

Approximation of the Spectrum of Closed Operators: The Determination of Normal Modes of a Rotating Basin

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Abstract. This paper gives a theory of spectral approximation for closed operators in Banach spaces. The perturbation theory developed in this paper is applied to the study of a finite element procedure for approximating the spectral properties of a differential system modeling a fluid in a rotating basin.

Introduction. In this paper, we give a theory of spectral approximation for closed operators in Banach spaces. We then apply this theory to an analysis of the approximation of the spectral properties of some differential systems by finite element methods.

Bramble and Osborn [1] and Osborn [14] developed a theory of spectral approximation for compact operators in Banach spaces. Their theory can be applied to the analysis of many numerical procedures for the spectral approximation of differential operators, T , such that $T + \lambda I$ has a compact inverse for some $\lambda \in \mathbb{C}$. Most of the differential systems in the theory of elasticity are in this class.

However, there are many differential systems of interest in mathematical physics which do not have compact resolvents. These operators can have continuous spectrum, eigenvalues of infinite multiplicity, and finite limit points of eigenvalues. Also, the eigenfunctions need not be smooth since the differential systems are not necessarily elliptic.

Descloux, Nassif, and Rappaz [4], [5] have studied the approximation of the spectrum of a differential system of interest in magnetohydrodynamics which has a bounded inverse, but not a compact inverse. They developed a theory of spectral approximation for bounded operators which treats this problem. An analysis of the approximation of the spectral properties of a class of bounded operators by finite element methods has also been done by Mills [12], [13].

The results in this paper apply to closed (not necessarily bounded) operators in Banach spaces. We apply the perturbation theory developed in this paper to the study of a finite element procedure for approximating the spectral properties of a differential system modeling a fluid in a rotating basin. We note that, unlike previous authors, we analyze the approximation of the differential operator directly and not through its inverse.

Received January 3, 1980.

1980 *Mathematics Subject Classification.* Primary 65N25, 65N30.

*Supported by AFOSR under Contract F49620-79-C-0149 and by a Faculty Research Fellowship from the Horace Rackham School of Graduate Studies, The University of Michigan.

**Supported by the Fonds National Suisse de la Recherche Scientifique.

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0025-5718/81/0000-0010/\$05.50

The time-dependent equations for the differential system modeling a fluid in a rotating basin are

$$(0.1) \quad \begin{aligned} \frac{\partial \zeta}{\partial t} &= -\nabla \cdot \vec{u}, & (x, t) \in \bar{\Omega} \times \mathbf{R}, \\ \frac{\partial \vec{u}}{\partial t} &= -\nabla \zeta - f\vec{u} - \omega R\vec{u}, & (x, t) \in \bar{\Omega} \times \mathbf{R}, \\ \vec{u} \cdot \vec{n} &= 0, & (x, t) \in \partial\Omega \times \mathbf{R}, \\ \int_{\Omega} \zeta \, dx &= 0, \end{aligned}$$

where $\Omega \subseteq \mathbf{R}^2$ is a bounded, connected open set with smooth boundary, $\partial\Omega$, $\vec{u} = (u_1, u_2)$ is the horizontal volume transport, ζ is the height of the fluid above equilibrium level, R is the linear operator $R\vec{u} = (-u_2, u_1)$, \vec{n} is the exterior normal to $\partial\Omega$, and $f \geq 0$ and ω are real constants representing friction and Coriolis terms.

Thus, we are concerned with approximating the spectral properties of the system

$$(0.2) \quad \begin{aligned} T(\zeta, \vec{u}) &= (-\nabla \cdot \vec{u}, -\nabla \zeta - f\vec{u} - \omega R\vec{u}), & x \in \Omega, \\ \vec{u} \cdot \vec{n} &= 0, & x \in \partial\Omega, \\ \int_{\Omega} \zeta \, dx &= 0. \end{aligned}$$

We note that $T - I$ is a symmetric, formally dissipative operator with maximal nonpositive boundary conditions [9].

If $f = \omega = 0$, then $T(0, R\nabla\psi) = 0$ for all smooth functions ψ such that $\psi = 0$ on $\partial\Omega$. Hence, in this case ($f = \omega = 0$), 0 is an eigenvalue of T of infinite multiplicity. Since the sum of an operator with noncompact inverse and a bounded operator has a noncompact inverse, it follows that $T - \lambda I$ has a noncompact inverse for the general case $f, \omega \in \mathbf{R}, \lambda \in \mathbf{C}, \lambda$ belonging to the resolvent set of T .

We give here error estimates for a finite element procedure proposed by Platzman [15] to approximate the spectral properties of (0.2). The selfadjoint case $f = 0$ was analyzed by Luskin [11] by techniques different from those used here.

In Section 1, we give a general theory for the approximation of a closed operator A by a family of finite-dimensional operators $\{A_h\}$. In our applications, A will be a differential operator, and A_h will be an approximation of A given by a finite element procedure. We propose two properties, (P1) and (P2), and show that these properties imply the convergence of the spectral properties of A_h to those of A . Error estimates in the applications will follow from Theorem 1.3.

We give special results in Section 2 for the case where A and A_h are selfadjoint operators in a Hilbert space. These results apply to the approximation of a continuous spectrum.

The operator theory developed in Sections 1 and 2 is applied, in Section 3 and 4, to examples of the approximation of spectral properties of differential operators by finite element methods. In Section 3, we apply our theory to obtain error estimates for the approximation of the spectral properties of scalar, second order, uniformly strongly elliptic operators by the standard finite element method. This example is included even though the results are not new since we believe that its inclusion will make it easier for the reader to understand our main example in Section 4.

In Section 4, we define and analyze the approximation of T . Theorem 1.3 gives

optimal order estimates for the approximation of eigenspaces. Optimal order eigenvalue estimates for this problem have been given in [11] for the selfadjoint case and are derived from the results in this paper and in [3] for the general case.

We note that we use, without explicit reference, the classical spectral theory; see, for example, Kato [6], Riesz-Nagy [16].

1. Approximation of the Spectrum of Closed Operators in Banach Spaces. We first introduce some notation. Let X be a complex Banach space with norm $\| \cdot \|$. Denote by $\mathfrak{B}(X)$ the set of bounded, linear operators $B: X \rightarrow X$. Also, denote by $\mathcal{C}(X)$ the set of closed, linear operators $C: \mathfrak{D}(C) \subset X \rightarrow X$, where the domain of C , $\mathfrak{D}(C)$, is not necessarily dense in X . For $C \in \mathcal{C}(X)$, $\rho(C)$ is the resolvent set of C defined by

$$\rho(C) = \{z \in \mathbb{C} \mid (z - C)^{-1} \in \mathfrak{B}(X)\}.$$

If $z \in \rho(C)$, we define the resolvent operator $R_z(C) = (z - C)^{-1}: X \rightarrow X$. The complement of $\rho(C)$ is $\sigma(C) = \{z \in \mathbb{C} \mid z \notin \rho(C)\}$, the spectrum of C .

Let Y and Z be closed subspaces of X and $x \in X$. We set

$$\delta(x, Y) = \inf_{y \in Y} \|x - y\|, \quad \delta(Y, Z) = \sup_{\substack{y \in Y \\ \|y\|=1}} \delta(y, Z),$$

$$\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y));$$

$\hat{\delta}(Y, Z)$ is called the gap between Y and Z and is a measure of the “distance” between these spaces. If C and D are in $\mathcal{C}(X)$, with graphs $G_C, G_D \subset X \times X$, then we define $\delta(C, D) = \delta(G_C, G_D)$, i.e.,

$$\delta(C, D) = \sup_{\substack{x \in \mathfrak{D}(C) \\ \|x\| + \|Cx\|=1}} \inf_{y \in \mathfrak{D}(D)} \{\|x - y\| + \|Cx - Dy\|\}.$$

Furthermore, we define

$$\hat{\delta}(C, D) = \max(\delta(C, D), \delta(D, C)).$$

Finally, if Y is a subspace of $\mathfrak{D}(C) \cap \mathfrak{D}(D)$, we set

$$\|C - D\|_Y = \sup_{\substack{y \in Y \\ \|y\|=1}} \|Cy - Dy\|.$$

Now, let $A \in \mathcal{C}(X)$ be a given operator. In order to approximate $\sigma(A)$, we consider a family $\{X_h\}$ of finite-dimensional subspaces of X parametrized by h and linear operators $A_h: X_h \rightarrow X_h$.

