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Calculation of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Function sn(x, k)

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Abstract. The Taylor series expansion coefficients of the Jacobian elliptic function $\mathrm{sn}(x, k)$ and its power $\mathrm{sn}^2(x, k)$ are studied. Recurrence formulae are given, and tables of the coefficients constructed. Using Lagrange's inversion formula, these coefficients can be expressed in terms of Legendre polynomials.

Introduction. Not much is known about the Taylor series expansion coefficients of the Jacobian elliptic functions $\mathrm{sn}(x, k)$, $\mathrm{cn}(x, k)$, and $\mathrm{dn}(x, k)$. In handbooks only the first four or five terms are given. (See for instance Abramowitz and Stegun [1, p. 575], Hancock's book on elliptic functions [4, p. 252 and p. 486], or Gradshteyn and Ryshik [5].) Recently, however, Alois Schett gave a "combinatorial" expression of the coefficients of the Taylor series expansion of $\mathrm{sn}(x, k)$ and evaluated explicitly the first eight, nontrivial, terms; see [6, p. 146]. His results were extended by Dominique Dumont [3], who gave a new combinatorial interpretation of the coefficients of $\mathrm{sn}(x, k)$. It is often said that no recurrence formulae exist for these coefficients ([1, p. 575] or [6, p. 143]). What is meant by that is probably that no simple recurrence formulae exist. Using differential and other equations, one can deduce a number of useful recurrence relations and, with the help of the Bürmann-Lagrange theorem, even a definite (although hard to handle) expression of the coefficients in question.

1. Derivation by Means of the Bürmann-Lagrange Theorem. The function $\mathrm{sn}(x, k)$ may be defined by (Bowman [2, p. 8])

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{dt}{((1 - t^2)(1 - k^2t^2))^{1/2}}, \qquad |k| \leqslant 1, -K < x < K, \tag{1.1}$$

where $K(k)$ is the complete elliptic integral of the first kind. Using an algebraic addition formula, one can define $\mathrm{sn}(x, k)$ for all real x and, with the help of imaginary transformations, for all complex x. Expanding the integrand in (1.1) in a Taylor series, one obtains

$$x = \sum_{n=0}^{\infty} C_{2n}(k) \frac{\mathrm{sn}^{2n+1}(x, k)}{2n + 1}, \tag{1.2}$$

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where

(1.3) $$C_{2n}(k) = k^n P_n\left(\frac{1}{2}\left(k + \frac{1}{k}\right)\right) = \frac{1}{4^n}\sum_{v=0}^{n}\binom{2v}{v}\binom{2n-2v}{n-v}k^{2v},$$

and $P_n(t)$ is the nth Legendre polynomial.

(1.2) may be used for computational purposes, at least for small k, because the polynomial $C_{2n}(k)$ satisfies a three term recurrence formula, viz.

(1.4) $$(2n+2)C_{2n+2}(k) - (1+k^2)(2n+1)C_{2n}(k) + 2nk^2C_{2n-2}(k) = 0,$$

with starting values $C_0(k) = 1$, $C_2(k) = \frac{1}{2}(1+k^2)$. (See Table I.)

Using Bürmann-Lagrange's inversion formula [1, art. 3.6.6], we may formulate the following theorem:

THEOREM 1. *The Taylor series expansion of* $\mathrm{sn}(x, k)$ *is*

$$\mathrm{sn}(x,k) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\sigma_{2n+1}(1,k),$$

where

$$\sigma_{2n+1}(1,k) = \left[\frac{d^{2n}}{da^{2n}}\left(\frac{1}{P(a)}\right)^{2n+1}\right]_{a=0},$$

and

$$P(a) = \sum_{v=0}^{\infty} C_{2v}(k)\frac{a^{2v}}{2v+1}.$$

To further exploit this theorem, we must know how to compute the nth derivative of a compound function. A useful formula is given in Gradshteyn and Ryshik [5]. We thus get

(1.5) $$\frac{d^{2n}}{da^{2n}}\frac{1}{(P(a))^{2n+1}} = \sum_{\substack{i_1+2i_2+\cdots+vi_v=2n \\ i_1+i_2+\cdots+i_v=m,\ m=1,2,\ldots,2n}} \frac{(-1)^m(2n+m)!}{P^{2n+m+1}(a)} \cdot \prod_{s=1}^{v}\left[\left(\frac{P^{(s)}(a)}{s!}\right)^{i_s}\frac{1}{i_s!}\right].$$

