_{L^2(I\times J_j)} + \|f\|^2_{L^2(I\times J_j)}\big\}.
\end{aligned} \tag{5.5}
$$

In order to obtain control over the full L^2-norm of $V(\cdot, t_j)$, we need to apply one other test function in (4.15). First let $\tilde{I}$ be the $n \times n$ matrix given by

$$
\tilde{I} = \begin{bmatrix} I^+ & 0 \\ 0 & -I^- \end{bmatrix},
$$

where I^+ and I^- are the identity matrices of dimension $m \times m$ and $(n-m) \times (n-m)$, respectively, and let $\tilde{\Lambda} = \tilde{I}\Lambda$. By using the function

$$
\chi(x,t) = \begin{cases} \tilde{I}DV_{xt}(x,t) & \text{if } t \in J_j, \\ 0 & \text{otherwise,} \end{cases}
$$

as a test function in (4.15), we obtain

$$
\int_{t_{j-1}}^{t_j}\int_0^1 \langle V_t + \Lambda V_x + \tilde{B}V + \omega - \tilde{f}, \tilde{I}DV_{xt}\rangle \, dx\, dt = 0. \tag{5.6}
$$

We observe that

$$
\begin{aligned}
\int_{t_{j-1}}^{t_j}\int_0^1 \langle \Lambda V_x, \tilde{I}DV_{xt}\rangle \, dx\, dt &= \frac{1}{2}\int_{t_{j-1}}^{t_j} \frac{d}{dt}\|(\tilde{\Lambda}D)^{1/2}V_x\|^2 \, dt \\
&= \frac{1}{2}\big\{\|(\tilde{\Lambda}D)^{1/2}V_x(\cdot, t_j)\|^2 - \|(\tilde{\Lambda}D)^{1/2}V_x(\cdot, t_{j-1})\|^2\big\}.
\end{aligned}
$$

Also, by (3.3), (4.13), (4.14), and (5.3),

$$
\int_{t_{j-1}}^{t_j}\int_0^1 \langle \tilde{B}V + \omega - \tilde{f}, \tilde{I}DV_{xt}\rangle \, dx\, dt \leqslant ch^{-2}\big(\|V\|^2_{L^2(I\times J_j)} + \|f\|^2_{L^2(I\times J_j)}\big),
$$

where the constant c is independent of V. In the same way as above, we also obtain from (5.2) that there is a constant c such that, for any $t \in J_j$,

$$
\begin{aligned}
\int_0^1 \langle V_t, \tilde{I}DV_{xt}\rangle \, dx &= \frac{1}{2}\int_0^1 \frac{d}{dx}\big(|(D^+)^{1/2}V_t^+(x,t)|^2 - |(D^-)^{1/2}V_t^-(x,t)|^2\big) \, dx \\
&\geqslant \frac{1}{2}\big\{(1-\varepsilon)|V_t^+(1,t)|^2 - \varepsilon|V_t^+(0,t)|^2 \\
&\qquad + (1-\varepsilon)|V_t^-(0,t)|^2 - \varepsilon|V_t^-(1,t)|^2\big\} \\
&\qquad - c\|V_t(\cdot, t)\|^2.
\end{aligned}
$$

From (5.6) and the estimates above, we derive that there is a constant c such that

$$
\begin{aligned}
&\|(\tilde{\Lambda}D)^{1/2}V_x(\cdot, t_j)\|^2 - \|(\tilde{\Lambda}D)^{1/2}V_x(\cdot, t_{j-1})\|^2 \\
&\qquad + (1-\varepsilon)\{\|V_t^+(1,\cdot)\|^2_{L^2(J_j)} + \|V_t^-(0,\cdot)\|^2_{L^2(J_j)}\} \\
&\qquad - \varepsilon\{\|V_t^+(0,\cdot)\|^2_{L^2(J_j)} + \|V_t^-(1,\cdot)\|^2_{L^2(J_j)}\} \\
&\qquad \leqslant ch^{-2}\big(\|V\|^2_{L^2(I\times J_j)} + \|f\|^2_{L^2(I\times J_j)}\big).
\end{aligned} \tag{5.7}
$$

Define now a new norm, $|||\cdot|||_\delta$ on $\mathfrak{N}_\delta^n$, by

$$|||W|||_\delta^2 = \|D^{1/2}(P_\delta W)\|^2 + h^2\|(\tilde{\Lambda}D)^{1/2}W_x\|^2.$$

We note that it follows from (3.3) and (5.3) that, for h sufficiently small, $|||\cdot|||_\delta$ is uniformly equivalent to $\|\cdot\|$ on $\mathfrak{N}_\delta^n$. If we add (5.5) and h^2 times (5.7) and use the boundary conditions in (4.15), then we obtain, by choosing ε sufficiently small, $\varepsilon = \varepsilon(S_{0,0}, S_{1,0}, \Lambda)$, that

$$
\begin{aligned}
&|||V(\cdot, t_j)|||_\delta^2 - |||V(\cdot, t_{j-1})|||_\delta^2 \\
&\qquad \leqslant c\{\|V\|^2_{L^2(I\times J_j)} + \|G_0\|^2_{L^2(J_j)} + \|G_1\|^2_{L^2(J_j)} + \|f\|^2_{L^2(I\times J_j)}\},
\end{aligned}
$$

where the constant c is independent of V. Hence, it follows from Lemma 4.2 that there is a constant c such that, for h sufficiently small,

$$
\begin{aligned}
|||V(\cdot, t_j)|||_\delta^2 &\leqslant (1 + ck_j)|||V(\cdot, t_{j-1})|||_\delta^2 \\
&\quad + c\{\|G_0\|^2_{L^2(J_j)} + \|G_1\|^2_{L^2(J_j)} + \|f\|^2_{L^2(I\times J_j)}\}.
\end{aligned} \tag{5.8}
$$

Since (5.8) holds for $j = 1, 2, \ldots, N$, (5.1) follows by the discrete analog of Gronwall's lemma. □

6. L^2-Convergence. In this section we shall use Theorem 5.1 to prove L^2-error estimates for the method (3.5). These estimates will be derived under certain smoothness assumptions on the solution u of (1.1), and hence the data in (1.1) has to satisfy certain compatibility conditions at $t = 0$. We shall, particularly, in the rest of this paper assume that $K_0u_0(0) = g_0(0)$ and $K_1u_0(1) = g_1(0)$ and that $Q, \Lambda, A \in (C^{(r+1)}(I))^{n\times n}$.

