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On Odd Perfect, Quasiperfect, and Odd Almost Perfect Numbers

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Abstract. We establish upper bounds for the six smallest prime factors of odd perfect, quasiperfect, and odd almost perfect numbers.

1. Suppose $N = \prod_{i=1}^{r} p_i^{a_i}$ is an odd perfect (OP) number, i.e. $\sigma(N) = 2N$, where p_i's are odd primes, $p_1 < \cdots < p_r$, and a_i's are positive integers. Grun [1] proved that

$$p_1 < 2 + 2r/3,$$

and Pomerance [5] proved that

$$(1) \qquad p_i < (4r)^{2^{i(i+1)/2}} \quad \text{for } 1 \leqslant i \leqslant r.$$

In [3] we showed that if N is an odd integer and the number $\omega(N)$ of distinct prime factors of N is 5, then

$$(2) \qquad |2 - \sigma(N)/N| > 10^{-14}.$$

From this it follows immediately that if M is an odd integer, $\sigma(M) = 2M + L$, and if $|L/M| < 10^{-14}$, then $\omega(M) \geqslant 6$. OP, quasiperfect (QP) numbers, i.e. $\sigma(N) = 2N + 1$, and odd almost perfect (OAP) numbers, i.e. $\sigma(N) = 2N - 1$, are such examples.

Also, it can be proved from (2) that if $M = \prod_{i=1}^{r} p_i^{a_i}$ is OP,

$$p_6 < 2 \cdot 10^{14}(r - 5).$$

However, if we consider only those $N = \prod_{i=1}^{5} p_i^{a_i}$ in (2) for which $\prod_{i=1}^{r} p_i^{a_i}$ is OP, then exponents a_i are restricted, and hence we have a better lower bound in (2). Consequently we have a better upper bound for p_6.

In this paper we prove

THEOREM. *Suppose $M = \prod_{i=1}^{r} p_i^{a_i}$. If M is OP or QP,*

$$p_i < 2^{2^{i-1}}(r - i + 1) \quad \textit{for } 2 \leqslant i \leqslant 6.$$

If M is OAP,

$$p_i < 2^{2^{i-1}}(r - i + 1) \quad \textit{for } 2 < i \leqslant 5, \quad \textit{and}$$
$$p_6 < 23775427335(r - 5).$$

Although our Theorem gives upper bounds for p_i only for $2 \leqslant i \leqslant 6$, they are better than (1). For example, if M is OP, then $p_5 < 65536(r - 4)$ by our Theorem

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and $p_r > 100110$ by Hargis and McDaniel [2]. Hence, we have another proof that $\omega(M) \geqslant 6$.

2. In order to prove our Theorem, we need three lemmas.

Definition. $S(N) = \sigma(N)/N$.

LEMMA 1. *Suppose* $M = \prod_{i=1}^r p_i^{a_i}$ *is OP. Then*

$$S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < \frac{3}{2}\,\frac{5}{4}\,\frac{17}{16}\,\frac{257}{256}\,\frac{65537}{65536} = \alpha \approx 2 - 4/10^{10}.$$

Proof. Since M is OP, by Euler,

(3) if $p_i \equiv 1\ (4)$, $a_i \equiv 0, 1, 2\ (4)$, and if $p_i \equiv 3\ (4)$, $a_i \equiv 0\ (2)$,

and if q is an odd prime factor of $\sigma(p_i^{a_i})$ for some i, then $q \mid M$. Suppose

$$\alpha \leqslant S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < 2, \tag{4}$$

and $q \neq p_i$ for $1 \leqslant i \leqslant 5$. If $q < 10^9$, then

$$\log 2 = \log S(M) \geqslant \log S\left(\prod_{i=1}^{5} p_i^{a_i}\right) + \sum_{i=6}^{r} \log S(p_i^{a_i})$$
$$> \log \alpha + \log(q+1)/q > \log \alpha + \log(10^9+1)/10^9 > \log 2,$$

a contradiction. Hence,

(5) If q is an odd prime factor of $\sigma(p_i^{a_i})$ for some i and $q \neq p_j$ for $1 \leqslant j \leqslant 5$, then $q > 10^9$.

As in [3], we used a computer (PDP11 at the University of Toledo) to find odd integers $\prod_{i=1}^5 p_i^{a_i}$ satifying (3) and (4). There were infinitely many such $\prod_{i=1}^5 p_i^{a_i}$. (However, there were finitely many (just over one hundred) $\prod_{i=1}^5 p_i^{a_i}$ if $a_i \leqslant a(p_i)$ where

$$a(p_i) = \min\{a_i \mid a_i \text{ satisfies (3) and } p_i^{a_i+1} > 10^{11}\}.$$

See [3].) In every case such $\prod_{i=1}^5 p_i^{a_i}$ had a component $p_i^{a_i}$ such that $a_i < a(p_i)$, q is an odd prime factor of $\sigma(p_i^{a_i})$, $q \neq p_j$ for $1 \leqslant j \leqslant 5$ and $q < 10^9$, contradicting (5). Q.E.D.

LEMMA 2. *Suppose* $M = \prod_{i=1}^r p_i^{a_i}$ *is QP. Then*

$$S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < \frac{3}{2}\,\frac{5}{4}\,\frac{17}{16}\,\frac{257}{256}\,\frac{65537}{65536} = \alpha \approx 2 - 4/10^{10}.$$

Proof. Since M is QP, by [3], $r \geqslant 6$, $S(\prod_{i=1}^5 p_i^{a_i}) < 2$, and

(6)
$a_i \equiv 0\ (2)$ for any i,
if $p_i = 3$, $a_i = 4$, 12 or $\geqslant 24$,
if $p_i = 5$, $a_i = 6$ or $\geqslant 16$,
if $p_i = 17$, $a_i = 2$ or $\geqslant 8$.