Applying (1.5) to the case $n = 2$, we get

$$\frac{d^4}{da^4}\frac{1}{P^5(a)} = -5\frac{P^{(4)}(a)}{P^6(a)} + 120\frac{P^{(3)}(a)\dot{P}(a)}{P^7(a)} + 90\frac{\ddot{P}^2(a)}{P^7(a)} -1260\frac{\ddot{P}(a)\dot{P}^2(a)}{P^8(a)} + 1680\frac{\dot{P}^4(a)}{P^9(a)}.$$

Since we are mainly interested in the case $a = 0$, we may formulate

THEOREM 2. *The coefficients* $\sigma_{2n+1}(1, k)$, *occurring in the Taylor series expansion of* $\mathrm{sn}(x, k)$, *are given by*

$$\sigma_{2n+1}(1,k) = \sum_{\substack{2i_2+4i_4+\cdots+2vi_{2v}=2n \\ i_2+i_4+\cdots+i_{2v}=m,\ m=1,2,\ldots,n}} (-1)^m(2n+m)!k^n \cdot \prod_{s=1}^{v}\frac{1}{i_{2s}!}\left(\frac{1}{2s+1}P_s\left(\frac{1}{2}\left(k+\frac{1}{k}\right)\right)\right)^{i_{2s}},$$

where $P_n(t)$ *is the nth Legendre polynomial.*

Proceeding in the same way, we can prove

THEOREM 3. *The coefficients* $\sigma_{2n+2}(2, k)$, *occurring in the Taylor series expansion of* $\operatorname{sn}^2(x, k)$, *are given by*

$$\sigma_{2n+2}(2, k) = 2 \sum_{\substack{2i_2+4i_4+\cdots+2vi_{2v}=2n \\ i_2+i_4+\cdots+i_{2v}=m,\ m=1,2,\ldots,n}} (-1)^m (2n+m+1)!\,k^n \cdot \prod_{s=1}^{v} \frac{1}{i_{2s}!}\left(\frac{1}{2s+1} P_s\left(\frac{1}{2}\left(k+\frac{1}{k}\right)\right)\right)^{i_{2s}}.$$

The use of Faà di Bruno's theorem to compute the coefficients of an inverted series is awkward, at best, and becomes prohibitively complex after a relatively few terms. The tenth one already contains 42 distinct terms. Therefore Theorems 2 and 3 are only of theoretical interest. A more practical approach is to note that $[(2n)!]^{-1}\sigma_{2n+1}(1, k)$ is simply the coefficient of x^{2n} in the power series expansion of $(P(x))^{-(2n+1)}$. The coefficients b_i of any power α of the series

$$f(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

can be computed recursively by

$$b_i = \alpha a_i + \frac{1}{i} \sum_{k=1}^{i-1} (k(\alpha+1) - i) a_k b_{i-k}, \tag{1.6}$$

which seems to be a computationally more tractable approach than the use of the multinomial theorem, even though the latter gives an explicit expression.

2. Recurrence Formulae for the Taylor Series Expansion Coefficients of $\operatorname{sn}^m(x, k)$, **Especially** $m = 1$ **and** $m = 2$, **Used to Calculate the Coefficients.** Using elementary properties of $\operatorname{sn}(x, k)$, $\operatorname{cn}(x, k)$, and $\operatorname{dn}(x, k)$, we may prove

$$\frac{d^2}{dx^2}\operatorname{sn}^n(x, k) = n(n-1)\operatorname{sn}^{n-2}(x, k) - n^2(1+k^2)\operatorname{sn}^n(x, k) + n(n+1)k^2 \operatorname{sn}^{n+2}(x, k). \tag{2.1}$$

(See Whittaker and Watson [7, p. 516].) We note the special cases $n = 1$ and $n = 2$ (Bowman [2, p.11]):

$$\frac{d^2}{dx^2}\operatorname{sn}(x, k) = -(1+k^2)\operatorname{sn}(x, k) + 2k^2\operatorname{sn}^3(x, k), \tag{2.2}$$

$$\frac{d^2}{dx^2}\operatorname{sn}^2(x, k) = 2 - 4(1+k^2)\operatorname{sn}^2(x, k) + 6k^2\operatorname{sn}^4(x, k). \tag{2.3}$$

(2.1) provides an alternative means of proving (1.4) by differentiating (1.2) twice with respect to x.