We denote by $\sigma(A_h)$, $\rho(A_h)$, and $R_z(A_h): X_h \rightarrow X_h$ the spectrum, resolvent set, and resolvent operator of A_h considered as a bounded operator in X_h . However, when used in connection with expressions of the type “ $\delta(A_h, A)$ ” or “ $\|A - A_h\|_{X_h}$ ”, A_h is considered a closed operator in X with nondense domain X_h .

Let $\Gamma \subset \rho(A)$ be a given Jordan closed curve. Then

$$E = \frac{1}{2\pi i} \int_{\Gamma} R_z(A) dz: X \rightarrow X$$

and, if $\Gamma \subset \rho(A_h)$,

$$E_h = \frac{1}{2\pi i} \int_{\Gamma} R_z(A_h) dz: X_h \rightarrow X_h$$

are the spectral projectors relative to A and A_h respectively. We shall also use the relations

$$AE = \frac{1}{2\pi i} \int_{\Gamma} zR_z(A) dz: X \rightarrow X$$

and

$$A_h E_h = \frac{1}{2\pi i} \int_{\Gamma} zR_z(A_h) dz: X_h \rightarrow X_h.$$

We now introduce the following two properties:

- (P1) $\lim_{h \rightarrow 0} \delta(A_h, A) \rightarrow 0$,
- (P2) $\forall x \in X, \lim_{h \rightarrow 0} \delta(x, X_h) = 0$.

The following theorems contain the main results of this section. We note that $c < \infty$ shall denote, in this paper, a positive constant which is independent of h , but which varies from estimate to estimate.

THEOREM 1.1. *Suppose (P1) is valid and let $K \subset \rho(A)$ be a compact set. Then there exists $h_0 > 0$ and $c = c(K)$ such that, for $h < h_0$, we have $K \subset \rho(A_h)$ and*

$$\|R_z(A) - R_z(A_h)\|_{X_h} \leq c\delta(A_h, A), \quad z \in K.$$

Theorem 1.1 shows that if (P1) is valid, then the approximation of $\sigma(A)$ by $\sigma(A_h)$ is upper semicontinuous. Furthermore, for $h < h_0$, $\|R_z(A_h)\|_{X_h}$ is uniformly bounded on K . This is a stability property.

THEOREM 1.2. *Suppose (P1) is valid. Then there exists $h_0 > 0$ and c so that, for $h < h_0$, we have the bounds*

$$(1.1) \quad \|E - E_h\|_{X_h} + \|AE - A_h E_h\|_{X_h} \leq c\delta(A_h, A),$$

$$(1.2) \quad \delta(E_h(X_h), E(X)) \leq c\delta(A_h, A),$$

$$(1.3) \quad \delta(x, E_h(X_h)) \leq c\{\delta(x, X_h) + \delta(A_h, A)\|x\|\}, \quad x \in E(x).$$

THEOREM 1.3. *Suppose that (P1) and (P2) are satisfied and that $E(X)$ is the finite-dimensional subspace corresponding to an isolated eigenvalue λ of algebraic multiplicity m of A . Let α be the ascent of $(\lambda - A)$ and let f be a holomorphic function defined in the neighborhood of λ . We set*

$$\gamma_h = \min\{\delta(A|_{E(X)}, A_h), \delta(A_h, A)\}.$$

Then, for h small enough, $A_h|_{E_h(X_h)}: E_h(X_h) \rightarrow E_h(X_h)$ has exactly m eigenvalues $\lambda_{1,h}, \dots, \lambda_{m,h}$, repeated according to multiplicity. Also, there exists $h_0 > 0$ and c such that the following bounds are valid for $h < h_0$:

$$(1.4) \quad \hat{\delta}(E_h(X_h), E(X)) \leq c\gamma_h,$$

$$(1.5) \quad \hat{\delta}(A_h|_{E_h(X_h)}, A|_{E(X)}) \leq c\gamma_h,$$

$$(1.6) \quad \left| f(\lambda) - \frac{1}{m} \sum_{i=1}^m f(\lambda_{i,h}) \right| \leq c\gamma_h,$$

$$(1.7) \quad \max_{i=1, \dots, m} |\lambda - \lambda_{i,h}|^\alpha \leq c\gamma_h.$$

Remarks. If (P1) and (P2) are satisfied, then (1.3) of Theorem 1.2 shows that $\lim_{h \rightarrow 0} \delta(x, E_h(X_h)) = 0, \forall x \in E(X)$, i.e., any $x \in E(X)$ can be approximated by vectors in $E_h(X_h)$. It is not true however that any $z \in \sigma(A)$ can be approximated by eigenvalues of A_h , i.e., there may exist $z \in \sigma(A)$ so that $\text{dist}(z, \sigma(A_h))$ fails to converge to zero. (The approximation of $\sigma(A)$ by $\sigma(A_h)$ is not necessarily lower semicontinuous.) The classical counterexample is the shift operator; see, for example, Kato [6, p. 210]. Note that if A and A_h are selfadjoint operators in a Hilbert space, then lower semicontinuity is true under weaker conditions than (P1), (P2); see, for example, Kato [6, p. 431].

In order to prove Theorems 1.1 and 1.2, we first prove a sequence of lemmas which are variants of results found in Kato [6, pp. 197–208].

LEMMA 1.1. *Let $B \in \mathfrak{B}(X), C \in \mathcal{C}(X)$. Then*

(a) $\delta(C, B) \leq \|C - B\|_{\mathfrak{D}(C)}$,

(b) $\|C - B\|_{\mathfrak{D}(C)} \leq (1 + \|B\|)^2 \delta(C, B) / (1 - (1 + \|B\|)\delta(C, B))$

if the denominator is positive.

Proof. Part (a) follows directly from the definitions. We prove part (b). Let $x \in \mathfrak{D}(C), \|x\| = 1$, and let $\epsilon > 0$ be arbitrary. It follows from the definition of $\delta(C, B)$ that there exists $y \in X$ such that

(1.8) $\|x - y\| + \|Cx - By\| \leq \delta(C, B)\{\|x\| + \|Cx\|\} + \epsilon.$

Consequently, we have

(1.9)
$$\begin{aligned} \|(C - B)x\| &\leq \|Cx - By\| + \|By - Bx\| \\ &\leq \|Cx - By\| + \|B\| \|y - x\| \\ &\leq (1 + \|B\|)\{\|Cx - By\| + \|y - x\|\} \\ &\leq (1 + \|B\|)[\delta(C, B)\{\|x\| + \|Cx\|\} + \epsilon]. \end{aligned}$$

Replacing $\|Cx\|$ by $\|(C - B)x\| + \|B\| \|x\|$ in the right-hand side of the above inequality and letting $\epsilon \rightarrow 0$ yields the estimate

(1.10) $\|(C - B)x\| \leq (1 + \|B\|)\delta(C, B)\{(1 + \|B\|)\|x\| + \|(C - B)x\|\}.$

The result now follows from (1.10). Q.E.D.

Remark. It follows from Lemma 1.1 that if $A \in \mathfrak{B}(X)$, then property (P1) is equivalent to $\lim_{h \rightarrow 0} \|A - A_h\|_{X_h} = 0$. This is the spectral approximation condition of Descloux-Nassif-Rappaz [4].

LEMMA 1.2. *Let $B \in \mathfrak{B}(X)$ and $C, D \in \mathcal{C}(X)$. Then*

$$\delta(C + B, D + B) \leq (1 + \|B\|)^2 \delta(C, D).$$

Proof. Let $x \in \mathfrak{D}(C), \|x\| + \|(C + B)x\| = 1$, and let $\epsilon > 0$ be arbitrary. We have then, by the triangle inequality, that

(1.11) $\|x\| + \|Cx\| \leq \|x\| + \|(C + B)x\| + \|Bx\| \leq 1 + \|B\|.$

Thus, it follows from the definition of $\delta(C, D)$ that we may choose $y \in \mathfrak{D}(D)$ such that

(1.12) $\|x - y\| + \|Cx - Dy\| \leq \delta(C, D)(1 + \|B\|) + \epsilon.$

Hence,

$$(1.13) \quad \begin{aligned} \|x - y\| + \|(C + B)x - (D + B)y\| \\ \leq \|x - y\| + \|Cx - Dy\| + \|B\| \|x - y\| \\ \leq (1 + \|B\|)^2 \delta(C, D) + (1 + \|B\|)\varepsilon. \end{aligned}$$

Hence, the result follows from (1.13) after letting $\varepsilon \rightarrow 0$. Q.E.D.