We used the computer to find odd integers $\prod_{i=1}^{5} p_i^{a_i}$ satisfying (6) and

$$\alpha < S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < 2,$$

but there were none. Q.E.D.

LEMMA 3. *Suppose* $M = \prod_{i=1}^{r} p_i^{a_i}$ *is OAP. Then*

$$S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < S(3^{12})\frac{5}{4}S(17^6)\frac{257}{256}\,\frac{62939}{62938} = \beta \approx 2 - 8/10^{11}.$$

Proof. Since M is OAP, by [3], $r \geqslant 6$ and

$$(7)\qquad \begin{aligned} &a_i \equiv 0\ (2) \text{ for all } i,\\ &\text{if } p_i = 3,\ a_i = 12,\ 16 \text{ or } \geqslant 24,\\ &\text{if } p_i = 5,\ a_i = 2,\ 10 \text{ or } \geqslant 16,\\ &\text{if } p_i = 257,\ a_i \geqslant 16. \end{aligned}$$

We used the computer to find odd integers $\prod_{i=1}^{5} p_i^{a_i}$ satisfying (7) and

$$\alpha < S\left(\prod_{i=1}^{5} p_i^{a_i}\right) < 2,$$

and the results were

$$3^{a_1}5^{10}17^{a_3}257^{a_4}65449^{a_5}, \quad \text{where } a_1 \geqslant 24,\ a_3 \geqslant 8,\ a_4 \geqslant 16,\ a_5 \geqslant 2, \text{ and}$$

$$3^{12}5^{a_2}17^{6}257^{a_4}62939^{a_5}, \quad \text{where } a_2 \geqslant 16,\ a_4 \geqslant 16,\ a_5 \geqslant 2.$$

Since

$$\frac{3}{2}S(5^{10})\frac{17}{16}\,\frac{257}{256}\,\frac{65449}{65448} < S(3^{12})\frac{5}{4}S(17^6)\frac{257}{256}\,\frac{62939}{62938} = \beta,$$

Lemma 3 follows. Q.E.D.

Proof of Theorem. We prove only the case $i = 5$. Suppose $M = \prod_{i=1}^{r} p_i^{a_i}$ is OP or QP, $N = \prod_{i=1}^{5} p_i^{a_i}$, and

$$\frac{2}{2-\alpha}(r-5) + 1 \leqslant p_6 < \cdots < p_r.$$

Since $\log(1+x) < x$ and $\log(1-x) < -x$ if $0 < x < 1$, we have, by Lemmas 1 and 2,

$$\begin{aligned} \log 2 \leqslant \log S(M) &= \log S(N) + \sum_{i=6}^{r} \log S(p_i^{a_i})\\ &< \log \alpha + (r-5)\log S(p_6^{a_6})\\ &< \log 2 + \log \alpha/2 + (r-5)\log p_6/(p_6-1)\\ &= \log 2 + \log(1-(2-\alpha)/2) + (r-5)\log(1+1/(p_6-1))\\ &< \log 2 - (2-\alpha)/2 + (r-5)/(p_6-1)\\ &< \log 2 - (2-\alpha)/2 + (2-\alpha)/2 = \log 2, \end{aligned}$$

a contradiction. Hence,

$$p_6 < \frac{2}{2-\alpha}(r-5) + 1 = 2^{2^5}(r-5) + 1.$$

Since p_6 is a prime, $p_6 < 2^{2^5}(r-5)$.

Suppose $M = \prod_{i=1}^{r} p_i^{a_i}$ is OAP, $N = \prod_{i=1}^{5} p_i^{a_i}$, and

$$\frac{2}{2-\beta}(r-5)+1 \leqslant p_6 < \cdots < p_r.$$

Since $M > 10^{30}$ by [4] and $\log(1-x) < -x - x^2/2$ if $0 < x < 1$, we have, by Lemma 3,

$$\begin{aligned}\log 2 - \tfrac{1}{2}\cdot 10^{30} &\approx \log 2 + \log\left(1 - \tfrac{1}{2}\cdot 10^{30}\right)\\ &= \log(2 - 1/10^{30}) < \log(2 - 1/M) = \log(S(M)/M)\\ &= \log S(N) + \sum_{i=6}^{r} \log S(p_i^{a_i}) < \log\beta + (r-5)\log p_6/(p_6 - 1)\\ &< \log 2 + \log(1 - (2-\beta)/2) + (r-5)/(p_6 - 1)\\ &< \log 2 - (2-\beta)/2 - (2-\beta)^2/8 + (2-\beta)/2\\ &= \log 2 - (2-\beta)^2/8 \approx \log 2 - 9\cdot 10^{-22},\end{aligned}$$

a contradiction. Hence

$$p_6 < \frac{2}{2-\beta}(r-5)+1 < 23775427335(r-5)+1.$$

Since p_6 is a prime, $p_6 < 23775427335(r-5)$. Q.E.D.

Finally, we (re)state the following

THEOREM. *Suppose $N = \prod_{i=1}^{r} p_i^{a_i}$ is an integer.*

(a) If $r = 5$, $|2 - S(N)| > 2 - S(3^7 5^6 17^2 233)\cdot 36550429/36550428 > 10^{-14}$.

(b) If $r = 4$, $|2 - S(N)| \geqslant 2 - S(3^7 5^6 17^2 233) > 5/10^8$.

(c) If $r = 3$, $|2 - S(N)| \geqslant S(3^5 5^2 13) - 2 > 3/10^4$.

(d) If $r = 2$, $|2 - S(N)| \geqslant 2 - \frac{3}{2}\,\frac{5}{4} = 0.125$.

(e) If $r = 1$, $|2 - S(N)| > 2 - \frac{3}{2} = 0.5$.

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