We now define the coefficients $\sigma_n(m, k)$ as follows $(m = 1, 2, \ldots)$

$$\operatorname{sn}^m(x, k) = \sum_{n=m}^{\infty} \frac{\sigma_n(m, k)}{n!} x^n; \qquad \sigma_m(m, k) = m!. \tag{2.4}$$

Using the definition of $\sigma_n(m, k)$ and (2.3) we get, identifying coefficients of corresponding powers of x and observing that $\sigma_{2n+1}(2, k) = 0$, THEOREM 4. *The Taylor series coefficients* $\sigma_{2n}(2, k)$ *of the function* $\mathrm{sn}^2(x, k)$ *satisfy the recurrence formula*

$$\sigma_{2n+2}(2, k) = -4(1 + k^2)\sigma_{2n}(2, k) + 6k^2 \sum_{v=1}^{n-1} \binom{2n}{2v} \sigma_{2v}(2, k)\sigma_{2n-2v}(2, k).$$

The coefficients $\sigma_2(2, k)$, $\sigma_4(2, k), \ldots, \sigma_{16}(2, k)$ are given in Table II. Writing $\mathrm{sn}^3(x, k) = \mathrm{sn}(x, k) \cdot \mathrm{sn}^2(x, k)$ and using (2.2) we get, observing that $\sigma_{2n}(1, k) = 0$,

THEOREM 5. *The Taylor series coefficients* $\sigma_{2n+1}(1, k)$ *of the Jacobian elliptic function* $\mathrm{sn}(x, k)$ *satisfy the recurrence formula*

$$\sigma_{2n+3}(1, k) = -(1 + k^2)\sigma_{2n+1}(1, k) + 2k^2 \sum_{v=0}^{n-1} \binom{2n+1}{2v+1} \sigma_{2v+1}(1, k)\sigma_{2n-2v}(2, k).$$

The coefficients $\sigma_1(1, k)$, $\sigma_3(1, k), \ldots, \sigma_{17}(1, k)$ are given in Table III. The first eight values are also given by Schett [6, p. 146].

Recurrence relations for the coefficients of $\mathrm{sn}^2(x, k)$ and $\mathrm{sn}(x, k)$ may be obtained alternatively in the following way:

For $\mathrm{sn}^2(x, k)$, we have, from (2.1)

$$\frac{d^2}{dx^2}\mathrm{sn}^2(x, k) = 2 - 4(1 + k^2)\mathrm{sn}^2(x, k) + 6k^2\mathrm{sn}^4(x, k),$$

$$\frac{d^2}{dx^2}\mathrm{sn}^4(x, k) = 12\,\mathrm{sn}^2(x, k) - 16(1 + k^2)\mathrm{sn}^4(x, k) + 20k^2\mathrm{sn}^6(x, k),$$

$$\frac{d^2}{dx^2}\mathrm{sn}^6(x, k) = 30\,\mathrm{sn}^4(x, k) - 36(1 + k^2)\mathrm{sn}^6(x, k) + 42k^2\mathrm{sn}^8(x, k),$$

etc.

Differentiating the first relation twice (and then twice again) and making use of the remaining ones, we get

$$\frac{d^4}{dx^4}\mathrm{sn}^2(x, k) = -8(1 + k^2) + 8(2 + 13k^2 + 2k^4)\mathrm{sn}^2(x, k)$$
$$-120k^2(1 + k^2)\mathrm{sn}^4(x, k) + 120k^4\mathrm{sn}^6(x, k),$$

$$\frac{d^6}{dx^6}\mathrm{sn}^2(x, k) = 32 + 208k^2 + 32k^4 - (64 + 1920k^2 + 1920k^4 + 64k^6)\mathrm{sn}^2(x, k)$$
$$+(2016k^2 + 8064k^4 + 2016k^6)\mathrm{sn}^4(x, k) - 6720k^4(1 + k^2)\mathrm{sn}^6(x, k)$$
$$+5040k^6\mathrm{sn}^8(x, k).$$

