

A Mesh Refinement Method for $Ax = \lambda Bx$

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Abstract. The aim of this paper is to introduce a simple but efficient mesh refinement strategy for use with inverse iteration for finding one or a few solutions of an ordinary or partial differential eigenproblem of the form $Ax = \lambda Bx$. The focus is upon the case where A and B are symmetric and B is positive definite, although the approaches have a very broad application. A discussion of the combined use of mesh refinement and a correction scheme multigrid technique is also provided. The methods are illustrated by numerical results from experiments with two-point boundary value problems.

1. Introduction. There are presently several areas of activity surrounding the development of numerical techniques for solving the generalized eigenproblem

$$(1) \quad Ax = \lambda Bx,$$

where A and B are $n \times n$ symmetric matrices and B is positive definite. A brief account of part of this history is given in [11], with more recent results reported in [7] and [10]. See also [9] for the case $B = I$. Of special interest is the large sparse eigenproblem where n is very large and only a small percentage of the entries of A and B are nonzero. Of further interest are problems of this type where the pattern of the nonzero entries in both A and B prohibits direct triangular factorization. For such problems, linear equations involving A and B must generally be solved indirectly, that is, by iteration.

Most problems of this latter type result from discretizing an infinite-dimensional eigenproblem, most notably from that of a partial differential operator. This is normally a major source of numerical difficulty for methods that attempt to solve (1) since it almost guarantees that the *relative* separation of the eigenvalues is pathologically small. Thus, even though only one or a few of the extreme eigenvalues of (1) are usually needed in such cases, "factorization-free" iterative methods will often fail to provide a good approximation without performing an intolerable amount of computational work; cf., [4]. The limitation of these methods is to be expected, however, since they rely on model assumptions on (1) that do not take much account of the problem origin. In fact, few practical methods presently in use do. (See, however, [6] and [13].) Yet, taking this origin into account can lead to techniques that are remarkably efficient. A scheme of this type is the subject of this paper.

The essence of the method described in the following sections is a combination of the techniques of mesh refinement, inverse iteration, and multigrid. On each grid in the progressive mesh refinement process, an outer loop inverse iteration equation

Received July 16, 1979; revised March 13, 1980.

1980 *Mathematics Subject Classification.* Primary 65L15, 65N25; Secondary 65F15, 65R99.

*This work was supported by the Air Force Office of Scientific Research under AFOSR Grant #76-3019 and the National Science Foundation under grant #MCS78-03847.

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0025-5718/81/0000-0058/\$04.50

is posed which has the effect of linearizing the eigenvalue problem. This allows for the use of an inner loop correction scheme version of multigrid iteration [5], for example. Dramatic steps from coarse to finer grids are made with the approximate eigenvector(s) and corresponding eigenvalue(s) "passed" to the finer grid to act as an initial guess for *one* outer loop iteration. The process is terminated when the truncation error on the present finest grid is smaller than a prescribed tolerance.

The mesh refinement process follows directly from earlier work (i.e., [1], [2], [8]) with Newton's method applied to nonlinear two-point boundary value problems. In fact, the eigenvalue problem can be recast as a nonlinear problem for which Newton's method reduces with some modification to RQI, the Rayleigh quotient iteration (cf., [11]). Some of this earlier work has its parallel in application to (1), although the present work is distinguished, as we shall see, by a difficulty with the Rayleigh quotient and an advantage of cubic convergence of RQI.

A closely related approach, which has been considered quite generally in [6] and is presently being investigated from a practical standpoint by A. Brandt and others, is to apply FAS, the full approximation multigrid scheme (cf., [5]), directly to the eigenproblem by treating it as a nonlinear operator equation. Quite roughly speaking, this can be viewed as an application of an iterative method that has been linearized at each step for computability. Thus, the roles of inner and outer loop iteration are just the reverse of the approach of this paper, although the multigrid involvement somewhat confuses this interpretation. In any event, the FAS approach is more complex and, at least for well-behaved problems, is probably a little less efficient. It is also, probably, somewhat harder to direct to the proper eigenvalue and is less amenable to modifications such as simultaneous iteration; cf., [7]. On the other hand, for problems that may benefit from adaptive refinement, the FAS approach should prove to be more versatile.

After the preliminaries of Section 2, Section 3 contains a discussion of the mesh refinement process, which can be viewed as a scheme for defining RQI by moving progressively from coarse to finer grids. The suitability for solving the RQI defining equation by the correction scheme multigrid method, which is a grid cycling scheme, is discussed briefly in Section 4. In Section 5 we present a theoretical basis for a variational formulation of the mesh refinement. The last section illustrates the mesh refinement scheme as it applies numerically to two-point boundary value problems.

2. Assumptions and Notations. We assume in this and the next section that \mathcal{A} and \mathcal{B} are mappings from some suitable function space $\mathcal{C}_1(\Omega)$ into another $\mathcal{C}_2(\Omega)$, with $\mathcal{C}_1(\Omega) \subset \mathcal{C}_2(\Omega) \subset \mathcal{L}_2(\Omega)$. The function domain is $\Omega \subset R^d$, where d is a positive integer. For each h in a nonempty indexing set $\mathcal{U} \subset R^+$, with $n = n(h)$, we assume the existence of $n \times n$ matrices ${}_hA$ and ${}_hB$ that, together with \mathcal{A} and \mathcal{B} , satisfy the following:

A1. The spectrum of

$$(2) \quad \mathcal{A}x = \lambda \mathcal{B}x, \quad x \in \mathcal{C}_1(\Omega) - \{0\},$$

consists of a discrete set of eigenvalues ordered, counting multiplicities, according to

$$(3) \quad \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots \nearrow \infty.$$

(The case where λ_1 is multiple is discussed briefly in Section 3.)