LEMMA 6.1. *Assume that $W \in \mathfrak{N}_\delta^n$ is such that $W(0) = 0$ and*

$$\int_0^1 \langle AW_x + Q\Lambda(Q^{-1})_xW, \chi\rangle\, dx = 0,$$

for all $\chi \in \mathfrak{N}_\delta^n$. Then, for h sufficiently small, $W \equiv 0$.

PROOF. Since $A = Q\Lambda Q^{-1}$, we have, for any $\chi \in \mathfrak{N}_\delta^n$, that

$$\int_0^1 \langle \Lambda(Q^{-1}W)_x, Q^*\chi\rangle\, dx = 0.$$

Therefore, since $(Q^{-1}W)^+(0) = 0$, there is a constant $c > 0$ such that

$$
\begin{aligned}
\frac{1}{c}\|(Q^{-1}W)^+\|_1^2 &\leqslant \int_0^1 \langle \Lambda(Q^{-1}W)_x, (Q^{-1}W)_x^+\rangle\, dx \\
&= \inf_{\chi\in\mathfrak{N}_\delta^n}\int_0^1 \langle \Lambda(Q^{-1}W)_x, (Q^{-1}W)_x^+ - Q^*\chi\rangle\, dx \\
&\leqslant \|\Lambda(Q^{-1}W)_x\| \inf_{\chi\in\mathfrak{N}_\delta^n}\|(Q^{-1}W)_x^+ - Q^*\chi\|.
\end{aligned}
$$

Note that, by (1.3), (3.1), and (3.3), there is a constant c such that

$$\begin{aligned}\inf_{\chi\in\mathcal{U}_\delta^n}\|(Q^{-1}W)_x^+ - Q^*\chi\| &= \inf_{\chi\in\mathcal{U}_\delta^n}\|Q^*\big((Q^*)^{-1}(Q^{-1}W)_x^+ - \chi\big)\| \\ &\leqslant ch\sum_{i=1}^{M} h_i^{r-1}\|(Q^*)^{-1}(Q^{-1}W)_x^+\|_{H^r(I_i)} \\ &\leqslant ch\|W\|_1.\end{aligned}$$

Hence, we have

$$\|(Q^{-1}W)^+\|_1^2 \leqslant ch\|W\|_1^2.$$

A similar result holds for $(Q^{-1}W)^-$, and therefore there is a constant c, independent of W, such that

$$\|Q^{-1}W\|_1 \leqslant ch\|Q^{-1}W\|_1.$$

This implies the desired result. □

Define $T_\delta\colon (H^1(I))^n \to \mathcal{U}_\delta^n$ by $(T_\delta w)(0) = w(0)$ and

$$\int_0^1 \langle A(w_x - (T_\delta w)_x) + Q\Lambda(Q^{-1})_x(w - T_\delta w), \chi\rangle\, dx = 0,$$

for $\chi \in \mathcal{U}_\delta^n$. We note that it follows from Lemma 6.1 that $T_\delta w \in \mathcal{U}_\delta^n$ is well-defined for h sufficiently small.

For any integer $\nu \geqslant 0$, let $H^\nu(\delta)$ denote the piecewise Sobolev space given by

$$H^\nu(\delta) = \big\{w \in L^2(I) | w_{|_{I_i}} \in H^\nu(I_i),\ 1 \leqslant i \leqslant M\big\},$$

and let $|||\cdot|||_{\nu,\delta}$ be the associated norm; i.e.,

$$|||w|||_{\nu,\delta}^2 = \sum_{i=1}^{M} \|w\|_{H^\nu(I_i)}^2.$$

If $\nu < 0$ is an integer, then we define $H^\nu(\delta)$ by duality with respect to the inner product on $L^2(I)$; i.e., $H^\nu(\delta)$ is the completion of $L^2(I)$ in the norm

$$|||w|||_{\nu,\delta} = \sup_{0\neq\varphi\in H^{-\nu}(\delta)} \frac{\int_0^1 w\varphi\, dx}{|||\varphi|||_{-\nu,\delta}}.$$

In the same way as above we define the spaces $H^\nu(\gamma)$, with norms $|||\cdot|||_{\nu,\gamma}$, consisting of functions defined on J. We observe that if $w \in H^\nu(\delta)$ for some $\nu < 0$, then $w \in H^\nu(I)$ and $\|w\|_\nu \leqslant |||w|||_{\nu,\delta}$.

The following error estimate for the "projection" T_δ now follows by a standard duality argument.

