Estimates Away From a Discontinuity for Dissipative Galerkin Methods for Hyperbolic Equations

By William J. Layton*

Abstract. We consider the approximate solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad u(x, 0) = v(x),$$

by a dissipative Galerkin method. When v is taken to have a jump discontinuity at zero, that discontinuity will propagate along $x + t = 0$, in the true solution u. Estimates in L_2 and L_∞ of the pollution effects of the discontinuity are found. These estimates show those effects to decay exponentially in h^{-1} in regions a fixed distance d from the discontinuity and exponentially in d for fixed h.

I. Introduction. In this paper the question of approximating, by Galerkin methods, discontinuous solutions to the initial value problem

(1)
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad u(x, 0) = v(x),$$

is considered. Here v is taken to be an $L_2(-\infty, \infty)$ function, smooth, except for a jump discontinuity at $x = 0$. This discontinuity will propagate along the characteristic $x + t = 0$ of the true solution $u(x, t) = v(x + t)$ of (1).

In the usual, continuous in time, Galerkin method for (1) a subspace S_h of $H^1(-\infty, \infty)$ is chosen and the approximate solution is computed as a differentiable map $u^h: [0, T] \to S_h$ by

(2)
$$\left(\frac{\partial u^h}{\partial t}, \phi\right) = \left(Du^h, \phi\right) \quad \text{for } \phi \in S_h, D = \frac{\partial}{\partial x},$$

(3)
$$u^h(0) \in S_h, \quad \text{an approximation to } v,$$

where $(\cdot, \cdot)$ denotes the usual inner product on $L_2(-\infty, \infty)$.

The usual Galerkin method is not appropriate for the solution of (1) with $v(x)$ as described for several reasons:

(i) The rate of convergence of the method (2), (3) depends upon the global smoothness of u. Here u is piecewise C^∞ but globally only L_2.

(ii) The usual Galerkin method is a strictly conservative method with an infinite domain of dependence. Experience with such methods indicates that the approximate solution will be hopelessly polluted by the discontinuity in u even far away from that discontinuity.

For these reasons a dissipative Galerkin method will be considered for the problem (1). Estimates of the error outside a neighborhood of the discontinuity will be derived for this method. These estimates imply, for example, that the error for (1), outside a neighborhood of $x + t = 0$, has two parts, the first being the error applied to smooth solutions of (1). The second component is the pollution effect of the discontinuity. This part is shown to decay exponentially in h^{-1} for regions a fixed distance d from the shock front and exponentially in d for fixed h.

II. The Method. We shall choose as S_h the space of splines of order μ of at most power growth. Define, following, e.g., Schoenberg [11] and Thomée [13], the B-splines of order $\mu \geq 2$. Let χ be the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. For $\mu \geq 2$ define ϕ and ϕ_l by $\phi = \chi^{*\mu}$, $\phi_l(x) = \phi(h^{-1}x - l)$. S_h is taken to be the space of splines of at most power growth

$$S_h = \left\{ \sum_l C_l \phi_l : C_l = O(|l|^q) \text{ as } |l| \to \infty \text{ for some } q \right\}.$$

We shall consider a nonstandard Galerkin method for the problem (1). This method is similar to a method first proposed by Dendy [5] and analyzed by Wahlbin, [17] and [18], who proved convergence for smooth solutions of some linear and nonlinear hyperbolic equations. This method adds dissipation to the usual method by use of a nonstandard variational principle.

We shall seek $u^h : [0, T] \to S_h$, as a differential map satisfying

$$(4) \qquad \left(\frac{\partial u^h}{\partial t} - Du^h, \phi_l - hD\phi_l \right) = 0, \qquad l \in \mathbf{Z},$$

$$(5) \qquad u^h(0) = I_h v \in S_h.$$

By I_h in (5) we denote the spline interpolation operator, that is, $I_h v$ is that element of S_h satisfying $(I_h v)(jh) = v(jh), j \in \mathbf{Z}$.

Next some results concerning the method (4), (5) shall be collected. Using techniques developed by Thomée [13] and Thomée and Wendroff [15], Wahlbin [17] has shown that the solution operator, $G_h(t)$, to the Galerkin equations (4), (5) factors as $G_h(t) = I_h F_k(t)$, where I_h is the spline interpolation operator and $F_k(t)$ is a finite difference operator.

For $\mu \geq 2$ and integer σ, $0 \leq \sigma \leq 2\mu - 2$, define the trigonometric polynomials

$$g_{\mu,\sigma}^{(\theta)} = h^{\sigma-1}(-i)^{\sigma-2\nu} \sum_{l=-\infty}^{\infty} (D^{\sigma-\nu}\phi_0, D^\nu \phi_l) e^{-il\theta},$$

with $\nu = [\sigma/2]$. Note that the factor $h^{\sigma-1}$ makes $g_{\mu,\sigma}$ independent of h. The next proposition collects some useful results of [17] concerning the method (4), (5).