A2. For each h in \mathcal{U} , ${}_hA$ and ${}_hB$ are symmetric and ${}_hB$ is positive definite. The eigenvalues of

$$(4_h) \quad {}_hAx = \lambda {}_hBx, \quad x \in R^n - \{0\},$$

are denoted by

$$(5_h) \quad {}_h\lambda_1 \leq {}_h\lambda_2 \leq \dots \leq {}_h\lambda_n.$$

For compatibility, on R^n let the B -inner product $\langle x, y \rangle_h$ and induced B -norm $\|x\|_h$ denote suitable discrete approximations to the B -norm and B -inner product in $\mathcal{L}_2(\Omega)$. (For example, if $\Omega = \prod_{i=1}^d [0, 1]$ and $h = n^{-1/d}$, we may choose $\langle x, y \rangle_h = h^d x^T {}_hBy$.) With this inner product, a ${}_hB$ -orthonormal set of eigenvectors for (4_h) is denoted by

$$(6_h) \quad \{ {}_hu_1, {}_hu_2, \dots, {}_hu_n \}.$$

A3. There exist $c_1, c_2, H \in R^+$, and a positive integer m such that

$$(7) \quad |{}_h\lambda_1 - \lambda_1| \leq c_1 h^m \quad \text{and} \quad |{}_h\lambda_2 - \lambda_2| \leq c_2 h^m$$

for all $h < H, h \in \mathcal{U}$.

A4. Let $h_1, h_2 \in \mathcal{U}$ and write $n_1 = n(h_1)$ and $n_2 = n(h_2)$. Denote by $I_{h_1}^{h_2}$ a mapping from R^{n_1} into R^{n_2} that preserves eigenvector approximations for (4_{h₁}) and (4_{h₂}). Specifically, we assume that for all $\epsilon > 0$ there exists an $H' = H'(\epsilon)$ so that, with any $x \in R^{n_1} - \{0\}$ such that $\|x\|_{h_1} = 1$ and with $y = I_{h_1}^{h_2}(x) / \|I_{h_1}^{h_2}(x)\|_{h_2}$, we have

$$(8) \quad \|y - {}_{h_2}u_1\|_{h_2} \leq \|x - {}_{h_1}u_1\|_{h_1} + \epsilon,$$

whenever $h_2 \leq h_1 \leq H', h_1, h_2 \in \mathcal{U}$. [Note that when (4_h) represents a variational discretization of (2) as in Section 5, then a natural embedding exists of each R^n into $\mathcal{L}_1(\Omega)$, for which $I_{h_1}^{h_2}$ is essentially the identity and (8) is satisfied by setting ϵ equal to the sum of the eigenvector truncation errors on grids h_1 and h_2 . For finite difference discretizations, (8) must also account for the interpolation error incurred in $I_{h_1}^{h_2}$. Since this can be made negligible when $I_{h_1}^{h_2}$ is chosen properly, we can roughly conclude that $\|y - {}_{h_2}u_1\|_{h_2} \sim \|x - {}_{h_1}u_1\|_{h_1}$.]

There are many examples of large, sparse eigenproblems that arise from a setting satisfying A1–A4 and where factorization of either A or B is undesirable. Often they are posed as partial differential eigenproblems such as those that arise in structural analysis (cf. [7]) and diffusion reaction modeling (cf. [4] and [13]). Note that all of the examples in Section 6 satisfy A1–A4.

The mesh refinement technique described in the next section applies to a much broader class of eigenvalue problems than the symmetric case treated here, including cases where the operators (differential or otherwise) are unsymmetric or nonlinear or where λ appears nonlinearly in (1). However, the method is discussed in the present simplified context because this simplicity highlights the essence of the technique and because many practical problems are of this type.

3. Mesh Refinement. It is natural to attempt to find the smallest eigenvalue of (2) by choosing a grid sequence $h_1 > h_2 > \dots > h_p$ in \mathcal{U} and performing some sort of iteration on each grid h_i initialized with $I_{h_{i-1}}^{h_i}$ applied to the accepted solution from grid $h_{i-1}, i = 2, 3, \dots, p$. However, this would be worth doing only in connection with an iterative process that converges at least superlinearly since the

essential difficulty with such methods is in obtaining good starting guesses. It is therefore natural to consider mesh refinement in connection with RQI which, ignoring the normalization step, is given by

$$(9) \quad ({}_h A - R(x^{(k)}) {}_h B)x^{(k+1)} = {}_h Bx^{(k)}, \quad k = 0, 1, 2, \dots,$$

where the Rayleigh quotient $R(x) = \langle x, {}_h Ax \rangle / \langle x, {}_h Bx \rangle$. (We postpone the discussion of how the linear equation in (9) is to be solved until Section 4.) This will usually fail to work because of the nature of $R(x)$, that is, even though an approximate eigenvector on a given grid may be fairly accurate, the Rayleigh quotient is biased well upward by the large eigenvalues present in (4_h) . The result is that, even for very modest jumps to finer grids, the approximation by (9) will usually shift from the smallest eigenvector to successively higher ones as the refinement progresses. The key in avoiding this unacceptable behavior is to pass from grid h_i to grid h_{i+1} not only the approximate eigenvector but the approximate eigenvalue as well. Except for this simple but critically important step, the mesh refinement strategy is similar to the approach in [8]. We therefore omit any background except to state that the bases for the refinement process include:

B1. for each i , compute an approximate smallest eigenvalue on grid h_i to within the truncation error $c_1 h_i^m$ of ${}_h \lambda_1$;

B2. attempt to design the refinement so that h_i/h_{i-1} is as small as possible (starting with $i = p$) yet insures that only one iteration in (9) is needed for each i ; and

B3. determine h_p so that the accepted approximation to ${}_h \lambda_1$ is within a prescribed tolerance ($\text{tol} > 0$) of λ_1 .