LEMMA 6.2. *There is a constant c such that, if h is sufficiently small and $w \in (H^{r+1}(I))^n$,*

$$|||w - T_\delta w|||_{-\nu,\delta} \leqslant ch^{r+\nu+1}\|w\|_{r+1} \quad \text{for } -1 \leqslant \nu \leqslant r-1.$$

Proof. Let $\xi = w - T_\delta w$. It follows from the definition of $T_\delta w$ that, for any $\chi \in \mathcal{U}_\delta^n$,

$$\int_0^1 \langle \Lambda(Q^{-1}\xi)_x, Q^*\chi\rangle\, dx = 0. \tag{6.1}$$

Assume that ν is an integer such that $0 \leqslant \nu \leqslant r-1$ and let $\eta \in (H^{\nu}(\delta))^n$. Define $\psi \in (H^{\nu+1}(\delta))^n$ by

$$(\Lambda\psi)_x = \eta \quad \text{and} \quad \psi(1) = 0.$$

Note that there is a constant c, independent of η, such that

$$|||\psi|||_{\nu+1,\delta} \leqslant c|||\eta|||_{\nu,\delta}. \tag{6.2}$$

From (3.1) and (6.1), we now obtain that

$$\begin{aligned}\int_0^1 \langle Q^{-1}\xi, \eta\rangle\, dx &= \int_0^1 \langle Q^{-1}\xi, (\Lambda\psi)_x\rangle\, dx \\ &= \inf_{\chi\in\mathcal{U}_\delta^n} \int_0^1 \langle \Lambda(Q^{-1}\xi)_x, Q^*\chi - \psi\rangle\, dx \\ &\leqslant ch^{\nu+1}\|(Q^{-1}\xi)_x\|\,|||\psi|||_{\nu+1,\delta}.\end{aligned}$$

Hence, it follows from (1.3) and (6.2) that

$$|||\xi|||_{-\nu,\delta} \leqslant ch^{\nu+1}\|\xi\|_1 \quad \text{for } 0 \leqslant \nu \leqslant r-1. \tag{6.3}$$

The proof will be completed by showing the desired result for $\nu = -1$.

By (1.3), (3.1), and (6.1), we obtain that there is a constant $c > 0$ such that

$$\begin{aligned}\frac{1}{c}\|(Q^{-1}\xi)^+\|_1^2 &\leqslant \int_0^1 \langle \Lambda(Q^{-1}\xi)_x, (Q^{-1}\xi)_x^+\rangle\, dx \\ &= \inf_{\chi_1\in\mathcal{U}_\delta^n} \int_0^1 \langle \Lambda(Q^{-1}\xi)_x, (Q^{-1}w)_x^+ - Q^*\chi_1\rangle\, dx \\ &\quad + \inf_{\chi_2\in\mathcal{U}_\delta^n} \int_0^1 \langle \Lambda(Q^{-1}\xi)_x, Q^*\chi_2 - (Q^{-1}(T_\delta w))_x^+\rangle\, dx \\ &\leqslant ch^r\|\xi\|_1\{\|w\|_{r+1} + |||T_\delta w|||_{r,\delta}\}.\end{aligned}$$

Observe that if we choose $\chi \in \mathcal{U}_\delta^n$ such that

$$\sum_{j=0}^{r} h^j|||w-\chi|||_{j,\delta} \leqslant ch^r\|w\|_r,$$

then it follows from (3.3) that

$$\begin{aligned}|||T_\delta w|||_{r,\delta} &\leqslant |||\chi|||_{r,\delta} + |||T_\delta w - \chi|||_{r,\delta} \\ &\leqslant |||\chi|||_{r,\delta} + ch^{-(r-1)}\{|||w-\chi|||_{1,\delta} + \|\xi\|_1\} \\ &\leqslant c\{\|w\|_r + h^{-(r-1)}\|\xi\|_1\}.\end{aligned}$$

Hence, it follows from the above that there is a constant c such that

$$\|(Q^{-1}\xi)^+\|_1^2 \leqslant c\|\xi\|_1(h^r\|w\|_{r+1} + h\|\xi\|_1).$$

By a similar argument we obtain an analog estimate for $(Q^{-1}\xi)^-$ and therefore there is a constant c such that

$$\|Q^{-1}\xi\|_1^2 \leqslant c\|\xi\|_1(h^r\|w\|_{r+1} + h\|\xi\|_1).$$

Note also that it follows from (6.3) that

$$\|\xi\|_1 = \|QQ^{-1}\xi\|_1 \leqslant c(\|Q^{-1}\xi\|_1 + \|\xi\|) \leqslant c(\|Q^{-1}\xi\|_1 + h\|\xi\|_1).$$

Hence, there is a constant c such that

$$\|\xi\|_1 \leqslant ch^r\|w\|_{r+1},$$

for h sufficiently small. Together with (6.3) this implies the desired result. □

The following superconvergence result for the operator T_δ now follows directly from the result above.

LEMMA 6.3. *There is a constant c such that, if h is sufficiently small and $w \in (H^{r+1}(I))^n$,*

$$\max_{\bar{x}\in\delta} |w(\bar{x}) - (T_\delta w)(\bar{x})| \leqslant ch^{2r}\|w\|_{r+1}.$$

PROOF. For a fixed $\bar{x} \in \delta$, define $G(\bar{x}, \cdot) \in (L^2(I))^n$ by

$$G(\bar{x}, x)_j = \begin{cases} (\lambda_j(x))^{-1} & \text{if } x \leqslant \bar{x}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for any $\psi \in (H^1(I))^n$, such that $\psi(0) = 0$, we have

$$\psi(\bar{x}) = \int_0^1 \langle \Lambda\psi_x, G(\bar{x}, x)\rangle \, dx.$$

Therefore, since $G(\bar{x}, \cdot) \in H^{r-1}(\delta)$, it follows from Lemma 6.2 that

$$\begin{aligned} |w(\bar{x}) - (T_\delta w)(\bar{x})| &= \left| \int_0^1 \langle \Lambda(w_x - (T_\delta w)_x), G(\bar{x}, x)\rangle \, dx \right| \\ &\leqslant ch^{2r}\|w\|_{r+1}. \quad \square \end{aligned}$$