LEMMA 1.3. *If C and $D \in \mathcal{C}(X)$ are invertible, then*

$$\delta(C, D) = \delta(C^{-1}, D^{-1}).$$

Proof. This result follows directly from the definitions. Q.E.D.

LEMMA 1.4. *Let $C, D \in \mathcal{C}(X)$ and suppose there exists κ such that*

$$(1.14) \quad \|Dx\| \geq \kappa \|x\|, \quad x \in \mathfrak{D}(D).$$

If $\delta(C, D) < \min(1, \kappa)$, then C is invertible.

Proof. We show that if C is not invertible, then $\delta(C, D) \geq \min(1, \kappa)$. Let $x \in \mathfrak{D}(C)$ be such that $\|x\| = 1$ and $Cx = 0$. Let $\varepsilon > 0$ be arbitrary. It follows from the definition of $\delta(C, D)$ that since $Cx = 0$ we can choose $y \in \mathfrak{D}(D)$ such that

$$\|x - y\| + \|Dy\| \leq \delta(C, D) + \varepsilon.$$

It then follows from (1.14) that

$$(1.15) \quad \begin{aligned} \delta(C, D) + \varepsilon \geq \|x - y\| + \kappa \|y\| \geq |1 - \|y\|| + \kappa \|y\| \\ \geq \min(1, \kappa) \{|1 - \|y\|| + \|y\|\} \geq \min(1, \kappa). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that we have reached the contradiction $\delta(C, D) \geq \min(1, \kappa)$. Q.E.D.

Proof of Theorem 1.1. In this proof, c depends on K but all estimates are uniform for $z \in K$. By Lemma 1.2,

$$(1.16) \quad \delta(z - A_h, z - A) \leq c\delta(A_h, A), \quad z \in K.$$

It follows from (1.16) and (P1) that

$$(1.17) \quad \lim_{h \rightarrow 0} \delta(z - A_h, z - A) \rightarrow 0 \quad \text{uniformly for } z \in K.$$

Since $K \subset \rho(A)$, there exists c_1 such that

$$(1.18) \quad \|(z - A)x\| \geq c_1 \|x\|, \quad x \in \mathfrak{D}(A), z \in K.$$

We can now conclude from (1.17), (1.18) and Lemma 1.4 that there exists $h_0 > 0$ such that $z - A_h$ is invertible for $h < h_0$ and $z \in K$. Since X_h is finite dimensional, we have that $K \subset \rho(A_h)$ for $h < h_0$.

Furthermore, it follows from (1.16) and Lemma 1.3 that

$$(1.19) \quad \delta(R_z(A_h), R_z(A)) \leq c\delta(A_h, A), \quad z \in K.$$

Hence, we can obtain from Lemma 1.1b the result

$$(1.20) \quad \begin{aligned} \|R_z(A) - R_z(A_h)\|_{X_h} &\leq c\delta(R_z(A_h), R_z(A)) \\ &\leq c\delta(A_h, A), \quad h < h_0, z \in K. \quad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem 1.2. The result (1.1) follows from Theorem 1.1 and the following estimates:

$$\begin{aligned}
 & \|E - E_h\|_{X_h} + \|AE - A_h E_h\|_{X_h} \\
 (1.21) \quad &= \left\| \frac{1}{2\pi i} \int_{\Gamma} (R_z(A) - R_z(A_h)) dz \right\|_{X_h} + \left\| \frac{1}{2\pi i} \int_{\Gamma} (zR_z(A) - zR_z(A_h)) dz \right\|_{X_h} \\
 &\leq (2\pi)^{-1} \max_{z \in \Gamma} \|R_z(A) - R_z(A_h)\|_{X_h} \int_{\Gamma} (1 + |z|) |dz|.
 \end{aligned}$$

The estimate (1.2) now follows directly from the estimate (1.1).

In order to prove (1.3), let $x \in E(X)$ and $x_h \in X_h$. Then $x - E_h x_h = E(x - x_h) + (E - E_h)x_h$. Consequently,

$$\begin{aligned}
 (1.22) \quad \delta(x, E_h(X_h)) &\leq (\|E\| \|x - x_h\| + \|E - E_h\|_{X_h} \|x_h\|) \\
 &\leq (\|E\| + \|E - E_h\|_{X_h}) \|x - x_h\| + \|E - E_h\|_{X_h} \|x\|.
 \end{aligned}$$

The result (1.3) follows by taking the infimum over $x_h \in X_h$ and using (1.1). Q.E.D.

It remains to prove Theorem 1.3. We first quote without proof the following simple result:

LEMMA 1.5. *Let Y and Z be two subspaces of X with the same finite dimension and let $P: Y \rightarrow Z$ be a linear operator such that*

$$(1.23) \quad \|Py - y\| \leq \frac{1}{2} \|y\|, \quad y \in Y.$$

Then P is bijective and

$$(1.24) \quad \|P^{-1}z\| \leq 2\|z\|, \quad z \in Z.$$

LEMMA 1.6. *Let Y and Z be two subspaces of X .*

(a) *If $\delta(Y, Z) < 1$, then $\dim Y = \dim Z$.*

(b) *If $\dim Y = \dim Z < \infty$, then*

$$(1.25) \quad \delta(Y, Z) \leq \delta(Z, Y) [1 - \delta(Z, Y)]^{-1}.$$

Proof. For (a), see Kato [6, p. 200]. For (b), see Kato [7]. Q.E.D.

In the rest of this section, we suppose that the hypotheses of Theorem 1.3 are satisfied. Also, c and h_0 will denote two generic positive constants which depend on Γ .

LEMMA 1.7. $\delta(A|_{E(X)}, A_h|_{E_h(X_h)}) \leq c\delta(A|_{E(X)}, A_h)$, $h \leq h_0$.

Proof. Let $x \in E(X)$, $\|x\| + \|Ax\| = 1$, and let $x_h \in X_h$ be such that

$$(1.26) \quad \|x - x_h\| + \|Ax - A_h x_h\| \leq \delta(A|_{E(X)}, A_h).$$

We have

$$\begin{aligned}
 & \|x - E_h x_h\| + \|Ax - A_h E_h x_h\| = \left\| \frac{1}{2\pi i} \int_{\Gamma} (R_z(A)x - R_z(A_h)x_h) dz \right\| \\
 (1.27) \quad &+ \left\| \frac{1}{2\pi i} \int_{\Gamma} (zR_z(A)x - zR_z(A_h)x_h) dz \right\| \\
 &\leq (2\pi)^{-1} \max_{z \in \Gamma} \|R_z(A)x - R_z(A_h)x_h\| \int_{\Gamma} (1 + |z|) |dz|.
 \end{aligned}$$

In order to conclude the proof of the lemma, it suffices to estimate $\|R_z(A)x - R_z(A_h)x_h\|$ for $z \in \Gamma$.

For $r_h \in X_h$, we have by Theorem 1 that

$$\begin{aligned}
 (1.28) \quad & \|R_z(A)x - R_z(A_h)x_h\| \leq \|R_z(A)x - r_h\| + \|r_h - R_z(A_h)x_h\| \\
 & = \|R_z(A)x - r_h\| + \|R_z(A_h)[(z - A_h)r_h - x_h]\| \\
 & \leq \|R_z(A)x - r_h\| + c\|(z - A_h)r_h - x_h\| \\
 & \leq \|R_z(A)x - r_h\| + c(\|(z - A_h)r_h - (z - A)R_z(A)x\| + \|x - x_h\|) \\
 & \leq c\|R_z(A)x - r_h\| + c\|AR_z(A)x - A_h r_h\| + c\|x - x_h\|.
 \end{aligned}$$

Since $R_z(A)x \in E(X)$, we can choose $r_h \in X_h$ so that

$$\begin{aligned}
 (1.29) \quad & \|R_z(A)x - r_h\| + \|AR_z(A)x - A_h r_h\| \\
 & \leq \delta(A|_{E(X)}, A_h)(\|R_z(A)x\| + \|AR_z(A)x\|) \\
 & \leq c\delta(A|_{E(X)}, A_h)(\|x\| + \|Ax\|) \leq c\delta(A|_{E(X)}, A_h).
 \end{aligned}$$

The proof now follows by utilizing (1.26) and (1.29) in (1.28). Q.E.D.