Simply stated, the preprocessing phase of the mesh refinement procedure begins by choosing initial coarse grids h_1 and $h_2 = h_1/2$ and examining the eigenvalues ${}_h \lambda_j$, $i, j = 1, 2$, in order to: approximate the truncation error expressions in (7); determine p and the final grid size h_p so that

$$c_1 h_p^m < \text{tol}$$

(as dictated by B1 and B3 above); choose as large an h_{p-1} as is possible so that h_{p-1}/h_p is a power of 2 and so that the Rayleigh quotient of the *first* RQI iterate is within $c_1 h_p^m$ of ${}_h \lambda_1$ (where RQI is initialized with the first grid h_{p-1} approximation interpolated to grid h_p); determine h_{p-2} in a similar manner with respect to h_{p-1} ; and proceed until h_2 satisfies these criteria with respect to h_3 . The execution phase of the mesh refinement procedure then begins by interpolating the grid h_2 approximation (to ${}_h u_1$) up to grid h_3 , to initialize *one* RQI, and continues by repeating this process until the grid h_p approximation (to ${}_h u_1$) is computed via *one* RQI.

There are many possible and advisable steps that should be taken in a fully implemented procedure that insure that the approximation is to the extreme eigenvalue, that the truncation error estimate is accurate, and that the final approximation is suitable, for example. We comment briefly on a few of them as they arise in the following description of the mesh refinement procedure ($m \geq 1$, $\gamma > 0$, and $\text{tol} > 0$ are assumed to be given):

Step 1. Choose $h_1 \in \mathcal{U}$ and let $h_2 = 2h_1$. Compute (to machine precision) ${}_h\lambda_j$ and ${}_h u_j$ and define the approximations to the truncation error coefficients in (7) by

$$\hat{c}_j = \gamma |{}_h\lambda_j - {}_{h_2}\lambda_j| h_1^{-m}, \quad i, j = 1, 2.$$

Comment 1. The quantity $\gamma > 0$ may be considered as a fudge factor to insure that $\hat{c}_j \geq c_j$ in (7). Note that for “monotonic” discretizations (for which, by definition, ${}_h\lambda_j$ converges monotonically to λ_j as $h \rightarrow 0^+$) we should choose $\gamma > 2^m / (2^m - 1)$.

The choice for h_1 should be made based upon some heuristics or by some specific knowledge of the problem (e.g., the number of zeros of u_1). With the constraint that ${}_h u_1$ is a sensible approximation to u_1 in this way, h_1 should be chosen so large as to guarantee that the computational effort expended in Step 1 is small compared to subsequent steps. (Note that it may be necessary to compute all of the eigenvalues of (4_{h_1}) and (4_{h_2}) to safely select the approximations for the two smallest. This will mandate a fairly large h_1 .)

For Step 1, we have assumed m is known, but c_1, c_2 are not. If c_1, c_2 are known, the computation on grid h_2 may be ignored. If m is known, a third grid h_3 may be used to “fit” the parameters m, c_1, c_2 in (7) to the “data” ${}_h\lambda_j, i = 1, 2, 3, j = 1, 2$.

Step 2. Perform a test to insure that the (computed) ${}_{h_2}\lambda_1$ is close enough to λ_1 to act as a shift for RQI, namely, that

$$\hat{c}_1 h_2^m < \frac{\lambda_2 - \lambda_1}{2}.$$

If h_2 is too large according to the test, reduce h_1 and repeat, beginning with Step 1.

Comment 2. A simple general test is

$$(3\hat{c}_1 + c_2)h_2^m < {}_{h_2}\lambda_2 - {}_{h_2}\lambda_1,$$

although knowledge of the problem setting (e.g., monotonicity as defined in Comment 1) may suggest a somewhat less stringent test.

Step 3. Let $p = 3$ and determine the final mesh size h_3 as the largest element of \mathcal{U} satisfying

$$2\hat{c}_1 h_p^m \leq \text{tol}.$$

(If $h_3 \geq h_2$, output ${}_{h_2}\lambda_1$ and ${}_{h_2}u_1$ and stop.)

Comment 3. The final value of p is the number of grids determined in the refinement process. The determination of h_p insures that an acceptable (see B1) approximation to ${}_h\lambda_1$ is within tol of λ_1 .

Step 4. Letting $i = 2$ and $q = c_1^{2/m} ({}_{h_2}\lambda_2 - {}_{h_2}\lambda_1)^{-2/m}$, determine whether or not

$$(10) \quad qh_i^3 < h_{i+1}.$$

If so, go to Step 6. Otherwise, go to Step 5.

Comment 4. We have made a few simplifying but reasonable assumptions here. For example, we assume that eigenvector truncation error is not significantly changed by interpolation. Moreover, we assume that \hat{c}_1, \hat{c}_2 bound c_1, c_2 and that the eigenvalue truncation error on grid $h_i, i > 2$, is negligible compared to $\lambda_2 - \lambda_1$. Fortunately, performance of the mesh refinement procedure is very insensitive to such assumptions and, only for badly behaved problems, need they be reconsidered.