Similar to the operator T_δ, define $T_\gamma: (H^1(J))^n \to \mathfrak{N}_\gamma^n$ by $(T_\gamma\varphi)(0) = \varphi(0)$ and

$$\int_0^{t^*} \langle \varphi_t - (T_\gamma\varphi)_t, \mu\rangle \, dt = 0 \quad \text{for } \mu \in \mathfrak{N}_\gamma^n.$$

It is easy to see that $T_\gamma\varphi \in \mathfrak{N}_\gamma^n$ is well-defined, and, by using arguments similar to the ones given in the proofs of Lemmas 6.2 and 6.3, we obtain that there is a constant c such that, for k sufficiently small,

$$|||\varphi - T_\gamma\varphi|||_{-\nu,\gamma} \leqslant ck^{s+\nu+1}\|\varphi\|_{H^{s+1}(J)} \quad \text{for } -1 \leqslant \nu \leqslant s-1, \tag{6.4}$$

and

$$\max_{\bar{t}\in\gamma} |\varphi(\bar{t}) - (T_\gamma\varphi)(\bar{t})| \leqslant ck^{2s}\|\varphi\|_{H^{s+1}(J)}. \tag{6.5}$$

Define $T_{\delta,\gamma}: (H^{1,0}(I \times J) \cap H^{0,1}(I \times J))^n \to \mathfrak{N}_{\delta,\gamma}^n$ by $T_{\delta,\gamma} = T_\delta \otimes T_\gamma$. We observe that it follows from the identity

$$I - T_{\delta,\gamma} = I - T_\delta + I - T_\gamma - (I - T_\delta) \otimes (I - T_\gamma) \tag{6.6}$$

and from Lemma 6.2, (6.4) and (6.5) that there is a constant c such that, for any $w \in (H^{r+1,0}(I \times J) \cap H^{1,s+1}(I \times J))^n$,

$$\|(I - T_{\delta,\gamma})w\|_{0,0} \leqslant c\{h^{r+1}\|w\|_{r+1,0} + k^{s+1}\|w\|_{1,s+1}\} \tag{6.7}$$

and

$$\max_{\bar{t}\in\gamma} \|((I - T_{\delta,\gamma})w)(\cdot, \bar{t})\| \leqslant c\{h^{r+1}\|w\|_{r+1,0} + k^{2s}\|w\|_{1,s+1}\}. \tag{6.8}$$

If u and U are the solutions of (1.1) and (3.5), respectively, then we let $e = u - U$. We have the following convergence result for method (3.5).

THEOREM 6.4. *Assume that* $u \in (H^{r+1,1}(I \times J) \cap H^{1,s+1}(I \times J))^n$. *There is a constant* c, *independent of* u, *such that*

$$\max_{\bar{t} \in \gamma} \|e(\bar{t})\| \leqslant c\{h^{r+1}\|u\|_{r+1,1} + k^{s+1}\|u\|_{1,s+1} + \|T_\delta u_0 - U_0\| + \|T_\gamma g_0 - G_0\|_{L^2(J)} + \|T_\gamma g_1 - G_1\|_{L^2(J)}\}.$$

Proof. Let $W = T_{\delta,\gamma}u$. Because of (6.8) it is enough to estimate $\theta = W - U$. Note that for any $\chi \in \mathcal{U}^n_{\delta,\gamma}$ we have

$$\int_0^{t^*}\int_0^1 \langle W_t + AW_x + BW + \rho - f, \chi\rangle \, dx \, dt = 0,$$

where

$$\rho = (I - T_\delta)u_t + (I - T_\gamma)(Au_x + Q\Lambda(Q^{-1})_x u) + (B - Q\Lambda(Q^{-1})_x)(I - T_{\delta,\gamma})u.$$

By Lemma 6.2, (6.4), and (6.7), there exists a constant c such that

$$\|\rho\|_{0,0} \leqslant c\{h^{r+1}\|u\|_{r+1,1} + k^{s+1}\|u\|_{1,s+1}\}. \tag{6.9}$$

Therefore, θ satisfies the system

$$\int_0^{t^*}\int_0^1 \langle \theta_t + A\theta_x + B\theta + \rho, \chi\rangle \, dx \, dt = 0 \quad \text{for } \chi \in \mathcal{U}^n_{\delta,\gamma},$$

$$K_0\theta(0, t) = (T_\gamma g_0)(t) - G_0(t),$$

$$K_1\theta(1, t) = (T_\gamma g_1)(t) - G_1(t) - (T_\gamma K_1(I - T_\delta)u)(1, t),$$

$$\theta(x, 0) = (T_\delta u_0)(x) - U_0(x).$$

Hence, it follows from Theorem 5.1, Lemma 6.3, and (6.9) that

$$\max_{\bar{t} \in \gamma} \|\theta(\cdot, \bar{t})\| \leqslant c\{h^{r+1}\|u\|_{r+1,1} + k^{s+1}\|u\|_{1,s+1} + \|T_\delta u_0 - U_0\| + \|T_\gamma g_0 - G_0\|_{L^2(J)} + \|T_\gamma g_1 - G_1\|_{L^2(J)}\},$$

and this implies the theorem. □

We observe that Theorem 6.4 implies that, if we choose $U_0 = T_\delta u_0$, $G_0 = T_\gamma g_0$ and $G_1 = T_\gamma g_1 - K_1((I - T_\delta)u_0)(1)$ in (3.5) and if u is sufficiently smooth, then

$$\max_{\bar{t} \in \gamma} \|e(\bar{t})\| = O(h^{r+1} + k^{s+1}).$$

In the next section we shall show that if G_0 and G_1 are chosen more carefully, then we can obtain a convergence estimate of the form

$$\max_{\bar{t} \in \gamma} \|e(\bar{t})\| = O(h^{r+1} + k^{2s}). \tag{6.10}$$

7. Superconvergence. The purpose of this section is to prove certain superconvergence results for the method (3.5). In addition to the result mentioned at the end of Section 6, we shall also show that, if U_0, G_0, and G_1 are all chosen properly, then we obtain that

$$\max_{\bar{t} \in \gamma}\left(\sum_{i=1}^{M} h_i(|e(x_i)|^2 + |e(x_{i-1})|^2)\right)^{1/2} = O(h^{2r} + k^{2s}).$$