Remark. Lemma 1.7 is still valid if the hypothesis (P1) is replaced by the uniform boundedness of $R_z(A_h)$ on Γ .

Proof of Theorem 1.3. From (1.2) and (1.3) of Theorem 1.2, it follows that $\lim_{h \rightarrow 0} \hat{\delta}(E_h(X_h), E(X)) = 0$. Consequently, by Lemma 1.6(a), $\dim E_h(X_h) = \dim E(X) = m$ for $h < h_0$. This proves that for $h < h_0$, $A_h|_{E_h(X_h)}: E_h(X_h) \rightarrow E_h(X_h)$ has exactly m eigenvalues $\lambda_{1,h}, \dots, \lambda_{m,h}$ repeated according to multiplicity.

By Lemma 1.6(b), we have that

$$(1.30) \quad \hat{\delta}(E(X), E_h(X_h)) \leq c \min\{\delta(E(X), E_h(X_h)), \delta(E_h(X_h), E(X))\}, \quad h < h_0.$$

It follows from Lemma 1.7 that

$$(1.31) \quad \delta(E(X), E_h(X_h)) \leq c\delta(A|_{E(X)}, A_h|_{E_h(X_h)}) \leq c\delta(A|_{E(X)}, A_h).$$

The result (1.31) and (1.2) of Theorem 1.2 yield (1.4) when substituted in (1.30).

We have by (1.1) of Theorem 1.2 that

$$(1.32) \quad \delta(A_h|_{E_h(X_h)}, A|_{E(X)}) \leq c\delta(A_h, A), \quad h < h_0.$$

Hypothesis (P1) and Lemmas 1.6 and 1.7 allow us to conclude the validity of (1.5).

Now let u_1, \dots, u_m be a basis for $E(X)$ with $\|u_i\| + \|Au_i\| = 1$, $i = 1, \dots, m$. We then choose $u_{1,h}, \dots, u_{m,h} \in E_h(X_h)$ so that

$$(1.33) \quad \|u_i - u_{i,h}\| + \|Au_i - A_h u_{i,h}\| \leq \delta(A|_{E(X)}, A_h|_{E_h(X_h)}), \quad i = 1, \dots, m.$$

Next, we define $\Lambda_h: E(X) \rightarrow E_h(X_h)$ as the linear operator such that $\Lambda_h u_i = u_{i,h}$, $i = 1, \dots, m$. It follows by (1.5) that

$$(1.34) \quad \|u - \Lambda_h u\| + \|Au - A_h \Lambda_h u\| \leq c\gamma_h \|u\|, \quad u \in E(X).$$

By (P1) $\lim_{h \rightarrow 0} \gamma_h = 0$, so by Lemma 1.5 and (1.34) Λ_h is a bijection whose inverse Λ_h^{-1} is uniformly bounded for $h < h_0$. Let $\hat{A} = A|_{E(X)}$ and $\hat{A}_h = \Lambda_h^{-1} A_h \Lambda_h: E(X) \rightarrow E(X)$. For $u \in E(X)$, we have

$$\begin{aligned}
 (1.35) \quad & \|(\hat{A} - \hat{A}_h)u\| = \|\Lambda_h^{-1}(\Lambda_h A - A_h \Lambda_h)u\| \\
 & \leq c\{\|(\Lambda_h - I)Au\| + \|Au - A_h \Lambda_h u\|\} \leq c\gamma_h \|u\|, \quad h < h_0.
 \end{aligned}$$

Consequently,

$$(1.36) \quad \|\hat{A} - \hat{A}_h\|_{E(X)} \leq c\gamma_h.$$

Now \hat{A}_h has eigenvalues $\lambda_{1,h}, \dots, \lambda_{m,h}$ and \hat{A} has the eigenvalue λ of algebraic multiplicity m . Also, α is the ascent of $(\lambda - \hat{A})$. We have reduced our problem to the matrix case. The results (1.6) and (1.7) of Theorem 1.3 now follow from the classical perturbation theory for finite-dimensional operators [17, pp. 80–81]. Q.E.D.

2. The Approximation of the Spectrum of a Selfadjoint Operator. In this section, we suppose that X is a Hilbert space, $A: \mathcal{D}(A) \subset X \rightarrow X$ is selfadjoint, and $A_h: X_h \rightarrow X_h$ is selfadjoint in X_h for all h . If $I \subset \mathbf{R}$ is an interval (finite or infinite), $E_I: X \rightarrow X$ will denote the spectral projector of A relative to I , and $E_{h,I}: X_h \rightarrow X_h$ will denote the spectral projector of A_h relative to I . We can prove the following theorems:

THEOREM 2.1. (P1) $\Leftrightarrow \forall$ closed intervals I, J , one of them bounded, $I \cap J = \emptyset$, we have

$$\lim_{h \rightarrow 0} \|E_I E_{h,J}\|_{X_h} = 0.$$

THEOREM 2.2. Suppose that (P1) is valid. Let $J \subset I$ where J is a closed bounded interval and I is an open interval. Then

$$(2.1) \quad \lim_{h \rightarrow 0} \delta(E_{h,J}(X_h), E_I(X)) = 0.$$

THEOREM 2.3. Suppose that (P1) and (P2) are valid.

(a) Let I be an open interval and $x \in E_I(X)$. Then

$$\lim_{h \rightarrow 0} \delta(x, E_{h,I}(X_h)) = 0.$$

(b) If $\lambda \in \sigma(A)$, then

$$\lim_{h \rightarrow 0} \text{dist}(\lambda, \sigma(A_h)) = 0.$$

Remark. Theorem 2.3(b) states that the approximation of $\sigma(A)$ by $\sigma(A_h)$ is lower semicontinuous.

Theorems 2.1, 2.2, and 2.3 have been proved in Section 3 of Descloux, Nassif, Rappaz [4] when A is bounded (recall our remarks after Lemma 1.1). Our proof will reduce the unbounded case to the bounded case. We shall restrict ourself to the proof of Theorem 2.2. Theorems 2.1 and 2.3 can be obtained by similar arguments. Furthermore, for the sake of simplicity and without much loss of generality, we shall suppose that $\rho(A) \cap \mathbf{R} \neq \emptyset$.

Proof of Theorem 2.2. As mentioned above, we suppose there exists $a \in \mathbf{R}$ with $a \in \rho(A)$. We introduce the function $\phi(\lambda) = (a - \lambda)^{-1}$ and set $B = \phi(A) = R_a(A)$. By Theorem 1.1, $B_h = \phi(A_h) = R_a(A_h)$ is well defined for h sufficiently small and

$$\lim_{h \rightarrow 0} \|B - B_h\|_{X_h} = 0.$$

For an interval M , $F_M: X \rightarrow X$ and $F_{h,M}: X_h \rightarrow X_h$ will denote, respectively, the spectral projectors of B and B_h relative to M . As a first case, suppose $a \notin I$ and set $K = \phi(J)$ and $L = \phi(I)$. Then K and L are, respectively, a compact and an open

interval with $K \subset L$. From Theorem 4 of Descloux, Nassif, Rappaz [4], we can conclude that

$$(2.2) \quad \lim_{h \rightarrow 0} \delta(F_{h,K}(X_h), F_L(X)) = 0.$$

However, the result (2.1) follows from (2.2) since $F_{h,K}(X_h) = E_{h,r}(X_h)$ and $F_L(X) = E_I(X)$.