It is clear that we will, in general, have

$$(-1)^{n+1}\frac{d^{2n}}{dx^{2n}}\mathrm{sn}^2(x, k) = \sum_{v=0}^{n+1} (-1)^v a_v(n, k)\mathrm{sn}^{2v}(x, k). \tag{2.5}$$

Differentiating (2.5) twice, we get

$$\begin{aligned}(-1)^{n+1}\frac{d^{2n+2}}{dx^{2n+2}}\mathrm{sn}^2(x, k) &= \sum_{v=1}^{n+1} (-1)^v a_v(n, k)\frac{d^2}{dx^2}\mathrm{sn}^{2v}(x, k) \\ &= -\sum_{v=0}^{n+2} (-1)^v a_v(n + 1, k)\mathrm{sn}^{2v}(x, k).\end{aligned} \tag{2.6}$$

Using (2.1), we now easily obtain the following recurrence relations for the coefficients $a_i(n, k)$:

$$\text{for } i = 0, \quad a_0(n+1, k) = 2a_1(n, k),$$

$$\text{for } i = 1, \quad a_1(n+1, k) = 4(1+k^2)a_1(n, k) + 12a_2(n, k), \text{ and,}$$

(2.7) generally, for $i \geqslant 2$,

$$a_i(n+1, k) = (2i-2)(2i-1)k^2 a_{i-1}(n, k) + 4i^2(1+k^2)a_i(n, k) + (2i+2)(2i+1)a_{i+1}(n, k),$$

where $a_0(1, k) = 2$, $a_1(1, k) = 4(1+k^2)$, $a_2(1, k) = 6k^2$, and $a_i(1, k) = 0$ when $i > 2$.

Since the Taylor series coefficients $\sigma_{2n}(2, k)$ are equal to

$$\left[\frac{d^{2n}}{dx^{2n}}\operatorname{sn}^2(x, k)\right]_{x=0},$$

it follows that

$$\sigma_{2n}(2, k) = (-1)^{n+1}a_0(n, k), \qquad \sigma_{2n+2}(2, k) = 2(-1)^n a_1(n, k).$$

We can thus formulate the following theorem:

THEOREM 6. *Define* $a_{-1}(n, k) = 0$, $a_0(1, k) = 2$, $a_1(1, k) = 4(1+k^2)$, $a_2(1, k) = 6k^2$, *and* $a_i(1, k) = 0$ *when* $i > 2$. *Furthermore, let*

$$a_i(n+1, k) = (2i-2)(2i-1)k^2 a_{i-1}(n, k) + 4i^2(1+k^2)a_i(n, k) + (2i+2)(2i+1)a_{i+1}(n, k).$$

Then

$$\sigma_{2n}(2, k) = (-1)^{n+1}a_0(n, k), \qquad \sigma_{2n+2}(2, k) = 2(-1)^n a_1(n, k),$$

where $\sigma_{2n}(2, k)$ *are the Taylor series expansion coefficients of* $\operatorname{sn}^2(x, k)$.

An advantage of this method is that it can also be used to obtain the Taylor series expansion coefficients of $\operatorname{sn}(x, k)$ directly, without first computing those for $\operatorname{sn}^2(x, k)$ as in Theorem 5.

For $\operatorname{sn}(x, k)$, we have, from (2.1),

$$\frac{d^3}{dx^3}\operatorname{sn}(x, k) = -(1+k^2)\frac{d}{dx}\operatorname{sn}(x, k) + 2k^2\frac{d}{dx}\operatorname{sn}^3(x, k),$$

$$\frac{d^3}{dx^3}\operatorname{sn}^3(x, k) = 6\frac{d}{dx}\operatorname{sn}(x, k) - 9(1+k^2)\frac{d}{dx}\operatorname{sn}^3(x, k) + 12k^2\frac{d}{dx}\operatorname{sn}^5(x, k),$$

$$\frac{d^3}{dx^3}\operatorname{sn}^5(x, k) = 20\frac{d}{dx}\operatorname{sn}^3(x, k) - 25(1+k^2)\frac{d}{dx}\operatorname{sn}^5(x, k) + 30k^2\frac{d}{dx}\operatorname{sn}^7(x, k),$$

etc.