The arguments that we shall use to obtain these superconvergence results are similar to the ones that were used by Douglas, Dupont, and Wheeler [3] in order to

prove certain superconvergence results for the standard Galerkin method for second order parabolic and hyperbolic equations. In the case of parabolic equations, some of these results were improved by Thomée [12].

We shall first show the estimate (6.10). Let $W^{(0)} = -(I - T_\gamma)u$ and for $j = 1, 2, \ldots$ define $W^{(j)} \in (L^2(I) \otimes \mathfrak{N}_\gamma)^n$ by $W^{(j)}(x, 0) = 0$ and for each $x \in I$

$$\int_0^{t^*} \langle W_t^{(j)}, \mu \rangle \, dt = -\int_0^{t^*} \langle \rho^{(j-1)}, \mu \rangle \quad \text{for } \mu \in \mathfrak{N}_\gamma^n, \tag{7.1}$$

where

$$\rho^{(j)} = AW_x^{(j)} + BW^{(j)}, \qquad j = 0, 1, \ldots. \tag{7.2}$$

For all integers $j \geqslant 0$, we also let

$$\theta^{(j)} = T_\gamma u + \sum_{i=1}^{j} W^{(i)} - U.$$

We observe that it follows from (1.1), (3.5), (7.1), and the definition of T_γ that $\theta^{(j)}$ satisfies the equation

$$\int_0^{t^*} \int_0^1 \langle \theta_t^{(j)} + A\theta_x^{(j)} + B\theta^{(j)} - \rho^{(j)}, \chi \rangle \, dx \, dt = 0 \quad \text{for } \chi \in \mathfrak{N}_{\delta,\gamma}^n. \tag{7.3}$$

LEMMA 7.1. *Let j be an integer such that $0 \leqslant j \leqslant s$. Then there is a constant c such that*

$$\|W^{(j)}\|_{L^2(I;H^{-\nu}(\gamma))} \leqslant ck^{s+\nu+j+1}\|u\|_{j,s+1}$$

for $-1 \leqslant \nu \leqslant s - j - 1$.

Proof. The result is obtained by induction on j. By (6.4), the result holds for $j = 0$. Assume now that the desired estimate holds for $j - 1$, where $1 \leqslant j \leqslant s$, and let ν be an integer such that $-1 \leqslant \nu \leqslant s - j - 1$. For a given $\eta \in (H^\nu(\gamma))^n$, define $\psi \in (H^{\nu+1}(\gamma))^n$ by $\psi_t = \eta$ and $\psi(t^*) = 0$.

We note that there exists a constant c, independent of η, such that

$$|||\psi|||_{\nu+1,\gamma} \leqslant c|||\eta|||_{\nu,\gamma}. \tag{7.4}$$

For each $x \in I$, we now have

$$\begin{aligned}\int_0^{t^*} \langle W^{(j)}, \eta \rangle \, dt &= \int_0^{t^*} \langle W^{(j)}, \psi_t \rangle \, dt \\ &= \inf_{\mu \in \mathfrak{N}_\delta^n} \Big\{ -\int_0^{t^*} \langle W_t^{(j)}, \psi - \mu \rangle \, dt - \int_0^{t^*} \langle \rho^{(j-1)}, \psi - \mu \rangle \, dt \\ &\qquad + \int_0^{t^*} \langle \rho^{(j-1)}, \psi \rangle \, dt \Big\}.\end{aligned}$$

Hence, we obtain from (3.3) and (7.4) that there is a constant c such that

$$\begin{aligned}\int_0^{t^*} \langle W^{(j)}, \eta \rangle \, dt \leqslant c\Big\{ &\big(\|W_t^{(j)}\|_{L^2(J)} + \|\rho^{(j-1)}\|_{L^2(J)}\big)h^{\nu+1}|||\eta|||_{\nu,\gamma} \\ &+ |||\rho^{(j-1)}|||_{-(\nu+1),\gamma}|||\eta|||_{\nu,\gamma}\Big\}.\end{aligned} \tag{7.5}$$

From (7.1) it follows that

$$\|W_t^{(j)}\|_{L^2(J)} \leqslant \|\rho^{(j-1)}\|_{L^2(J)}$$

and, by (7.2) and the induction hypothesis, we have

$$\|\rho^{(j-1)}\|_{L^2(I;H^{-i}(\gamma))} \leqslant ck^{s+i+j}\|u\|_{j,s+1}$$

for $-1 \leqslant i \leqslant s-j$. Hence, the desired result follows from (7.5). □

From Lemma 7.1 we immediately obtain the following superconvergence estimate for the functions $W^{(j)}$.