The case $a \in I$ can be reduced to the preceding one by noting that we can find compact intervals $J1, J2$ and open intervals $I1, I2$ with the following properties (for h sufficiently small):

$$\begin{aligned} a &\notin I1 \cup I2, \\ J1 &\subset I1, \quad J2 \subset I2, \\ E_I(X) &= E_{I1}(X) \oplus E_{I2}(X), \\ E_{h,r}(X_h) &= E_{h,r1}(X_h) \oplus E_{h,r2}(X_h). \quad \text{Q.E.D.} \end{aligned}$$

3. Application to Scalar Elliptic Boundary Value Problems. We shall need the following notation to discuss the application of our theory to the approximation of the spectral properties of scalar elliptic boundary value problems. Denote by Ω an open set in \mathbf{R}^n with a smooth boundary, $\partial\Omega$. As usual, we denote by $L^2(\Omega)$ the Hilbert space of square integrable, complex-valued functions with inner product and norm

$$(3.1) \quad \langle F, G \rangle = \int_{\Omega} F\bar{G} \, dx, \quad \|F\|^2 = \langle F, F \rangle.$$

We denote by $H^r(\Omega)$ the space of complex-valued functions whose distribution derivatives of order less than or equal to r , r a nonnegative integer, are in $L^2(\Omega)$ with norm

$$(3.2) \quad \|F\|_r^2 = \sum_{|\alpha| \leq r} \|D^\alpha F\|^2.$$

We wish to consider the spectral approximation of the operator

$$(3.3) \quad \begin{aligned} Lu &= - \sum_{i,j=1}^n D_i(a_{ij}D_j u) + \sum_{i=1}^n a_i D_i u + au, \quad x \in \Omega, \\ \mathfrak{B}u &= \sum_{i,j=1}^n a_{ij} \nu_i D_j u = 0, \quad x \in \partial\Omega, \end{aligned}$$

where we assume that L is a uniformly, strongly elliptic operator with real-valued coefficients in $C^\infty(\bar{\Omega})$ and $\nu = (\nu_1, \dots, \nu_n)$ is the unit exterior normal to $\partial\Omega$. We associate with L the continuous, sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$,

$$(3.4) \quad B(\phi, \psi) = \sum_{i,j=1}^n \langle a_{ij}D_j \phi, D_i \psi \rangle + \sum_{i=1}^n \langle a_i D_i \phi, \psi \rangle + \langle a\phi, \psi \rangle.$$

We may assume (by replacing L by $L + \mu$, $\mu \in \mathbf{R}$) that, for some $b > 0$,

$$(3.5) \quad \text{Re } B(\phi, \phi) \geq b \|\phi\|_1^2, \quad \phi \in H^1(\Omega).$$

Let $\mathfrak{S}_h \subset H^1(\Omega)$ be a family of finite-dimensional subspaces, parametrized by h , $0 < h \leq 1$, with r a positive integer and c a positive constant, independent of h ,

such that for $1 \leq s \leq r + 1$, $u \in H^s(\Omega)$, we have

$$(3.6) \quad \inf_{\chi \in \mathfrak{S}_h} (\|u - \chi\| + h\|u - \chi\|_1) \leq ch^s \|u\|_s.$$

Many finite element spaces are known to satisfy (3.6) [2].

We define the operator $L_h: \mathfrak{S}_h \rightarrow \mathfrak{S}_h$ by the relation

$$(3.7) \quad B(U_h, W) = \langle L_h U_h, W \rangle, \quad W \in \mathfrak{S}_h.$$

Note that L_h is well defined since \mathfrak{S}_h is finite dimensional. We shall study the approximation of the spectral properties of L by the spectral properties of L_h .

Define the space

$$H_{\mathfrak{B}}^2(\Omega) = \{u \in H^2(\Omega) \mid \mathfrak{B}u = 0 \text{ for } x \in \partial\Omega\}.$$

We consider L as a closed operator from $L^2(\Omega)$ to $L^2(\Omega)$, i.e., $X = L^2(\Omega)$, with domain $D(L) = H_{\mathfrak{B}}^2(\Omega)$. Also, in the notation of Section 1, we set $X_h = \mathfrak{S}_h$. It follows from (3.6) that if $u \in C^\infty(\bar{\Omega})$, then

$$\delta(u, \mathfrak{S}_h) = \inf_{\chi \in \mathfrak{S}_h} \|u - \chi\| \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Since $C^\infty(\bar{\Omega})$ is dense in $L^2(\Omega)$, it is clear that property (P2) is valid.

We now turn to the verification of (P1). To that end, let $P_h^0: L^2(\Omega) \rightarrow \mathfrak{S}_h$ be the $L^2(\Omega)$ projection,

$$(3.8) \quad \langle P_h^0 u - u, W \rangle = 0, \quad W \in \mathfrak{S}_h,$$

and let $P_h^1: H^1(\Omega) \rightarrow \mathfrak{S}_h$ be the $H^1(\Omega)$ projection defined by

$$(3.9) \quad B(P_h^1 u - u, W) = 0, \quad W \in \mathfrak{S}_h.$$

It follows from (3.5) that P_h^1 is well defined.

The following estimates for P_h^0 and P_h^1 are well known [2]. There exists $c < \infty$ such that, for $0 \leq s \leq r + 1$ and $u \in H^s(\Omega)$,

$$(3.10) \quad \|P_h^0 u - u\| \leq ch^s \|u\|_s$$

and such that, for $1 \leq s \leq r + 1$ and $u \in H^s(\Omega)$,

$$(3.11) \quad \|P_h^1 u - u\| + h\|P_h^1 u - u\|_1 \leq ch^s \|u\|_s.$$

We now verify that

$$(3.12) \quad \delta(L_h, L) \rightarrow 0.$$

For $U_h \in \mathfrak{S}_h$, let $u \in D(L)$ satisfy

$$(3.13) \quad Lu = L_h U_h.$$

It follows by elliptic regularity that

$$(3.14) \quad \|u\|_2 \leq c\|L_h U_h\|.$$

Now it is easily verified that $U_h = P_h^1 u$. Hence, by (3.11) and (3.14),

$$(3.15) \quad \|u - U_h\| = \|u - P_h^1 u\| \leq ch^2 \|u\|_2 \leq ch^2 \|L_h U_h\|.$$

Since $Lu = L_h U_h$, we can conclude that for each $U_h \in \mathfrak{S}_h$, there exists $u \in D(L)$ such that

$$(3.16) \quad \|u - U_h\| + \|Lu - L_h U_h\| \leq ch^2 (\|U_h\| + \|L_h U_h\|).$$

Hence, we can conclude that

$$\delta(L_h, L) \leq ch^2.$$

It is well known that the spectrum of L consists of isolated eigenvalues with finite-dimensional generalized eigenspaces. Let $\lambda \in \sigma(L)$ and let $E(X)$ be the generalized eigenspace corresponding to λ . Thus, $\dim E(X) < \infty$. We shall show that

$$(3.17) \quad \delta(L|_{E(X)}, L_h) \leq ch^{r+1}.$$

The result (3.17) will imply that the conclusions of Theorem 1.3 are valid with $\gamma_h \leq ch^{r+1}$.

If $u \in E(X)$, set $U_h = P_h^1 u \in \mathcal{S}_h$. It follows from elliptic regularity that $E(X) \subset C^\infty(\bar{\Omega})$. Hence, by (3.11),

$$(3.18) \quad \|u - P_h^1 u\| \leq ch^{r+1} \|u\|_{r+1}.$$

Also, note that if $u \in H^2(\Omega)$, then

$$(3.19) \quad P_h^0 Lu = L_h P_h^1 u.$$

Thus, it follows from (3.10) that

$$(3.20) \quad \|Lu - L_h U_h\| = \|Lu - L_h P_h^1 u\| = \|Lu - P_h^0 Lu\| \leq ch^{r+1} \|Lu\|_{r+1}.$$

Note that since $\dim E(X) < \infty$ and all norms on finite-dimensional spaces are equivalent, there exists a constant $c < \infty$ such that, for $u \in E(X)$,

$$\|Lu\|_{r+1} \leq c \|u\|, \quad \|u\|_{r+1} \leq c \|u\|.$$

So, (3.17) follows from (3.18) and (3.20).