Differentiating the first relation twice and making use of the second one, we get

$$\frac{d^5}{dx^5}\operatorname{sn}(x, k) = (1 + 14k^2 + k^4)\frac{d}{dx}\operatorname{sn}(x, k) - 20k^2(1+k^2)\frac{d}{dx}\operatorname{sn}^3(x, k) + 24k^4\frac{d}{dx}\operatorname{sn}^5(x, k).$$

We find, in general, that

$$(2.8)\qquad (-1)^{n+1}\frac{d^{2n-1}}{dx^{2n-1}}\operatorname{sn}(x,k) = \sum_{v=1}^{n}(-1)^{v+1}b_v(n,k)\frac{d}{dx}\operatorname{sn}^{2v-1}(x,k),$$

and, using the same methods as in (2.6) ($i = 1, 2, \ldots$),

$$(2.9)\qquad \begin{aligned} b_i(n+1,k) &= (2i-3)(2i-2)k^2b_{i-1}(n,k) + (2i-1)^2(1+k^2)b_i(n,k) \\ &\quad + (2i+1)(2i)b_{i+1}(n,k), \end{aligned}$$

with $b_1(1,k) = 1$, and $b_i(1,k) = 0$ when $i > 1$.

Since the Taylor series coefficients $\sigma_{2n-1}(1,k)$ are equal to

$$\left[\frac{d^{2n-1}}{dx^{2n-1}}\operatorname{sn}(x,k)\right]_{x=0},$$

it follows that $\sigma_{2n-1}(1,k) = (-1)^{n+1}b_1(n,k)$. We can then formulate

THEOREM 7. *Define* $b_1(1,k) = 1$ *and* $b_i(1,k) = 0$ *when* $i > 1$. *Furthermore, let*

$$\begin{aligned} b_i(n+1,k) &= (2i-3)(2i-2)k^2b_{i-1}(n,k) + (2i-1)^2(1+k^2)b_i(n,k) \\ &\quad + (2i+1)2ib_{i+1}(n,k). \end{aligned}$$

Then

$$\sigma_{2n-1}(1,k) = (-1)^{n+1}b_1(n,k), \qquad n = 1, 2, \ldots,$$

where $\sigma_{2n-1}(1,k)$ *are the Taylor series expansion coefficients of* $\operatorname{sn}(x,k)$.

Similar schemes can be worked out for the Taylor series coefficients of $\operatorname{cn}^2(x,k)$ and $\operatorname{cn}(x,k)$, using the formula

$$(2.10)\qquad \begin{aligned} \frac{d^2}{dx^2}\operatorname{cn}^n(x,k) &= n(n-1)(1-k^2)\operatorname{cn}^{n-2}(x,k) + n^2(2k^2-1)\operatorname{cn}^n(x,k) \\ &\quad -n(n+1)k^2\operatorname{cn}^{n+2}(x,k). \end{aligned}$$

3. Some Other Relations and Check Formulae. A lot of formulae relating the coefficients $\sigma_{2n}(2,k)$ and $\sigma_{2n+1}(1,k)$ to themselves or each other may be deduced.

We start with Jacobi's imaginary transformation (Bowman [2, p. 37]), i.e.

$$(3.1)\qquad \operatorname{sn}(ix,k) = i\frac{\operatorname{sn}(x,k')}{\operatorname{cn}(x,k')}, \qquad k' = (1-k^2)^{1/2}.$$

Squaring the identity (3.1), we obtain, after a little algebra,

$$(3.2)\qquad \sigma_{2n}(2,k') + (-1)^n\sigma_{2n}(2,k) = \sum_{v=0}^{n}\binom{2n}{2v}(-1)^v\sigma_{2v}(2,k)\sigma_{2n-2v}(2,k').$$

The most obvious formula relating $\sigma_{2n}(2,k)$ to $\sigma_{2n+1}(1,k)$ is

$$(3.3)\qquad \sigma_{2n}(2,k) = \sum_{v=1}^{n}\binom{2n}{2v-1}\sigma_{2v-1}(1,k)\sigma_{2n-2v+1}(1,k),$$

but also several more intricate relations may be deduced. Starting from Landen's second transformation of $\operatorname{sn}(x,k)$ (Bowman [2, p. 72])

$$(3.4)\qquad \operatorname{sn}\left((1+k)x, \frac{2k^{1/2}}{1+k}\right) = \frac{(1+k)\operatorname{sn}(x,k)}{1+k\operatorname{sn}^2(x,k)},$$

we easily get (3.5), i.e.