LEMMA 7.2. *Let j be an integer such that $0 \leqslant j \leqslant s$. For each integer $\nu \geqslant 0$, there is a constant c such that*

$$\max_{\bar{t}\in\gamma} \|W^{(j)}(\cdot, \bar{t})\|_\nu \leqslant ck^{2s}\|u\|_{j+\nu,s+1}.$$

Proof. By (7.1) we have for $i \geqslant 0$

$$\left(\frac{\partial}{\partial x}\right)^i W^{(j)}(x, \bar{t}) = -\int_0^{\bar{t}} \left(P_\gamma\left(\frac{\partial}{\partial x}\right)^i \rho^{(j-1)}\right)(x, t)\, dt.$$

Therefore, there is a constant c such that

$$\left\|\left(\frac{\partial}{\partial x}\right)^i W^{(j)}(\cdot, \bar{t})\right\| \leqslant c\left\|\left(\frac{\partial}{\partial x}\right)^i \rho^{(j-1)}\right\|_{L^2(I;H^{j-s}(\gamma))}$$

for all $i \geqslant 0$ and $\bar{t} \in \gamma$. But from Lemma 7.1 and (7.2) we obtain

$$\left\|\left(\frac{\partial}{\partial x}\right)^i \rho^{(j-1)}\right\|_{L^2(I;H^{j-s}(\gamma))} \leqslant ck^{2s}\|u\|_{j+i,s+1},$$

and this implies the lemma. □

We also note that it follows from Lemma 7.1 that, for all integers $\nu \geqslant 0$, there is a constant c such that

$$\|W^{(j)}\|_{\nu,0} \leqslant ck^{s+j+1}\|u\|_{j+\nu,s+1} \quad \text{for } 0 \leqslant j \leqslant s. \tag{7.6}$$

The following superconvergence result will now be derived for the method (3.5).

THEOREM 7.3. *Assume that $u \in (H^{r+s,1}(I \times J) \cap H^{s,s+1}(I \times J))^n$. Then there is a constant c, independent of u, such that*

$$\max_{\bar{t}\in\gamma} \|e(\bar{t})\| \leqslant c\left\{h^{r+1}\|u\|_{r+s,1} + k^{2s}\|u\|_{s,s+1} + \|T_\delta u_0 - U_0\| + \sum_{i=0}^{1}\left\|T_\gamma g_i + \sum_{j=1}^{s-1} K_i W^{(j)}(i, \cdot) - G_i\right\|_{L^2(J)}\right\}.$$

Proof. Write the error in the form

$$e = (I - T_{\delta,\gamma})u - T_\delta\left(\sum_{j=1}^{s-1} W^{(j)}\right) + T_\delta\theta^{(s-1)}.$$

Note that the term $(I - T_{\delta,\gamma})u$ can be estimated by (6.8), and from Lemma 7.2 we obtain

$$\max_{\bar{t}\in\gamma}\left\|T_\delta\left(\sum_{j=1}^{s-1} W^{(j)}\right)\right\| \leqslant c\max_{\bar{t}\in\gamma}\sum_{j=1}^{s-1}\|W^{(j)}\|_1 \leqslant ck^{2s}\|u\|_{s,s+1}. \tag{7.7}$$

In order to estimate $T_\delta\theta^{(s-1)}$, let

$$\rho = \rho^{(s-1)} + (T_\delta - I)\Big(u_t + \sum_{j=1}^{s-1} W_t^{(j)}\Big) + \big(B - Q\Lambda(Q^{-1})_x\big)(T_\delta - I)\Big(T_\gamma u + \sum_{j=1}^{s-1} W^{(j)}\Big).$$

First, observe that Lemma 7.1 and (7.2) imply that

$$\|\rho^{(s-1)}\|_{0,0} \leqslant ck^{2s}\|u\|_{s,s+1}. \tag{7.8}$$

From (7.1) it follows that, for each integer $\nu \geqslant 0$, there is a constant c such that

$$\|W^{(j)}\|_{\nu,0} \leqslant c\|u\|_{\nu+j,1}. \tag{7.9}$$

Therefore, since $W_t^{(j)} = -P_\gamma\rho^{(j-1)}$, we obtain from Lemma 6.2 that

$$\|\rho\|_{0,0} \leqslant c\big\{h^{r+1}\|u\|_{r+s,1} + k^{2s}\|u\|_{s,s+1}\big\}. \tag{7.10}$$

Note also that, by the definition of T_δ and (7.3), we have

$$\int_0^{t^*}\int_0^1 \langle \big(T_\delta\theta^{(s-1)}\big)_t + A\big(T_\delta\theta^{(s-1)}\big)_x + B\big(T_\delta\theta^{(s-1)}\big) - \rho, \chi\rangle\, dx\, dt = 0, \tag{7.11}$$

for all $\chi \in \mathfrak{N}_{\delta,\gamma}^n$. Finally, we observe that $T_\delta\theta^{(s-1)}$ satisfies the boundary conditions

$$K_0\big(T_\delta\theta^{(s-1)}\big)(0, t) = (T_\gamma g_0)(t) + \sum_{j=1}^{s-1} K_0 W^{(j)}(0, t) - G_0(t)$$

and

$$K_1\big(T_\delta\theta^{(s-1)}\big)(1, t) = (T_\gamma g_1)(t) + \sum_{j=1}^{s-1} K_1 W^{(j)}(1, t) - G_1(t) + (T_\gamma K_1(T_\delta - I)u)(1, t) + \sum_{j=1}^{s-1} K_1\big((T_\delta - I)W^{(j)}\big)(1, t).$$

Since Lemma 6.3 and (7.9) imply that

$$\Big\|(T_\gamma K_1(T_\delta - I)u)(1, \cdot) + \sum_{j=1}^{s-1} K_1\big((T_\delta - I)W^{(j)}\big)(1, \cdot)\Big\|_{L^2(J)} \leqslant ch^{2r}\|u\|_{r+s,1}, \tag{7.12}$$

it now follows from Theorem 5.1, (7.10), (7.11), and the boundary conditions above that

$$\max_{\bar t\in\gamma}\|\big(T_\delta\theta^{(s-1)}\big)(\cdot, \bar t)\| \leqslant c\Big\{h^{r+1}\|u\|_{r+s,1} + k^{2s}\|u\|_{s,s+1} + \|T_\delta u_0 - U_0\| + \sum_{i=0}^{1}\Big\|T_\gamma g_i + \sum_{j=1}^{s-1} K_i W^{(j)}(i, \cdot) - G_i\Big\|_{L^2(J)}\Big\}.$$