PROPOSITION 1. *Let $G_h(t)$ be the solution operator to (4), (5). For $\lambda > 0$ fixed, $k = \lambda h$, consider t of the form $t = nk$. Then $G_h(t)$ factors as*

$$G_h(t) = I_h(F_k)^n, \qquad t = nk > 0,$$

where I_h is the spline interpolation operator and F_k is a finite difference operator with symbol

$$a(\theta) = \exp\left(\lambda \frac{ig_{\mu,1}^{(\theta)} - g_{\mu,2}^{(\theta)}}{g_{\mu,0}^{(\theta)} + ig_{\mu,1}^{(\theta)}} \right) = \exp(\lambda P(\theta)),$$

where $\operatorname{Re} P(\theta) \leq -c\theta^{2\mu}$, *for* $0 \leq |\theta| \leq \pi$, *and* $\operatorname{Im} P(\theta) = \theta(1 + O(\theta^{2\mu}))$ *as* $\theta \to 0$.

Proof. See Wahlbin [17]. □

From this one obtains that F_k is accurate of order $2\mu - 1$, dissipative of order 2μ, and stable in L_2 and L_∞. Further, L_2-optimality of the error follows; for more details see Wahlbin [17] or [18]:

THEOREM 1. *Suppose the approximate solution to the initial value problem* (1) *is computed using the method* (4), (5). *Then there is a positive constant c such that, for* $v \in H^\mu(-\infty, \infty)$,

$$\max_{0 < t < T} \|u(t) - u^h(t)\|_{L_2} \leqslant ch^\mu \max_{0 < t < T} \|u(t)\|_{H^\mu}.$$

Proof. See Wahlbin [17] or [18]. □

In the following, the $l_{2,h}$ norm shall denote

$$\|v\|_{l_{2,h}}^2 = h \sum_{j=-\infty}^{\infty} v(jh)^2.$$

Also for $t = nk$ we shall write $(F_k)^n$ as $F_k(t)$.

III. Discontinuous Initial Data. In this section the problem outlined in the introduction is considered. The initial data is taken to be an L_2 function vanishing for positive x with a jump discontinuity at zero and smooth on $(-\infty, 0)$. The case of a function smooth on $(-\infty, 0)$ and $(0, \infty)$ with a jump at zero may be reduced to a function vanishing for positive x by subtracting a smooth function agreeing with the initial data on $(0, \infty)$. The analogous work on this question in finite difference theory was carried out in [1], [3], [6], and [12]. These estimates are shown to hold in the finite element case as well. This work depends upon using spline interpolation theory to derive estimates upon the error in interchanging spline interpolation and multiplication by a characteristic function.

To be more specific, the true solution of (1) for such a $v(x)$ vanishes for $x > -t$. Therefore $u^h(x, t)$ is estimated in terms of the distance d of x, parallel to the x-axis, to the shock front $x + t = 0$. If χ_y denotes the characteristic function of (y, ∞), we estimate $\|\chi_{d-t}u^h(t)\|_{L_2}$ and $\|\chi_{d-t}u^h(t)\|_{L_\infty}$. The basic result is that these two quantities decay exponentially in d for fixed h and exponentially in h^{-1} for fixed d. An analogous result can be shown for functions $v(x)$ vanishing on the other side of the shock front. These results, with Theorem 1, are then combined to give the result announced in the introduction.

THEOREM 2. *Suppose the approximate solution of* (1) *is computed using method* (4), (5) *with μ even. Suppose also $v(x) \in l_{2,h} \cap L_2 \cap L_\infty$ and that v vanishes for positive x. Then there are positive constants C, c, and α (C and α depending upon μ) such that*

(6) $\quad \|\chi_{d-t}u^h(t)\|_{L_2} \leqslant C\|v\|_{L_\infty}h^{1/2}\exp(-\alpha h^{-1}d) + \exp(-cn\min\{d_t^\kappa, d_t\})\|v\|_{l_2}$,

with $t = nk > 0$, $d_t = dt^{-1}$. Also $\kappa = 2\mu/(2\mu - 1)$.

THEOREM 3. *Suppose the hypotheses of Theorem 2 hold. Suppose also $|x|^{-a}v(x) \in L_\infty$ for some $a \geqslant 0$. Then*

(7)
$$\|\chi_{d-t}u^h(t)\|_{L_\infty} \leqslant C\|v\|_{L_\infty}\exp(-\alpha h^{-1}d)$$
$$+ Ch^{a/\kappa}\exp(-cn\min\{d_t^\kappa, d_t\})\| |x|^{-a}v\|_{L_\infty}$$

holds for $t \leqslant T$, with d_t, C, c, α, and κ as in Theorem 3.1.

Proof of the Theorems. Note that $\chi_{d-t}u^h(t) = \chi_{d-t}I_hF_k(t)v$. We would like to interchange I_h and multiplication by χ_{d-t}. To do so the error in interchanging these two operations must be estimated. To this end, consider the cardinal spline interpolation problem:

For μ even, we seek $L_h(x) \in S_h$ satisfying $L_h(x) \in L_\infty(-\infty, \infty)$ and

$$L_h(jh) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j \neq 0. \end{cases}$$

The following basic result is known about this problem [10], [11].