$$(1+k)^{2n+1}\sigma_{2n+1}\left(1, \frac{2k^{1/2}}{1+k}\right) + k\sum_{v=0}^{n}\binom{2n+1}{2v}\sigma_{2v}(2,k)(1+k)^{2n+1-2v}\sigma_{2n+1-2v}\left(1, \frac{2k^{1/2}}{1+k}\right) = (1+k)\sigma_{2n+1}(1,k).$$

The duplication formula (Bowman [2, p. 14]) may also be used. From the identity

$$\operatorname{sn}(2x,k) = \frac{\dfrac{d}{dx}\operatorname{sn}^2(x,k)}{1-k^2\operatorname{sn}^4(x,k)}, \tag{3.6}$$

we get, using (2.3),

$$\frac{\operatorname{sn}(2x,k)}{6}\frac{d^2}{dx^2}\operatorname{sn}^2(x,k) + \frac{d}{dx}\operatorname{sn}^2(x,k) + \frac{2}{3}(1+k^2)\operatorname{sn}(2x,k)\operatorname{sn}^2(x,k) = \frac{4}{3}\operatorname{sn}(2x,k). \tag{3.7}$$

In terms of the coefficients studied, (3.7) yields (3.8), i.e.

$$\frac{1}{6}\sum_{v=0}^{n}\binom{2n+1}{2v+1}2^{2v+1}\sigma_{2v+1}(1,k)\left[\sigma_{2n+2-2v}(2,k) + 4(1+k^2)\sigma_{2n-2v}(2,k)\right] + \sigma_{2n+2}(2,k) = \frac{2^{2n+3}}{3}\sigma_{2n+1}(1,k).$$

A final example comes from combining (2.2) and (2.3) into

$$\frac{d^2}{dx^2}\operatorname{sn}^2(x,k) = 2-(1+k^2)\operatorname{sn}^2(x,k) + 3\operatorname{sn}(x,k)\frac{d^2}{dx^2}\operatorname{sn}(x,k), \tag{3.9}$$

which yields

$$\sigma_{2n+2}(2,k) = -(1+k^2)\sigma_{2n}(2,k) + 3\sum_{v=1}^{n}\binom{2n}{2v-1}\sigma_{2v-1}(1,k)\sigma_{2n+3-2v}(1,k). \tag{3.10}$$

Using (3.3), (3.10) may be transformed into a recurrence formula for the coefficients $\sigma_{2n+1}(1,k)$. We may formulate the following theorem:

Theorem 8. *The Taylor series coefficients* $\sigma_{2n+1}(1,k)$ *of the Jacobian elliptic function* $\operatorname{sn}(x,k)$ *satisfy the recurrence formula*

$$(2n-4)\sigma_{2n+1}(1,k) = \begin{cases} 2n(1+k^2)\sigma_{2n-1}(1,k) \\ + \displaystyle\sum_{v=2}^{n}\sigma_{2v-1}(1,k)\sigma_{2n+3-2v}(1,k)\left[\binom{2n+2}{2v-1} - 3\binom{2n}{2v-1}\right] \\ + \displaystyle\sum_{v=2}^{n}(1+k^2)\binom{2n}{2v-1}\sigma_{2v-1}(1,k)\sigma_{2n+1-2v}(1,k). \end{cases}$$

A similar formula (i.e. of the same degree of complexity) may be deduced combining (2.2) and (1.6) with $\alpha = 3$.

4. A Note on Bernoulli Numbers. It is easy to prove that

$$\operatorname{sn}(x, 1) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x). \tag{4.1}$$

The Taylor series expansion of $\tanh(x)$ is well known; see [1, p. 85]. Therefore

$$\operatorname{sn}(x, 1) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_{2n} x^{2n-1}. \tag{4.2}$$

Moreover

$$\begin{aligned} \operatorname{sn}^2(x, 1) &= \tanh^2(x) = 1 - \frac{d}{dx}\tanh(x) \\ &= -\sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} x^{2n-2}. \end{aligned} \tag{4.3}$$