Step 5. Rename each h_j , $j = 3, 4, \dots, p$, by calling it h_{j+1} (h_3 becomes h_4 , h_4 becomes h_5 , etc.) and increment p by one. Now, determine as large an h_3 as is possible so that h_3/h_4 is a power of 2 and so that (10) is satisfied with $i = 2$. Return to Step 4.

Comment 5. When the loop in Steps 4–5 is completed, we have determined a grid sequence $h_1, h_2, h_3, \dots, h_p$, so that (10) is satisfied for each $i = 1, 2, \dots, p - 1$, so that h_i/h_{i+1} is a power of 2 for each $i = 3, 4, \dots, p - 1$, and so that the “jump” from grid h_i to h_{i+1} is as large as possible, $i = p - 1, p - 2, \dots, 4$. Incidentally, because of the cubic convergence of RQI, often in practice $p = 3$. Note that $i = 2$ the first time through Step 6.

Step 6. Let $\lambda^{(i)}$ denote the Rayleigh quotient of the final approximation on grid h_i and let $x^{(0)}$ denote the result of applying $I_{h_i}^{h_{i+1}}$ to this approximate eigenvector. Perform one outerloop inverse iteration according to

$$(11) \quad \left({}_{h_{i+1}}A - \lambda_{h_{i+1}}^{(i)} B \right) x^{(1)} = {}_{h_{i+1}}B x^{(0)}.$$

(This is not strictly RQI, although we refer to it as such for our convenience.)

Comment 6. The computation of $x^{(1)}$ may be done either directly by a factorization scheme or indirectly by some iteration method; see Sections 4 and 6. Note that the mesh refinement process can thus be viewed as a means for linearizing the problem safely in the sense of providing an initial guess that requires only one linearization per grid.

Step 7. Replace $x^{(1)}$ by $x^{(1)}/\|x^{(1)}\|_{h_{i+1}}$. If $i < p$, increment i by one and go to Step 6. Otherwise, output $R(x^{(1)})$ and $x^{(1)}$ and stop.

Comment 7. Several tests may be required here to insure that the results are appropriate. For example, examination of the problem behavior may be observed on grid $2h_p$ to insure that the truncation error estimates are acceptable and that $x^{(1)}$ on grid h_p is sufficiently accurate. This is facilitated by using multigrid as the innerloop iteration but is expensive when other techniques are used to solve (11).

The assumption in (3), that λ_1 is simple, is not strictly necessary, although for multiple λ_1 we can expect that the discretization in effect splits λ_1 into two or more very close eigenvalues. Thus, the mesh refinement process will be confused in its determination of c_1 and c_2 , unless a precaution is taken to recognize the condition that $|\lambda_1 - \lambda_2|$ is tending to zero. An alternative is to implement simultaneous iteration by carrying several vectors at once in order to approximate several of the lowest eigenvalues in (3). The Ritz steps should then be performed after each application of (11), although the solution process in connection with (11) must now account for several vectors together. In terms of the multigrid iteration, this will result in a conservative algorithm by requiring that the smoothing scheme account for all of the vectors at once. In terms of the mesh refinement process, (7) must then be modified to account for the subspace truncation error estimates which must then be incorporated into the algorithm.

The mesh refinement strategy developed here is very efficient, but perhaps a greater attribute is the control that it provides in the solution process. Thus, as long as the target solution is even vaguely apparent on the coarsest grid, this strategy allows for a complete examination of the coarse grid problem to direct the procedure on the finer grid to the proper target. It is probably best then to think of

this approach as a means of providing an acceptable starting guess for the fine grid iteration, a perennial problem for fast converging processes such as RQI.

This mesh refinement strategy is independent of the way in which (11) is to be solved. Certainly for two-point boundary value problems, direct factorization is usually best. A numerical example in this connection is given in Section 6. But for more complex problems (e.g., $d \geq 2$), it is often more efficient to use a multigrid approach. It has been suggested [6] that multigrid methods for solving inverse iteration equations (first proposed in [13]) will be unacceptably slow because of the inherent numerical conditioning of the defining equations. However, multigrid methods are apparently insensitive to the condition of the linear system and in fact work remarkably well in application to (11), as we shall see in Section 6.

4. The Multigrid Process. There have been several recent and essentially unsuccessful attempts to develop algorithms for solving (1) based upon defining outer-loop RQI and using an innerloop iterative scheme like conjugate gradients. Although such methods may need suitable modification to account for possible indefiniteness of the RQI defining matrix, the major difficulty with these methods is rather the matrix condition. These methods fail uniformly for most eigenproblems, especially those stemming from differential operators. The potential for success with the multigrid process (cf. [5]) therefore lies in its insensitivity to the condition of the linear problem and its ability to solve most problems in $O(n(h))$ arithmetic operations.

The correction scheme [5] is a slightly more efficient technique than FAS and other multigrid approaches since it makes use of the linearity of the differential operator. Moreover, it is computationally advantageous for the multigrid process in general if the operator is also positive definite or, more precisely, if the operator approximation on the coarsest grid used in the grid cycling scheme is nonnegative definite. This insures that the relaxation iterations do not inordinately magnify the error components in the eigenvectors belonging to negative eigenvalues and thus eliminates the need to change to a different and, perhaps, direct solution process on the coarsest grid. These two conditions, namely, linearity and coarse grid nonnegative definiteness, can be guaranteed by proper choice of the shifts in (11).