Together with (6.8) and (7.7) this implies the theorem. □

We note that for $i = 0, 1$ the functions $K_i u(i, \cdot) = g_i$ are given and hence the functions $K_i W^{(j)}(i, \cdot)$ can be computed from (7.1) and (7.2) (by using the fact that u satisfies the equation (1.1)). Therefore, if we choose

$$U_0 = T_\delta u_0, \quad G_0 = T_\gamma g_0 + \sum_{j=1}^{s-1} K_0 W^{(j)}(0, \cdot)$$

and

$$G_1 = T_\gamma g_1 + \sum_{j=1}^{s-1} K_1 W^{(j)}(1, \cdot) - K_1((I - T_\delta)u_0)(1),$$

then Theorem 7.3 implies (6.10).

Finally, we want to show the superconvergence property at the knots of the partition δ mentioned in the beginning of this section. Let $Z^{(0)} = -(I - T_\delta)u$ and for each integer $j \geqslant 1$ define $Z^{(j)} \in (\mathcal{M}_\delta \otimes L^2(J))^n$ by $Z^{(j)}(0, t) = 0$ and for each $t \in J$

$$\int_0^1 \langle AZ_x^{(j)} + Q\Lambda(Q^{-1})_x Z^{(j)}, \chi\rangle \, dx = -\int_0^1 \langle \psi^{(j-1)}, \chi\rangle \, dx, \tag{7.13}$$

where

$$\psi^{(j)} = Z_t^{(j)} + (B - Q\Lambda(Q^{-1})_x)Z^{(j)} \quad \text{for } j = 0, 1, \ldots. \tag{7.14}$$

We note that it follows from Lemma 6.1 that $Z^{(j)}$ is well-defined for h sufficiently small. Now let ψ denote the function

$$\psi = \rho^{(s-1)} + (T_\delta - I)\left(\sum_{j=1}^{s-1} W_t^{(j)}\right) + (B - Q\Lambda(Q^{-1})_x)(T_\delta - I)\left((T_\gamma - I)u + \sum_{j=1}^{s-1} W^{(j)}\right).$$

Observe that it follows from Lemma 6.2 and (7.6) that, for $0 \leqslant j \leqslant s - 1$,

$$\begin{aligned}\|(T_\delta - I)W^{(j)}\|_{0,0} &\leqslant ch^{r+1}\|W^{(j)}\|_{r+1,0}^{\cdot} \\ &\leqslant c(h^{2(r+1)} + k^{2(s+1)})\|u\|_{r+s,s+1}.\end{aligned}$$

Hence, since $W_t^{(j)} = -P_\gamma \rho^{(j-1)}$, it follows from Lemma 6.2 and (7.8) that there is a constant c such that

$$\|\psi\|_{0,0} \leqslant c(h^{2r} + k^{2s})\|u\|_{r+s,s+1}. \tag{7.15}$$

Define for each integer $j \geqslant 0$

$$\xi^{(j)} = T_\delta \theta^{(s-1)} + \sum_{i=1}^{j} Z^{(i)}$$

and let $\omega^{(j)} = \psi + \psi^{(j)}$. Note that it follows from (7.11), (7.13), and (7.14) that

$$\int_0^{t^*}\int_0^1 \langle \xi_t^{(j)} + A\xi_x^{(j)} + B\xi^{(j)} - \omega^{(j)}, \chi\rangle \, dx \, dt = 0, \tag{7.16}$$

for all $\chi \in \mathcal{M}_{\delta,\gamma}^n$. In exactly the same way as the corresponding results were proved for the functions $W^{(j)}$, we have that if j is an integer such that $0 \leqslant j \leqslant r$, then the functions $Z^{(j)}$ satisfy the estimates

$$\|Z^{(j)}\|_{L^2(J;H^{-\nu}(\delta))} \leqslant ch^{r+\nu+j+1}\|u\|_{r+1,j}, \qquad -1 \leqslant \nu \leqslant r - j - 1, \tag{7.17}$$

$$\max_{\bar{x}\in\delta} \|Z^{(j)}(\bar{x}, \cdot)\|_{H^\nu(J)} \leqslant ch^{2r}\|u\|_{r+1,j+\nu}, \qquad \nu \geqslant 0, \tag{7.18}$$

and

$$\|Z^{(j)}\|_{0,\nu} \leqslant ch^{r+j+1}\|u\|_{r+1,j+\nu}, \qquad \nu \geqslant 0. \tag{7.19}$$

First note that, since $r, s \geqslant 1$, it follows from (7.15) and (7.17) that

$$\|\omega^{(r-1)}\|_{0,0} \leqslant c(h^{2r} + k^{2s})\|u\|_{r+s,r+s}. \tag{7.20}$$

Observe also that (7.18) implies that, for $0 \leqslant j \leqslant r-1$,

$$\max_{(\bar{x},\bar{t})\in\delta\times\gamma} \left|\left(T_\gamma Z^{(j)}\right)(\bar{x},\bar{t})\right| \leqslant c\|Z^{(j)}(\bar{x},\cdot)\|_{H^1(J)} \leqslant ch^{2r}\|u\|_{r+1,r}. \tag{7.21}$$