THEOREM 4. *For μ even, there is a unique L_h in S_h. Furthermore, there are positive constants A and α, depending upon μ, such that*

$$|L_h(x)| < A \exp(-\alpha h^{-1}|x|), \qquad x \in \mathbf{R}.$$

Also, for $C = (c_j) \in l_\infty$, the unique function $S_h(x)$ in S_h satisfying $S_h(jh) = c_j$, $S_h(x) \in L_\infty$ is given by

$$S_h(x) = \sum_{j=-\infty}^{\infty} c_j L_h(x - jh),$$

the series converging absolutely and uniformly on each compact interval. $\quad\square$

This shall be used to produce the desired estimate. For a function $f(x) \in L_\infty$, expand I_hf and $I_h(\chi_y f)$ using Theorem 4:

$$(I_hf)(x) = \sum_{j=-\infty}^{\infty} f(jh)L_h(x - jh),$$

$$I_h(\chi_y f)(x) = \sum_{j=J}^{\infty} f(jh)L_h(x - jh),$$

where $J = \min\{j: jh > y\}$. Thus,

$$|(I_hf)(x) - I_h(\chi_y f)(x)| = \left| \sum_{j=-\infty}^{J-1} f(jh)L_h(x - jh) \right|$$

$$\leqslant A\|f\|_{L_\infty} \sum_{j=-\infty}^{J-1} \exp(-\alpha|xh^{-1} - j|).$$

Setting $y = d/2 - t$ and considering only $x > d - t$ gives

(8)
$$|I_hf(x) - I_h(\chi_{d/2-t}f)(x)| \leqslant C\|f\|_{L_\infty} \exp(-\alpha xh^{-1}) \sum_{j=-\infty}^{\infty} \exp(\alpha j)$$

$$\leqslant C\|f\|_{L_\infty} \exp(-\alpha h^{-1}(x - (d/2 - t))).$$

Thus, (8) gives a pointwise estimate and an L_2 estimate

(9)
$$\|I_hf - I_h(\chi_{d/2-t}f)\|_{L_\infty(d-t,\infty)} \leqslant C\|f\|_{L_\infty} \exp\left(-\frac{\alpha}{2}h^{-1}d\right),$$

$$\|I_hf - I_h(\chi_{d/2-t}f)\|_{L_2(d-t,\infty)} \leqslant C\|f\|_{L_\infty}h^{1/2} \exp\left(-\frac{\alpha}{2}h^{-1}d\right).$$

Now return to the problem at hand. For $d > 0$ and $p = 2, \infty$, there follows

(10)
$$\|I_h\chi_{d-t}F_k(t)v\|_{L_p(-\infty,\infty)} \leqslant \|I_h(\chi_{d-t}F_k(t)v)\|_{L_p(d-t,\infty)}$$

$$+ \|\chi_{d/2-t}(I_hF_k(t)v) - I_h(\chi_{d/2-t}F_k(t)v)\|_{L_p(d-t,\infty)}.$$

These two terms shall be estimated separately. Since spline interpolation is a bounded linear operator between L_∞ and L_∞ and between L_2 and $l_{2,h}$ (but not L_2 and L_2) [8], [11] we have

(11)
$$\|I_h(\chi_{d-t}F_k(t)v)\|_{L_2} \leqslant C\|\chi_{d-t}F_k(t)v\|_{l_{2,h}},$$
$$\|I_h(\chi_{d-t}F_k(t)v)\|_{L_\infty} \leqslant C\|\chi_{d-t}F_k(t)v\|_{L_\infty}.$$

Finite difference theory shall be used to estimate the above two quantities.

The finite difference problem has been addressed in [1], [3], [6] or [12]. In [3] estimates on $\|\chi_{d-t}F_k(t)v\|_{L_p}$ are derived for $p = 2$ and ∞ using inequalities for the symbol $a(\theta)$ of F_k in a neighborhood of the real axis. If $F_k(t)$ is considered as a Toeplitz matrix active between l_2 and l_2 instead of L_2 and L_2, these estimates on $a(\theta)$ may be used to derive l_2 estimates on $\chi_{d-t}F_k(t)v$. Specifically, using the notation of Theorems 1 and 2, there follows

(12)
$$\|\chi_{d-t}F_k(t)v\|_{l_{2,h}} \leqslant \exp(-cn \min\{d_t^\kappa, d_t\})\|v\|_{l_{2,h}}.$$

The corresponding L_∞ estimate in [3] reads

(13)
$$\|\chi_{d-t}F_k(t)v\|_{L_\infty} \leqslant Ch^{\alpha/\kappa} \exp(-cn \min\{d_t^\kappa, d_t\})\||x|^{-\alpha}v\|_{L_\infty}.$$

The theorems now follow by setting $f = F_k(t)v$ in (9), (10), using the fact that F_k is stable in L_∞ and using inequalities (9), (10), (11), (12), and (13). $\square$

Remarks. A corresponding analysis may be carried out for fully discrete schemes and for other methods of introducing dissipation into the Galerkin method. For an analysis of either case or more details upon the present method see Chapter 3 of [7]. Also, corresponding results may be shown for μ odd by posing the interpolation operator and the finite difference operator at midpoints rather than at the knots.

Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37916

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