Another important point is that the solution of (11) on grid h_{i+1} should proceed by first starting the multigrid iterations on grid $h_i/2$ using the approximation interpolated from grid h_i . The multigrid scheme would then progress in turn on grids $2^{-2}h_i$, $2^{-3}h_i$, and so forth, until grid h_{i+1} is reached. Unless the solution is acceptable at this point, the cycling would then begin with the use of grids h_i , $2^{-1}h_i$, $2^{-2}h_i$, \dots , h_{i+1} . This suggests that the residual equation be solved by direct means on grid h_i , provided it is not too fine. However, care should be taken here since the defining matrix on grid h_i is now numerically singular and will result in a correction that will dominate the approximation on grid h_{i+1} . One of several ways to avoid this behavior is to replace the shift by the Rayleigh quotient of the fine grid approximation, assuming that the relaxation sweeps have sufficiently damped out the higher eigenvalue contributions. In the experiments reported in Section 6, we succeeded by simply ignoring this problem. We were able to do this because the components that were magnified, because of the lack of positive definiteness, were very close to the eigenvectors being sought. For more complex problems, it is

suggested that the Rayleigh quotient be computed after a few relaxation sweeps on grid h_{i+1} , that it be used as the new RQI shift, and that the grid h_i correction be computed by a technique that does not require positive definiteness (e.g., by factorization).

5. Theory for Variational Methods. The purpose of this section is to develop a theoretical footing for the mesh refinement process developed in Section 3. Since the method relies on *estimates* of upper bounds for $|\lambda_{h_i} - \lambda_{h_{i+1}}|$, then, unless rigorous bounds are known from analysis, a complete theory would require assumptions on the existence of positive *lower* bounds for these quantities. Indeed, without proper safeguards, accidentally small values of these quantities on coarse grids could easily trip up the refinement process. This is one reason for possibly implementing a conservative fudge factor in (10) or using more than two initial grids to improve the estimate for c_1 . Nevertheless, we will assume c_1 has been properly estimated and focus our attention on motivating the mesh transition expression depicted in (10). For simplicity, we assume a variational formulation of the discretization in (4_h) with a setting somewhat more general than that described in Section 2.

Suppose $\mathcal{Q} : D(\mathcal{Q}) \rightarrow H$ and $\mathfrak{B} : D(\mathfrak{B}) \rightarrow H$ are selfadjoint linear operators with domains $D(\mathcal{Q}) \subset D(\mathfrak{B})$ that are dense subsets of the real separable Hilbert space H . Suppose also that \mathfrak{B} is positive definite on $D(\mathfrak{B})$. Assume that $S_1 \subset S_2 \subset \dots \subset D(\mathcal{Q})$ represents an infinite sequence of closed subspaces for which $\lim_{n \rightarrow \infty} S_n \equiv \cup_{n=1}^{\infty} S_n$ is dense in H . We assume, for definiteness, that $\dim S_n = n$. Let $\mathcal{P}_n : H \rightarrow S_n$ denote the canonical orthogonal projector of H onto S_n and write $A_n = \mathcal{P}_n \mathcal{Q} \mathcal{P}_n$ and $B_n = \mathcal{P}_n \mathfrak{B} \mathcal{P}_n$. Then the eigenvalues of

$$(12) \quad A_n x = \lambda B_n x$$

are real and may be written, counting multiplicities, as

$$(13) \quad \lambda_{1n} \leq \lambda_{2n} \leq \dots \leq \lambda_{nn}.$$

The corresponding B -unit eigenvectors are written as

$$(14) \quad u_{1n}, u_{2n}, \dots, u_{nn}.$$

(By B -unit vectors we mean vectors of unit length in the inner-product $\langle x, y \rangle_{\mathfrak{B}} \equiv \langle \mathfrak{B}x, y \rangle$, $x, y \in D(\mathfrak{B})$. The corresponding \mathfrak{B} -norm is given by $\|x\|_{\mathfrak{B}} = \langle \mathfrak{B}x, x \rangle^{1/2}$.)

Suppose $\mathcal{Q}x = \lambda \mathfrak{B}x$ has a complete set of \mathfrak{B} -orthonormal eigenvectors $\{u_1, u_2, \dots\}$ that correspond to the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$ (assuming, for simplicity, that the spectrum is bounded from below). Define

$$\epsilon_n = \lambda_{1n} - \lambda_1$$

for $1 \leq i \leq n$ and note that our variational setting implies that ϵ_n is a nonincreasing sequence of reals bounded from below by zero. We, naturally, assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and, further, that λ_1 is simple. We also require $\lim_{n \rightarrow \infty} \lambda_{2n} = \lambda_2$. Let $m = \lambda_2 - \lambda_1$, $m_n = \lambda_{2n} - \lambda_{1n}$, and $R(x) = \langle \mathcal{Q}x, x \rangle / \langle \mathfrak{B}x, x \rangle$, $x \in D(\mathfrak{B})$. In what follows, note that $R(x) = \langle \mathcal{Q}_n x, x \rangle / \langle \mathfrak{B}_n x, x \rangle$, whenever $x \in S_n$.

The first lemma provides an estimate for the error in $x \in S_n$ as an approximation to u_{1n} in terms of the error $R(x) - \lambda_{1n}$.