Similarly, it follows from Lemma 7.2 that, for $0 \leqslant j \leqslant s-1$,

$$\max_{(\bar{x},\bar{t})\in\delta\times\gamma} |\left(T_\delta W^{(j)}\right)(\bar{x},\bar{t})| \leqslant ck^{2s}\|u\|_{s,s+1}, \tag{7.22}$$

and, by Lemma 6.3, (6.5), and (6.6), we obtain

$$\max_{(\bar{x},\bar{t})\in\delta\times\gamma} |\left((I - T_{\delta,\gamma})u\right)(\bar{x},\bar{t})| \leqslant c(h^{2r} + k^{2s})\|u\|_{r+1,s+1}. \tag{7.23}$$

Now write the error $e = u - U$ in the form

$$e = (I - T_{\delta,\gamma})u - \sum_{j=1}^{s-1} T_\delta W^{(j)} - \sum_{j=1}^{r-1} T_\gamma Z^{(j)} + T_\gamma \xi^{(r-1)}. \tag{7.24}$$

By (7.21), (7.22), and (7.23), all terms in this expansion have been estimated at the knots, except for $T_\gamma \xi^{(r-1)}$. Let $\xi = T_\gamma \xi^{(r-1)}$. Then $\xi \in \mathcal{M}^n_{\delta,\gamma}$ and, by (7.16) and the definition of T_γ, ξ satisfies the equation

$$\int_0^{t^*}\int_0^1 \langle \xi_t + A\xi_x + B\xi - \omega, \chi\rangle \, dx\, dt = 0 \quad \text{for } \chi \in \mathcal{M}^n_{\delta,\gamma},$$

where

$$\omega = \omega^{(r-1)} + (T_\gamma - I)\sum_{j=1}^{r-1} \left(AZ_x^{(j)} + BZ^{(j)}\right).$$

The relations (6.4), (7.13), and (7.19) imply that

$$\left\|(T_\gamma - I)\sum_{j=1}^{r-1}\left(AZ_x^{(j)} + BZ^{(j)}\right)\right\|_{0,0} \leqslant c(h^{2(r+1)} + k^{2(s+1)})\|u\|_{r+1,r+s}.$$

Hence, from (7.20) we obtain

$$\|\omega\|_{0,0} \leqslant c(h^{2r} + k^{2s})\|u\|_{r+s,r+s}.$$

We also note that ξ satisfies the initial condition

$$\xi(x,0) = (T_\delta u_0)(x) + \sum_{j=1}^{r-1} Z^{(j)}(x,0) - U_0(x)$$

and the boundary conditions

$$K_0\xi(0,t) = (T_\gamma g_0)(t) + \sum_{j=1}^{s-1} K_0 W^{(j)}(0,t) - G_0(t)$$

and

$$K_1\xi(1, t) = (T_\gamma g_1)(t) + \sum_{j=1}^{s-1} K_1 W^{(j)}(1, t) - G_1(t) + (T_\gamma K_1(T_\delta - I)u)(1, t)$$
$$+ \sum_{j=1}^{s-1} K_1((T_\delta - I)W^{(j)})(1, t) + K_1 \sum_{j=1}^{r-1} (T_\gamma Z^{(j)})(1, t).$$

Since (7.18) implies

$$\left\| K_1 \sum_{j=1}^{r-1} (T_\gamma Z^{(j)})(1, \cdot) \right\|_{L^2(J)} \leqslant ch^{2r}\|u\|_{r+1,r},$$

it follows from Theorem 5.1 and (7.12) that there is a constant c such that

$$\max_{t\in\gamma} \|\xi(\cdot, t)\| \leqslant c\Bigg\{ (h^{2r} + k^{2s})\|u\|_{r+s,r+s}$$
$$+ \left\| T_\delta u_0 + \sum_{j=1}^{r-1} Z^{(j)}(\cdot, 0) - U_0 \right\| \tag{7.25}$$
$$+ \sum_{i=0}^{1} \left\| T_\gamma g_i + \sum_{j=1}^{s-1} K_i W^{(j)}(i, \cdot) - G_i \right\|_{L^2(J)} \Bigg\}.$$

Finally, note that, since $\xi \in \mathfrak{M}^n_{\delta,\gamma}$, there is a constant c, independent of δ, such that

$$\max_{t\in\gamma} \left(\sum_{i=1}^{M} h_i\big(|\xi(x_i, t)|^2 + |\xi(x_{i-1}, t)|^2\big) \right)^{1/2} \leqslant c \max_{t\in\gamma} \|\xi(\cdot, t)\|.$$

Hence, the following superconvergence result follows from (7.21)–(7.25).

THEOREM 7.4. *Assume that* $u \in (H^{r+s,r+s}(I \times J))^n$. *Then there is a constant* c, *independent of* u, *such that*

$$\max_{t\in\gamma} \left(\sum_{i=1}^{M} h_i\big(|e(x_i, t)|^2 + |e(x_{i-1}, t)|^2\big) \right)^{1/2}$$
$$\leqslant c\Bigg\{ (h^{2r} + k^{2s})\|u\|_{r+s,r+s} + \left\| T_\delta u_0 + \sum_{j=1}^{r-1} Z^{(j)}(\cdot, 0) - U_0 \right\|$$
$$+ \sum_{i=0}^{1} \left\| T_\gamma g_i + \sum_{j=1}^{s-1} K_i W^{(j)}(i, \cdot) - G_i \right\|_{L^2(J)} \Bigg\}.$$

In the same way as above, we observe that, since u satisfies (1.1), the functions $Z^{(j)}(\cdot, 0)$ can be computed from (7.13) and (7.14). However, this requires that the matrix $Q\Lambda(Q^{-1})_x$ be computed. We also remark that the regularity assumption in Theorem 7.4 is stronger than necessary. Weaker conditions can be derived if we are more careful in obtaining some of the estimates above.

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