We conclude that

$$\sigma_{2n}(2, 1) = -\frac{2^{2n+2}(2^{2n+2} - 1)}{(2n+2)} B_{2n+2} = -\sigma_{2n+1}(1, 1). \tag{4.4}$$

(4.4) was used to check the polynomials given in Tables II and III. Combining (4.4) and Theorem 2, we finally get

$$\begin{aligned} B_{2n+2} = \frac{2n+2}{2^{2n+2}(2^{2n+2}-1)} & \sum_{\substack{2i_2+4i_4+\cdots+2vi_{2v}=2n \\ i_2+i_4+\cdots+i_{2v}=m,\ m=1,2,\ldots,n}} (-1)^m(2n+m)! \\ & \cdot \prod_{s=1}^{v} \frac{1}{i_{2s}!}\left(\frac{1}{2s+1}\right)^{i_{2s}}. \end{aligned} \tag{4.5}$$

5. Tables.

TABLE I

The polynomials $C_{2n}(k) = k^n P_n\left(\frac{1}{2}(k + 1/k)\right)$

$$C_0 = 1$$

$$C_2 = \frac{1 + k^2}{2}$$

$$C_4 = \frac{3 + 2k^2 + 3k^4}{8}$$

$$C_6 = \frac{5 + 3k^2 + 3k^4 + 5k^6}{16}$$

$$C_8 = \frac{35 + 20k^2 + 18k^4 + 20k^6 + 35k^8}{128}$$

$$C_{10} = \frac{63 + 35k^2 + 30k^4 + 30k^6 + 35k^8 + 63k^{10}}{256}$$

$$C_{12} = \frac{231 + 126k^2 + 105k^4 + 100k^6 + 105k^8 + 126k^{10} + 231k^{12}}{1024}$$

$$C_{14} = \frac{429 + 231k^2 + 189k^4 + 175k^6 + 175k^8 + 189k^{10} + 231k^{12} + 429k^{14}}{2048}$$

TABLE II

Taylor series expansion coefficients of $\mathrm{sn}^2(x, k)$

$$\sigma_2(2, k) = 2$$
$$\sigma_4(2, k) = -8 - 8k^2$$
$$\sigma_6(2, k) = 32 + 208k^2 + 32k^4$$
$$\sigma_8(2, k) = -128 - 3840k^2 - 3840k^4 - 128k^6$$
$$\sigma_{10}(2, k) = 512 + 64256k^2 + 224256k^4 + 64256k^6 + 512k^8$$
$$\sigma_{12}(2, k) = -2048 - 1042432k^2 - 10139648k^4 - 10139648k^6 - 1042432k^8 - 2048k^{10}$$
$$\sigma_{14}(2, k) = 8192 + 16748544k^2 + 408870912k^4 + 1052502016k^6 + 408870912k^8 + 16748544k^{10} + 8192k^{12}$$
$$\sigma_{16}(2, k) = -32768 - 268304384k^2 - 15590621184k^4 - 89073713152k^6 - 89073713152k^8 - 15590621184k^{10} - 268304384k^{12} - 32768k^{14}$$

TABLE III

Taylor series expansion coefficients of $\mathrm{sn}(x, k)$

$$\sigma_1(1, k) = 1$$
$$\sigma_3(1, k) = -1 - k^2$$
$$\sigma_5(1, k) = 1 + 14k^2 + k^4$$
$$\sigma_7(1, k) = -1 - 135k^2 - 135k^4 - k^6$$
$$\sigma_9(1, k) = 1 + 1228k^2 + 5478k^4 + 1228k^6 + k^8$$
$$\sigma_{11}(1, k) = -1 - 11069k^2 - 165826k^4 - 165826k^6 - 11069k^8 - k^{10}$$
$$\sigma_{13}(1, k) = 1 + 99642k^2 + 4494351k^4 + 13180268k^6 + 4494351k^8 + 99642k^{10} + k^{12}$$
$$\sigma_{15}(1, k) = -1 - 896803k^2 - 116294673k^4 - 834687179k^6 - 834687179k^8 - 116294673k^{10} - 896803k^{12} - k^{14}$$
$$\sigma_{17}(1, k) = 1 + 8071256k^2 + 2949965020k^4 + 47152124264k^6 + 109645021894k^8 + 47152124264k^{10} + 2949965020k^{12} + 8071256k^{14} + k^{16}.$$

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