The results on eigenvalue approximation implied by (3.17) are not the optimal $O(h^{2r})$ results for this problem that have been obtained by other methods [1], [14] unless $r = 1$. However, it can be shown that optimal $O(h^{2r})$ eigenvalue results for the problem can be derived from Theorem 1.3 for this problem if the approximation of L by L_h is analyzed in a space X which is taken to be an appropriate negative order Sobolev space.

4. Application to the Linearized Shallow Water Equations. We turn now to the description and analysis of an approximation procedure for the spectral properties of the operator, T , associated with the linearized shallow water equations. Recall that

$$(4.1) \quad \begin{aligned} T(\zeta, \vec{u}) &= (-\nabla \cdot \vec{u}, -\nabla \zeta - f\vec{u} - \omega R\vec{u}), & x \in \Omega, \\ \vec{u} \cdot \vec{n} &= 0, & x \in \partial\Omega, \\ \int_{\Omega} \zeta \, dx &= 0, \end{aligned}$$

where $\Omega \subseteq \mathbf{R}^2$ is a bounded, connected open set with smooth boundary, $\partial\Omega$, $\vec{u} = (u_1, u_2)$, R is the linear operator $R\vec{u} = (-u_2, u_1)$, \vec{n} is the exterior normal to $\partial\Omega$, and $f \geq 0$ and ω are real constants representing friction and Coriolis terms. We assume that $\partial\Omega$ has a finite number of connected components, $\{\partial\Omega_i\}_{i=0}^s$. Also, assume that the sets $\partial\Omega_i$ are smooth arcs in \mathbf{R}^2 .

We shall need the definitions of the following spaces of scalar-valued functions:

$$L_*^2(\Omega) = \left\{ w \in L^2(\Omega) \mid \int_{\Omega} w \, dx = 0 \right\},$$

$$(4.2) \quad H_*^1(\Omega) = H^1(\Omega) \cap L_*^2(\Omega),$$

$$H_c^1(\Omega) = \left\{ w \in H^1(\Omega) \mid w(x) = 0 \text{ for } x \in \partial\Omega_0, \text{ and there exist constants, } (c_i)_{i=1}^s, \text{ such that } w(x) = c_i \text{ for } x \in \partial\Omega_i, i = 1, \dots, s \right\}.$$

For $k \geq 1$, set

$$(4.3) \quad H_*^k(\Omega) = H_*^1(\Omega) \cap H^k(\Omega), \quad H_c^k(\Omega) = H_c^1(\Omega) \cap H^k(\Omega).$$

We give these subspaces of $L^2(\Omega)$ and $H^1(\Omega)$ the inner product and norms defined in (3.1) and (3.2).

We also define the following spaces of vector-valued functions:

$$L^2(\Omega)^2 = \{ \vec{u} = (u_1, u_2) \mid u_1, u_2 \in L^2(\Omega) \},$$

$$H^k(\Omega)^2 = \{ \vec{u} = (u_1, u_2) \mid u_1, u_2 \in H^k(\Omega) \} \quad \text{for } k \geq 1.$$

With these spaces we associate the inner product and norms

$$\langle \vec{u}, \vec{v} \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle,$$

$$\| \vec{u} \|^2 = \langle \vec{u}, \vec{u} \rangle, \quad \text{for } \vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2) \in L^2(\Omega)^2,$$

$$\| \vec{u} \|_k^2 = \| u_1 \|_k^2 + \| u_2 \|_k^2, \quad \text{for } \vec{u} = (u_1, u_2) \in H^k(\Omega)^2.$$

Finally, for functions $(\zeta, \vec{u}), (\xi, \vec{v}) \in L_*^2(\Omega) \times L^2(\Omega)^2$ we define the inner product

$$\langle (\zeta, \vec{u}), (\xi, \vec{v}) \rangle = \langle \zeta, \xi \rangle + \langle \vec{u}, \vec{v} \rangle$$

and norm

$$\| (\zeta, u) \|^2 = \langle (\zeta, \vec{u}), (\zeta, \vec{u}) \rangle,$$

and for functions $(\zeta, \vec{u}) \in H_*^1(\Omega) \times L^2(\Omega)^2$ we define the norm

$$\| (\zeta, u) \|_X^2 = \| \zeta \|_1^2 + \| \vec{u} \|^2.$$

If $\vec{u} \in L^2(\Omega)^2$, then there exist unique scalar functions $\phi \in H_*^1(\Omega)$ and $\psi \in H_c^1(\Omega)$ such that

$$(4.4) \quad \vec{u} = -\nabla\phi + R\nabla\psi.$$

The functions $\{\phi, \psi\}$ are known as Stokes-Helmholtz potentials for \vec{u} [8], [15], and we define the functions $S\vec{u} = \phi, H\vec{u} = \psi$. It follows from elliptic regularity [10] that for $k \geq 0$, there exists $c = c(k), c < \infty$, such that

$$(4.5) \quad \| S\vec{u} \|_{k+1} \leq c \| \vec{u} \|_k, \quad \| H\vec{u} \|_{k+1} \leq c \| \vec{u} \|_k, \quad \forall \vec{u} \in H^k(\Omega)^2.$$

It is easily verified that

$$(4.6) \quad \langle \nabla\phi, R\nabla\psi \rangle = 0, \quad \phi \in H_*^1(\Omega), \psi \in H_c^1(\Omega).$$

Hence, (4.4) is an orthogonal direct sum decomposition.

Let $\{\mathfrak{N}_h\}, 0 < h \leq 1$, be a family of finite-dimensional subspaces of $H^1(\Omega)$ parametrized by h . Set

$$(4.7) \quad \mathfrak{N}_h^c = \mathfrak{N}_h \cap H_c^1(\Omega),$$

$$(4.8) \quad \mathfrak{N}_h^* = \mathfrak{N}_h \cap H_*^1(\Omega).$$

We assume that there exists a positive constant c , independent of h , and a positive integer r such that, for $1 \leq k \leq r + 1$ and $w \in H_*^k(\Omega)$,

$$(4.9) \quad \inf_{\chi \in \mathfrak{N}_h^*} (\|w - \chi\| + h\|w - \chi\|_1) \leq ch^k \|w\|_k,$$

and, for $1 \leq k \leq r + 1$ and $w \in H_c^k(\Omega)$,

$$(4.10) \quad \inf_{\chi \in \mathfrak{N}_h^c} (\|w - \chi\| + h\|w - \chi\|_1) \leq ch^k \|w\|_k.$$

We also wish to assume that the spaces $\{\mathfrak{N}_h\}$ satisfy the ‘‘inverse property’’ that there exists a positive constant c , independent of h , such that

$$\|\chi\|_1 \leq ch^{-1} \|\chi\|_0, \quad \chi \in \mathfrak{N}_h.$$

We shall use the spaces of vector-valued functions

$$\mathfrak{U}_h = \{ \vec{U} \mid \vec{U} = -\nabla\Phi + R\nabla\Psi, \Phi \in \mathfrak{N}_h^*, \Psi \in \mathfrak{N}_h^c \}.$$

It follows from (4.5), (4.9), and (4.10) that, for $0 \leq k \leq r$, $\vec{u} \in H^k(\Omega)^2$,

$$(4.11) \quad \inf_{\vec{\chi} \in \mathfrak{U}_h} \|\vec{u} - \vec{\chi}\| \leq ch^k \|\vec{u}\|_k.$$

We shall need the result that if $\Psi \in \mathfrak{N}_h^c$, then

$$(4.12) \quad \nabla\Psi \in \mathfrak{U}_h.$$

It is easy to see that (4.12) is valid if \mathfrak{N}_h contains the constant functions.