LEMMA 1. Let $n \geq 1$ and $x \in S_n$ such that $\varepsilon \equiv R(x) - \lambda_{1n} < m_n/2$. Then $\langle x, u_{1n} \rangle_{B_n} \neq 0$ and

$$(15) \quad \left\| \frac{x}{\langle x, u_{1n} \rangle_{B_n}} - u_{1n} \right\|_{B_n}^2 \leq \frac{\varepsilon}{m_n - \varepsilon}.$$

Proof. Write $x = \langle x, u_{1n} \rangle_{B_n}(u + \delta h)$, where $h \in S_n$, $\langle B_n u_{1n}, h \rangle = 0$, and $\langle B_n h, h \rangle = 1$. Then δ^2 is the left side of the inequality in (15). This inequality now follows from the observation

$$\begin{aligned} \varepsilon = R(x) - \lambda_{1n} &= \frac{\lambda_{1n} + \delta^2 \langle A_n h, h \rangle}{1 + \delta^2} - \lambda_{1n} \\ &= \frac{\delta^2}{1 + \delta^2} \langle (A_n - \lambda_{1n} B_n) h, h \rangle \geq \frac{m_n \delta^2}{1 + \delta^2}. \quad \square \end{aligned}$$

The next lemma yields upper bounds for the error after one iteration of (11) performed on S_n .

LEMMA 2. Let λ satisfy $0 < \lambda - \lambda_{1n} < m_n/2$. Given x in S_n such that $\langle x, u_{1n} \rangle_{B_n} \neq 0$, let $y \in S_n$ be the solution to

$$(16) \quad (A_n - \lambda B_n)y = B_n x.$$

We then have

$$(17) \quad R(y) - \lambda_{1n} \leq \frac{\delta^2 (\lambda - \lambda_{1n})^2 m_n}{(\lambda_{2n} - \lambda)^2 + \delta^2 (\lambda - \lambda_{1n})^2},$$

where $\delta = \|x / \langle x, u_{1n} \rangle_{B_n} - u_{1n}\|_{B_n}$.

Proof. We first assume $\mathfrak{B} = \mathfrak{G}$, the identity on H , so that $B_n = \mathfrak{P}_n$ and $B_n/S_n = \mathfrak{G}/S_n$. Assuming, without loss of generality, that $\lambda = 0$ so that $\lambda_{1n} < 0 < \lambda_{2n}$, then

$$R(y) - \lambda_{1n} = \frac{\langle A_n y, y \rangle}{\langle y, y \rangle} - \lambda_{1n} = \frac{\langle A_n A_n^+ x, A_n^+ x \rangle}{\langle A_n^+ x, A_n^+ x \rangle} - \lambda_{1n},$$

where A_n^+ is the inverse of A_n/S_n . Hence, writing $x = \langle x, u_{1n} \rangle_{B_n}(u_{1n} + \delta h)$, as in the proof of Lemma 1, we have

$$(18) \quad R(y) - \lambda_{1n} = \frac{\delta^2 \lambda_{1n}^2 \langle A_n^+ h, h \rangle - \lambda_{1n}^3 \delta^2 \langle A_n^+ h, A_n^+ h \rangle}{\langle h, h \rangle + \delta^2 \lambda_{1n}^2 \langle A_n^+ h, A_n^+ h \rangle}.$$

The maximum of this quantity over unit $h \in \{u_{1n}\}^\perp \cap S_n$ occurs as a zero of the gradient of the right-hand side of (18), considered as a function of this h . But this implies that h is the zero of a second-degree polynomial in A^+ . Hence, h must be a linear combination of at most two eigenvectors of A^+ and, therefore, A . Writing $h = \alpha_i u_{in} + \alpha_j u_{jn}$, where $\alpha_i^2 + \alpha_j^2 = 1$, $i, j \geq 2$, and setting $\alpha = \alpha_i^2$, we have

$$R(y) - \lambda_{1n} \leq \frac{\delta^2 \lambda_{1n}^2 [(\lambda_{in}^{-1} - \lambda_{jn}^{-1})\alpha + \lambda_{jn}^{-1} - \lambda_{1n}(\lambda_{in}^{-2} - \lambda_{jn}^{-2})\alpha - \lambda_{1n} \lambda_{jn}^{-2}]}{1 + \delta^2 \lambda_{1n}^2 [(\lambda_{in}^{-2} - \lambda_{jn}^{-2})\alpha + \lambda_{jn}^{-2}]}$$

for some $i, j \geq 2$ and $0 \leq \alpha \leq 1$. We must now maximize this expression over i, j , and α , which we first do for α . Noting that the right-hand side is simply a quotient

of two linear terms in α , the maximum over α must occur either at $\alpha = 0$ or at $\alpha = 1$. Thus,

$$(19) \quad R(y) - \lambda_{1n} \leq \max_{2 \leq i \leq n} \frac{\delta^2 \lambda_{1n}^2 (\lambda_{in} - \lambda_{1n})}{\lambda_{in}^2 + \delta^2 \lambda_{1n}^2}.$$

If we view λ_i as an arbitrary real in the interval $[\lambda_{2n}, \lambda_{nn}]$, then the maximum in (19) is attained at either λ_{nn} , λ_{2n} , or $\lambda_{1n} \pm \sqrt{\lambda_{in}^2 - \delta^2}$. The latter are the zeros of the derivative of the right-hand side, considered as a function of λ_i . But $\lambda_{1n} \pm \sqrt{\lambda_{in}^2 - \delta^2} < \lambda_{2n}$, so the maximum in (19) is attained at $i = 2$ and (17) now follows.

The case for general \mathfrak{B} rests on establishing (17) by performing a Choleski factorization of B_n/S_n and transforming (12) into a symmetric eigenproblem with B_n/S_n becoming the identity. This is straightforward and is therefore omitted. \square

THEOREM 1. *Let N be given so that*

$$\epsilon_k < m_l/2$$

for all $l, k \geq N$. Suppose, for a given $k \geq N$, that an $x \in S_k$ has been determined so that

$$|\lambda_{1k} - R(x)| \leq \epsilon_k.$$

Let $l \geq k$ satisfy the transition condition

$$(20) \quad \frac{(2\epsilon_k)^3 m_l}{(m_l - 2\epsilon_k)^3 + (2\epsilon_k)^3} \leq \epsilon_l.$$

Then y , determined by (16) with $n = k$ and $\lambda = R(x)$, satisfies

$$|\lambda_{1l} - R(y)| \leq \epsilon_l.$$

That is, one iteration is sufficient for convergence to λ_{1l} to within the truncation error ϵ_l .

