MATHEMATICS OF COMPUTATION
VOLUME 36, NUMBER 153
JANUARY 1981

Hyperperfect Numbers With Three Different Prime Factors

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Abstract. The existence of hyperperfect numbers with more than two different prime factors is shown by five examples.

Recently, Minoli [2] has defined n-hyperperfect numbers as positive integers m such that there is some positive integer n with

$$m = 1 + n[\sigma(m) - m - 1]. \tag{1}$$

1-hyperperfect numbers are the classical perfect numbers. Minoli gives a list of all n-hyperperfect numbers $< 1{,}500{,}000$ with $n > 1$, and these numbers have the form $p^\alpha q$, where p and q are prime numbers, $p < q$ and $\alpha \in \mathbf{N}$. Minoli wonders whether *all* hyperperfect numbers might have this form. By using a well-known technique, which was used, for instance, by Euler [1] to compute amicable number pairs, we have computed five hyperperfect numbers, each with three different prime factors.

Let $m = pqr$, $p < q < r$ prime numbers, be an n-hyperperfect number. By (1) we have

$$pqr = 1 + n(pq + pr + qr + p + q + r).$$

Now, if we assume that p and n are given, this is a quadratic equation in q and r. We write it as

$$(p - n)qr - n(p + 1)q - n(p + 1)r = 1 + np.$$

Multiplying by $(p - n)$, and adding $n^2(p + 1)^2$ to both sides yields

$$[(p - n)q - n(p + 1)][(p - n)r - n(p + 1)] = (p - n)(1 + np) + n^2(p + 1)^2.$$

If AB, $A < B$, is a factorization of the known right-hand side, then we can write

$$q = [n(p + 1) + A]/(p - n), \qquad r = [n(p + 1) + B]/(p - n).$$

If now both q and r are integers *and* prime, then pqr is an n-hyperperfect number. Clearly, a *small* value of $(p - n)$ will facilitate finding integers q and r. The simplest choice is $p - n = 1$; this gives

$$q = p^2 - 1 + A, \quad r = p^2 - 1 + B, \quad AB = p^4 - p^2 - p + 2, \quad A < B. \tag{2}$$

If $A = 1$, then $q = p^2$, not a prime. If $p \equiv 2 \pmod 3$, then $p^2 - 1 \equiv 0 \pmod 3$ and $AB = p^4 - p^2 - p + 2 \equiv 0 \pmod 3$, so that $3 \mid A$ or $3 \mid B$; hence, at least one of q

Received June 5, 1980.
1980 *Mathematics Subject Classification*. Primary 10A20.
Key words and phrases. Hyperperfect numbers.

0025-5718/81/0000-0029/$01.50

and r is composite. Excluding these cases we have checked (2) for all primes $p < 300$ (and $n = p - 1$), and all possible factorizations AB. We found the following five hyperperfect numbers, each with three different prime factors:

$$\begin{aligned}
1570153 &= 13 \cdot 269 \cdot 449 & n &= 12,\\
3675965445337 &= 229 \cdot 67187 \cdot 238919 & n &= 228,\\
8898807853477 &= 283 \cdot 112087 \cdot 280537 & n &= 282,\\
72315968283289 &= 277 \cdot 78541 \cdot 3323977 & n &= 276,\\
348231627849277 &= 223 \cdot 49807 \cdot 31352557 & n &= 222.
\end{aligned}$$

Remark. All n-hyperperfect numbers in Minoli's table with *odd* n are instances of the following rule: if both $p = 6k - 1$ and $q = 12k + 1$ are prime numbers for some $k \in \mathbf{N}$, then p^2q is an n-hyperperfect number with $n = 4k - 1$. We conjecture that there are infinitely many hyperperfect numbers.

Note added in proof. By generalizing the technique described in this note, we have constructed seven more hyperperfect numbers (for details, see [3]):

$$\begin{aligned}
601\ 10701 &= 7^2 \cdot 383 \cdot 3203 & n &= 6,\\
1\ 35441\ 68521 &= 13^2 \cdot 2347 \cdot 34147 & n &= 12,\\
899\ 21651\ 19733 &= 19^2 \cdot 6871 \cdot 3625243 & n &= 18,\\
21715\ 85816\ 00773 &= 43^2 \cdot 84319 \cdot 1392883 & n &= 42,\\
7972\ 29919\ 68160\ 43329 &= 97^2 \cdot 913571 \cdot 927465611 & n &= 96,\\
37\ 36320\ 97872\ 70370\ 68273 &= 73^3 \cdot 31293799 \cdot 306914431 & n &= 72,\\
1\ 60510\ 81329\ 59576\ 12416\ 00025\ 71981 &= 1327 \cdot 6793 \cdot 10020547039 \cdot 17769709449589 & &\\
& & n &= 1110.
\end{aligned}$$

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3. H. J. J. TE RIELE, *Hyperperfect Numbers With More Than Two Different Prime Factors*, Report NW 87/80, Mathematical Centre, Amsterdam, August 1980.