In order to define our approximation procedure, we introduce the continuous, sesquilinear form $B(\cdot, \cdot)$ on $X \times X = [H_*^1(\Omega) \times L^2(\Omega)^2] \times [H_*^1(\Omega) \times L^2(\Omega)^2]$ by

$$(4.13) \quad B((\zeta, \vec{u}), (\xi, \vec{v})) = \langle \vec{u}, \nabla\xi \rangle - \langle \nabla\zeta, \vec{v} \rangle - \langle f\vec{u} + \omega R\vec{u}, \vec{v} \rangle, \\ (\zeta, \vec{u}), (\xi, \vec{v}) \in H_*^1(\Omega) \times L^2(\Omega)^2.$$

We regard T in (4.1) as an unbounded, closed operator, $T: H_*^1(\Omega) \times L^2(\Omega)^2 \rightarrow H_*^1(\Omega) \times L^2(\Omega)^2$, with domain

$$(4.14) \quad D(T) = \{ (\zeta, \vec{u}) \mid \zeta \in H_*^1(\Omega), \vec{u} \in L^2(\Omega)^2, \nabla \cdot \vec{u} \in H^1(\Omega), \vec{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega \}.$$

Thus, if $(\zeta, \vec{u}) \in D(T)$,

$$(4.15) \quad B((\zeta, \vec{u}), (\xi, \vec{v})) = \langle T(\zeta, \vec{u}), (\xi, \vec{v}) \rangle, \quad (\xi, \vec{v}) \in H_*^1(\Omega) \times L^2(\Omega)^2.$$

We now define $T_h: \mathfrak{N}_h^* \times \mathfrak{U}_h \rightarrow \mathfrak{N}_h^* \times \mathfrak{U}_h$ by

$$(4.16) \quad B((Z, \vec{U}), (Y, \vec{V})) = \langle T_h(Z, \vec{U}), (Y, \vec{V}) \rangle, \quad (Y, \vec{V}) \in \mathfrak{N}_h^* \times \mathfrak{U}_h.$$

The operator T_h is well defined since the spaces $\mathfrak{N}_h^* \times \mathfrak{U}_h$ are finite dimensional.

We now show that the spectral properties of T_h approximate those of T by verifying (P1) and (P2) in this case for $X = H_*^1(\Omega) \times L^2(\Omega)^2$ and $X_h = \mathfrak{N}_h^* \times \mathfrak{U}_h$. The verification of (P2) follows directly from (4.9) and (4.11). To see this, we note that for $(\zeta, \vec{u}) \in [L_*^2(\Omega) \cap C^\infty(\bar{\Omega})] \times C^\infty(\bar{\Omega})^2$ it follows from (4.9) and (4.11) that

$$\delta((\zeta, \vec{u}), X_h) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

The validity of (P2) now follows from the density of $[L_*^2(\Omega) \cap C^\infty(\bar{\Omega})] \times C^\infty(\bar{\Omega})^2$ in $H_*^1(\Omega) \times L^2(\Omega)^2$.

We next turn to the verification of (P1). In order to prove this result, we introduce and analyze some projection operators. First, we define the $H_*^1(\Omega)$

projection $P_h^*: H_*^1(\Omega) \rightarrow \mathfrak{N}_h^*$ by

$$(4.17) \quad \langle \nabla(P_h^* w - w), \nabla \chi \rangle = 0, \quad \chi \in \mathfrak{N}_h^*,$$

and the $H_c^1(\Omega)$ projection $P_h^c: H_c^1(\Omega) \rightarrow \mathfrak{N}_h^c$ by

$$(4.18) \quad \langle \nabla(P_h^c w - w), \nabla \chi \rangle = 0, \quad \chi \in \mathfrak{N}_h^c.$$

It is well known [2] that there exists $c < \infty$ such that, for $1 \leq k \leq r + 1$ and $w \in H_*^k(\Omega)$,

$$(4.19) \quad \|P_h^* w - w\|_1 \leq ch^{k-1} \|w\|_k$$

and, for $1 \leq k \leq r + 1$ and $w \in H_c^k(\Omega)$,

$$(4.20) \quad \|P_h^c w - w\|_1 \leq ch^{k-1} \|w\|_k.$$

It follows from (4.11) that if $\vec{Q}_h: L^2(\Omega)^2 \rightarrow \mathfrak{N}_h$ is the $L^2(\Omega)^2$ projection defined by

$$(4.21) \quad \langle \vec{Q}_h \vec{u} - \vec{u}, \vec{\chi} \rangle = 0, \quad \vec{\chi} \in \mathfrak{N}_h,$$

then there exists a constant $c < \infty$ such that $0 \leq k \leq r$ and $\vec{u} \in H^k(\Omega)^2$ imply that

$$(4.22) \quad \|\vec{Q}_h \vec{u} - \vec{u}\| \leq ch^k \|\vec{u}\|_k.$$

It is easily checked that

$$(4.23) \quad \vec{Q}_h \vec{u} = -\nabla P_h^* S \vec{u} + R \nabla P_h^c H \vec{u}.$$

Finally, define the $L^2(\Omega)$ projection $\hat{Q}_h: L_*^2(\Omega) \rightarrow \mathfrak{N}_h^*$ by

$$(4.24) \quad \langle \hat{Q}_h w - w, \chi \rangle = 0, \quad \chi \in \mathfrak{N}_h^*.$$

It follows from (4.9) and the inverse property of \mathfrak{N}_h that there exists $c < \infty$ such that if $1 \leq k \leq r + 1$ and $w \in H_*^k(\Omega)$, then

$$(4.25) \quad \|\hat{Q}_h w - w\|_1 \leq ch^{k-1} \|w\|_k.$$

The following lemmas will help us analyze T_h .

LEMMA 4.1. (a) Let $\vec{u} \in L^2(\Omega)^2$ and $\chi \in \mathfrak{N}_h^*$. Then

$$(4.26) \quad \langle \vec{u} - \vec{Q}_h \vec{u}, \nabla \chi \rangle = 0.$$

(b) Let $\zeta \in H_*^1(\Omega)$ and $\vec{\chi} \in \mathfrak{N}_h$. Then

$$(4.27) \quad \langle \nabla(\zeta - P_h^* \zeta), \vec{\chi} \rangle = 0.$$

Proof. The proof follows easily from (4.6) and the definitions of P_h^* and \vec{Q}_h .
Q.E.D.

Now let $\mathfrak{P}_h = P_h^* \oplus \vec{Q}_h: H_*^1(\Omega) \times L^2(\Omega)^2 \rightarrow \mathfrak{N}_h^* \times \mathfrak{N}_h$.

LEMMA 4.2. If $(\zeta, \vec{u}) \in D(T)$ and $(Y, \vec{V}) \in \mathfrak{N}_h^* \times \mathfrak{N}_h$, then

$$(4.28) \quad \begin{aligned} B((\zeta, \vec{u}) - \mathfrak{P}_h(\zeta, \vec{u}), (Y, \vec{V})) &= \langle T(\zeta, \vec{u}) - T_h \mathfrak{P}_h(\zeta, \vec{u}), (Y, \vec{V}) \rangle \\ &= -\langle \omega R(u - \vec{Q}_h \vec{u}), \vec{V} \rangle. \end{aligned}$$

Proof. The proof follows directly from the definition of B and Lemma 4.1.
Q.E.D.

Hence, it follows from Lemma 4.2 that if $Q_h = \hat{Q}_h \oplus \vec{Q}_h$, $(\zeta, \vec{u}) \in D(T)$, and $(Y, \vec{V}) \in \mathfrak{N}_h^* \times \mathfrak{N}_h$, then

$$(4.29) \quad \langle Q_h T(\zeta, \vec{u}) - T_h \mathfrak{P}_h(\zeta, \vec{u}), (Y, \vec{V}) \rangle = -\langle \omega R(\vec{u} - \vec{Q}_h \vec{u}), \vec{V} \rangle.$$

THEOREM 4.1. $\delta(T_h, T) \leq ch$.

Proof. Let $(Z, \vec{U}) \in \mathfrak{N}_h^* \times \mathfrak{N}_h$ and denote $T_h(Z, \vec{U})$ by (D, \vec{E}) . We must find $(\zeta, \vec{u}) \in D(T)$ such that

$$(4.30) \quad \begin{aligned} \|(Z, \vec{U}) - (\zeta, \vec{u})\|_X + \|T_h(Z, \vec{U}) - T(\zeta, \vec{u})\|_X \\ \leq ch(\|(Z, \vec{U})\|_X + \|T_h(Z, \vec{U})\|_X). \end{aligned}$$

We choose (ζ, \vec{u}) as follows. Let $\phi \in H_*^2(\Omega)$ be the solution to

$$(4.31) \quad \begin{cases} \Delta\phi = D, & x \in \Omega, \\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $\partial/\partial n$ is the exterior normal derivative. (Note that $\int_\Omega D \, dx = 0$ since $D \in L_*^2(\Omega)$.) Now set $\zeta = Z$, $\vec{u} = -\nabla\phi + R\nabla H\vec{U}$. It is clear that since $Z \in H_*^1(\Omega)$, $\nabla \cdot \vec{u} = -\Delta\phi = -D \in H^1(\Omega)$, $\vec{u} \cdot \vec{n} = -\partial\phi/\partial n = 0$ for $x \in \partial\Omega$ that $(\zeta, \vec{u}) \in D(T)$.