Proof. With $\lambda = R(x)$, note that

$$0 \leq \lambda - \lambda_{1l} \leq \lambda - \lambda_1 \leq \lambda - \lambda_{1k} + \lambda_{1k} - \lambda_1 \leq 2\epsilon_k.$$

Hence, by Lemma 1, viewing x as an element of δ_l yields

$$\left\| \frac{x}{\langle x, u_{1l} \rangle_{B_l}} - x_{1l} \right\|_{B_l}^2 \leq \frac{2\epsilon_k}{m_l - 2\epsilon_k}.$$

Hence, by Lemma 2,

$$R(y) - \lambda_l \leq \frac{\left(\frac{2\epsilon_k}{m_l - 2\epsilon_k}\right)(2\epsilon_k)^2 m_l}{(m_l - 2\epsilon_k)^2 + \left(\frac{2\epsilon_k}{m_l - 2\epsilon_k}\right)(2\epsilon_k)^2} \leq \frac{(2\epsilon_k)^3 m_l}{(m_l - 2\epsilon_k)^3 + (2\epsilon_k)^3}.$$

The theorem follows from this and (20). \square

Remark 1. For sufficiently large N , the expression in (20) is essentially the same as

$$\frac{(2\epsilon_k)^3}{m_l^2} \leq \epsilon_l.$$

With $\epsilon_k = c_1 h_k^m$, this is just the mesh refinement transition condition depicted in (10).

Remark 2. The theory of this section can be applied to $\lambda_i, i > 1$, with a few obvious modifications.

Remark 3. There are several special contexts to which the general assumptions of this section apply, such as eigenproblems for integral operators (cf. [3]) and differential operators (cf. [6]).

6. Numerical Results. In this section, we report on the results of experiments with the methods presented here applied to four specific examples: two two-point boundary value eigenproblems ($d = 1$) and two partial differential eigenproblems ($d = 2$). Note that $B = I$ in each case. Since the performance of these methods is invariant under similarity transformations, it is sufficient, from a numerical point of view, to test these problems with $B = I$. That is, a Choleski decomposition of ${}_h B = {}_h L {}_h L^T$ can be used to transform the case in which ${}_h B$ is not the identity to one in which it is. The numerical performance of the methods of this paper is invariant under such a transformation. Note that this viewpoint is in the same spirit as that in using only diagonal matrices (cf. [12]) to test the Lanczos process. The first two examples are the following two-point boundary value eigenproblems:

E1. (*Simple Harmonic Motion Equation*).

$$\begin{aligned}
 -x'' &= \lambda x, \\
 x(0) &= x(1) = 0, \\
 \int_0^1 x^2(t) dt &= 1, \\
 (\text{actual } \lambda_1 &= \pi^2 \doteq 9.8969044).
 \end{aligned}$$

E2. (*Mathieu's Equation*).

$$\begin{aligned}
 -x'' + 20\pi^2 \cos(2\pi t)x &= \lambda \pi^2 x, \\
 x(0) &= x(1) = 0, \\
 \int_0^1 x^2(t) dt &= 1, \\
 (\text{actual } \lambda_1 &\doteq -13.9365525).
 \end{aligned}$$

To, in effect, ignore the special properties of these problems, we have implemented them in a rather naive way. For example, we discretized both by standard central differences on a uniform grid on $[0, 1]$ (so that $m = 2$ in (7)), we made no use of any accelerating techniques such as extrapolation, and we restricted $I_{h_1}^{h_2}$, $h_1 > h_2$, to simple linear interpolation. We also focused on the computation of λ_1 , although higher eigenvalues may be computed by these methods. All tests were done on a CYBER 172.

The mesh refinement process was first implemented by solving for each RQI iterate on each grid directly by elimination. Table 1 depicts ${}_h \hat{\lambda}_1$ and ${}_{h_2} \hat{\lambda}_1$ (which are the computed values of ${}_h \lambda_1$ and ${}_{h_2} \lambda_1$, respectively, with $h_1 = 2^{-4}$ and $h_2 = 2^{-5}$), the estimate $\hat{c}_1 = 16^2 |{}_h \hat{\lambda}_1 - {}_{h_2} \hat{\lambda}_1|$, the predicted acceptable mesh size h_3 for which $|{}_h \lambda_1 - \lambda_1| < \text{tol}/2$ (here we use $\text{tol} = 2^{-18}$), the approximation ${}_{h_3} \hat{\lambda}_1$ computed via one RQI on grid h_3 , and $r = |{}_h \hat{\lambda}_1 - \lambda_1| \text{tol}^{-1}$. The erroneous digits in the

eigenvalue approximations are underlined. Note that success is indicated by the condition that $r < 1$.

This choice of $\text{tol} (\doteq 3.8 \times 10^{-6})$ was made for convenience in our tests. Note that the relative error in each λ_1 is therefore less than 5×10^{-7} .

Convergence on the fine grid was achieved, as expected, for each problem by the first RQI iterate. In our experiments, we were actually able to do much better than is indicated. In fact, we were able to jump from grid $h_1 = 2^{-2}$ to $h_2 = 2^{-14}$ for both problems without incurring the need for additional RQI steps. This is not surprising since the analysis of RQI is naturally pessimistic, but it can be fortunate since it will tend to counteract the errors in determining an acceptable value for c_1 .