It follows by the definition of ϕ that $P_h^*\phi = S\vec{U}$. By elliptic regularity for (4.31),

$$\|\phi\|_2 \leq c\|D\| \leq c\|D\|_1.$$

Hence, we can conclude from (4.19) that

$$(4.32) \quad \|\nabla(S\vec{u} - S\vec{U})\| = \|\nabla(\phi - P_h^*\phi)\| \leq ch\|\phi\|_2 \leq ch\|D\|_1.$$

So,

$$(4.33) \quad \begin{aligned} \|(Z, \vec{U}) - (\zeta, \vec{u})\|_X &= \|\nabla(S\vec{u} - S\vec{U})\| \leq ch\|D\|_1 \\ &\leq ch\|T_h(Z, \vec{U})\|_X. \end{aligned}$$

Denote $T(\zeta, \vec{u})$ by (d, \vec{e}) . It follows from (4.31) that $d = D$. Since $\mathfrak{P}_h(\zeta, \vec{u}) = (Z, \vec{U})$, we obtain from (4.29)

$$(4.34) \quad \langle \vec{Q}_h \vec{e} - \vec{E}, \vec{V} \rangle = \langle \omega R(\nabla\phi - \nabla P_h^*\phi), \vec{V} \rangle.$$

Thus, if we set $\vec{V} = \vec{Q}_h \vec{e} - \vec{E}$ in (4.34) and use the Cauchy-Schwarz inequality, we can derive the estimate

$$(4.35) \quad \begin{aligned} \|\vec{Q}_h \vec{e} - \vec{E}\| &\leq c\|\nabla\phi - \nabla P_h^*\phi\| \leq ch\|\phi\|_2 \\ &\leq ch\|D\|_1 \leq ch\|T_h(Z, \vec{U})\|_X. \end{aligned}$$

We must now estimate

$$(4.36) \quad \|\vec{e} - \vec{Q}_h \vec{e}\| = \inf_{\vec{\chi} \in \mathfrak{N}_h} \|\vec{e} - \vec{\chi}\|.$$

By the definition of T ,

$$(4.37) \quad \vec{e} = -\nabla\zeta - f\vec{u} - \omega R\vec{u} = -\nabla Z - f(-\nabla\phi + R\nabla H\vec{U}) - \omega(-R\nabla\phi - \nabla H\vec{U}).$$

However,

$$(4.38) \quad -\nabla Z - fR\nabla H\vec{U} + \omega\nabla H\vec{U} \in \mathfrak{N}_h$$

and

$$(4.39) \quad \|f\nabla\phi + \omega R\nabla\phi\|_1 \leq c\|D\|_1.$$

So,

$$(4.40) \quad \begin{aligned} \|\vec{e} - \vec{Q}_h \vec{e}\| &= \inf_{\vec{\chi} \in \mathfrak{N}_h} \|f\nabla\phi + \omega R\nabla\phi - \vec{\chi}\| \leq ch\|f\nabla\phi + \omega R\nabla\phi\|_1 \\ &\leq ch\|D\|_1 \leq ch\|T_h(Z, \vec{U})\|_X. \end{aligned}$$

The triangle inequality, (4.35), and (4.40) yield

$$(4.41) \quad \|T(\zeta, \vec{u}) - T_h(Z, \vec{U})\|_X = \|\vec{e} - \vec{E}\| \leq ch \|T_h(Z, \vec{U})\|_X.$$

Thus, (4.30) follows from (4.33) and (4.41). Q.E.D.

Now let $\lambda \in \sigma(T)$ be an isolated eigenvalue of T with finite algebraic multiplicity m . Let $E(X)$ be the generalized eigenspace corresponding to λ . It follows from Theorem 1.3 that we are interested in an estimate for $\delta(T|_{E(X)}, T_h)$.

THEOREM 4.2. *Let $\lambda \in \sigma(T)$ be an isolated eigenvalue of T and let $E(X)$ be its associated generalized eigenspace. Assume $\dim E(X) = m < \infty$, and assume that $(\zeta, \vec{u}) \in E(X)$ implies that $\zeta \in H_*^{r+1}(\Omega)$, $\vec{u} \in H^r(\Omega)^2$, $\nabla \cdot \vec{u} \in H^r(\Omega)$. Then there exists $c < \infty$, independent of h , such that*

$$(4.42) \quad \delta(T|_{E(X)}, T_h) \leq ch^r.$$

Proof. Let $(\zeta, \vec{u}) \in E(X)$. Set $(Z, \vec{U}) = \mathcal{P}_h(\zeta, \vec{u})$. It follows from (4.19) and (4.22) that

$$(4.43) \quad \|(\zeta, \vec{u}) - \mathcal{P}_h(\zeta, \vec{u})\|_X \leq ch^r (\|\zeta\|_{r+1} + \|\vec{u}\|_r).$$

Also, setting $(Y, \vec{V}) = Q_h T(\zeta, \vec{u}) - T_h \mathcal{P}_h(\zeta, \vec{u})$ in (4.29) and using the Cauchy-Schwarz inequality, we obtain

$$(4.44) \quad \|Q_h T(\zeta, \vec{u}) - T_h \mathcal{P}_h(\zeta, \vec{u})\|_X = \|Q_h T(\zeta, \vec{u}) - T_h \mathcal{P}_h(\zeta, \vec{u})\| \leq c \|\vec{u} - \vec{Q}_h \vec{u}\| \leq ch^r \|\vec{u}\|_r.$$

Finally, from (4.22) and (4.24) we obtain

$$(4.45) \quad \|T(\zeta, \vec{u}) - Q_h T(\zeta, \vec{u})\|_X \leq ch^r (\|\zeta\|_{r+1} + \|\vec{u}\|_r + \|\nabla \cdot \vec{u}\|_r).$$

The result of the theorem now follows from (4.43), (4.44), (4.45), and the finite dimensionality of $E(X)$. Q.E.D.

Finally, we note that our results can be combined with the results in [3] to obtain optimal order estimates on the convergence of eigenvalues. Assume that the conditions of Theorem 4.2 are valid. Let $E(X^*)$ denote the generalized eigenspace associated with the eigenvalue $\bar{\lambda}$ of T^* . We note that it is well known that $\dim E(X^*) = m$ and that $\bar{\lambda}$ is an isolated eigenvalue.

Let

$$\hat{w}_h = \delta(E(X), E(X_h)), \quad \hat{w}_h^* = \delta(E(X^*), E(X_h^*)).$$

Let α be the ascent of λ . Then it follows from Proposition 3.2 of [3] that for h_0 sufficiently small

$$\max_{i=1, \dots, m} |\lambda - \lambda_{i,h}|^\alpha \leq c \hat{w}_h \hat{w}_h^*, \quad \left| \lambda - \frac{1}{m} \sum_{i=1}^m \lambda_{i,h} \right| \leq c \hat{w}_h \hat{w}_h^*, \quad h \leq h_0.$$

Under the hypotheses of Theorem 4.2 we have proven that $\hat{w}_h \leq ch^r$. Now assume, in addition, that $(\zeta, \vec{u}) \in E(X^*)$ implies that $\zeta \in H_*^{r+1}(\Omega)$, $\vec{u} \in H^r(\Omega)^2$, $\nabla \cdot \vec{u} \in H^r(\Omega)$. Then we can conclude, from applying the above arguments to the adjoint problem, that $\hat{w}_h^* \leq ch^r$. Hence, under the above conditions, we obtain the optimal order eigenvalue estimates

$$\max_{i=1, \dots, m} |\lambda - \lambda_{i,h}|^\alpha \leq ch^{2r}, \quad \left| \lambda - \frac{1}{m} \sum_{i=1}^m \lambda_{i,h} \right| \leq ch^{2r}, \quad h \leq h_0.$$

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