TABLE 1. *Mesh Refinement / Elimination*

Example	$h_1 \hat{\lambda}_1$	$h_2 \hat{\lambda}_1$	\hat{c}_1	h_3	$h_3 \hat{\lambda}_1$	$r = h_3 \hat{\lambda}_1 - \lambda_1 \text{tol}^{-1}$
E1	<u>9.838</u>	<u>9.8617</u>	6	2^{-11}	<u>9.8696026</u>	.47
E2	<u>-14.022</u>	<u>-13.9575</u>	16	2^{-12}	<u>-13.9365538</u>	.33

TABLE 2. *Mesh Refinement / Multigrid*
(with coarse grid elimination)

Example	No. of Cycles	$h_3 \hat{\lambda}_1$	$r = h_3 \hat{\lambda}_1 - \lambda_1 \text{tol}^{-1}$
E1	0	<u>9.8696027</u>	.45
E2	0	<u>-13.9365482</u>	1.13
	1	<u>-13.9365496</u>	.76

The next set of experiments involved the same technique as before, except that the problem on grid h_3 was solved by a fixed correction scheme multigrid method [5]. Thus, the only difference between these and the first set of tests was that the direct process of solving for the RQI iterate on grid h_3 was replaced by multigrid cycles initiated on grid $h_2/2$. (The multigrid scheme was of fixed type in that the cycling was predetermined to move progressively from grid h_2 to h_3 and back. In progressing to grid h_3 , three Gauss-Seidel relaxation sweeps were used while one was used in the return to grid h_2 . Although this is a fairly primitive multigrid scheme, it serves to illustrate the performance of methods of this type.) Each time the multigrid iteration arrived at grid h_3 , the Rayleigh quotient of the iterate was computed and compared to the true λ_1 for test purposes to determine performance. (Of course, a computable convergence criteria is necessary in practice. A safe test is to compute the norm of the gradient of $R(x)$ which can be used to bound the error in x .)

Table 2 depicts the results of tests on both equations. The cycle count was initially set to zero when the multigrid iteration first arrived at grid h_3 and was

incremented by one each time it returned from coarse grid cycling. Note the remarkable success of this scheme especially for equation E1. In fact, this suggests that combined mesh refinement multigrid iteration will solve most eigenvalue problems to which it applies in an equivalent of just a few fine grid matrix multiplies, regardless of whether or not $\mathfrak{B} = \mathfrak{G}$. This is underscored by the fact that a *relatively* major expense in producing an acceptable eigenvalue by this approach is in the computation of the Rayleigh quotient of the fine grid approximation.

The last two examples are two-dimensional partial differential boundary value eigenproblems:

E3. (*Helmholtz Equation in the Unit Circle*).

$$\begin{aligned}
 -\Delta x &= \lambda x & \text{in } \Omega &= \{(s, t): s^2 + t^2 < 1\}, \\
 x &= 0 & \text{on } \partial\Omega &= \{(s, t): s^2 + t^2 = 1\}, \\
 \int \int_{\Omega} x(s, t) \, ds \, dt &= 1, \\
 (\text{actual } \lambda_1 &\doteq 5.78306).
 \end{aligned}$$

E4. (*Helmholtz Equation in an Egg-Shaped Region*).

$$\begin{aligned}
 -\Delta x &= \lambda x & \text{in } \Omega &= \{(s, t): s^2 + t^2 < 1, t \leq 0\} \cup \{(s, t): 4s^2 + t^2 < 4, t \geq 0\}, \\
 x &= 0 & \text{on } \partial\Omega &= \{(s, t): s^2 + t^2 = 1, t \leq 0\} \cup \{(s, t): 4s^2 + t^2 = 4, t \geq 0\}, \\
 \int \int_{\Omega} x(s, t) \, ds \, dt &= 1, \\
 (\text{actual } \lambda_1 &\doteq 4.2080).
 \end{aligned}$$

The tests with these examples were with multigrid only. The tol values were chosen to reflect different criteria of accuracy so that the eigenproblem for E4 is larger and produces greater accuracy. In each case, $h_1 = 2^{-1}$ so that $h_2 = 2^{-2}$. Note that the performance, as expected, is similar to that of the method applied to the one-dimensional examples E1 and E2.

These examples are typical of the behavior we have observed in experiments with the mesh refinement/multigrid process. In all of the tests we have run, the eigenvalue approximation was acceptable after one multigrid cycle from the finest grid. In most cases, it was acceptable even before the cycle.

TABLE 3. *Mesh Refinement / Multigrid*
(with coarse grid elimination)

Example	$h_1 \hat{\lambda}_1$	$h_2 \hat{\lambda}_1$	\hat{c}_1	tol	h_3	No. of Cycles	$h_3 \hat{\lambda}_1$	$r = h_3 \hat{\lambda}_1 - \lambda_1 ^{\text{tol}^{-1}}$
E3	<u>5.439</u>	<u>5.681</u>	1.3	10^{-2}	2^{-4}	0	<u>-5.778</u>	.5
E4	<u>4.0162</u>	<u>4.1524</u>	.73	10^{-3}	2^{-5}	0	<u>-4.2071</u>	.9

Acknowledgements. The author wishes to thank Achi Brandt, Paddy Prenter, and John Locker for several helpful discussions, Steve Schaffer and John Ruge for the

numerical implementation and tests reported in the last section, and the referee for suggesting several improvements in the exposition.

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