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Scheme-Independent Stability Criteria for Difference Approximations of Hyperbolic Initial-Boundary Value Problems. II

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Abstract. Convenient stability criteria are obtained for difference approximations to initial-boundary value problems associated with the hyperbolic system $\mathbf{u}_t = A\mathbf{u}_x + B\mathbf{u} + \mathbf{f}$ in the quarter plane $x > 0$, $t > 0$. The approximations consist of arbitrary basic schemes and a wide class of boundary conditions. The new criteria are given in terms of the outflow part of the boundary conditions and are independent of the basic scheme. The results easily imply that a number of well-known boundary treatments, when used in combination with arbitrary stable basic schemes, always maintain stability. Consequently, many special cases studied in recent literature are generalized.

0. Introduction. In this paper we extend the results of [2] to obtain easily checkable stability criteria for difference approximations of initial-boundary value problems associated with the linear hyperbolic differential system $\mathbf{u}_t = A\mathbf{u}_x + B\mathbf{u} + \mathbf{f}$ in the quarter plane $x \geqslant 0$, $t \geqslant 0$. The difference approximations, introduced in Section 1, consist of arbitrary basic schemes–explicit or implicit, dissipative or unitary, two-level or multi-level–and boundary conditions of a rather general type.

The first step in our stability analysis is made in Section 2, where we prove that the approximation is stable if and only if the scalar outflow components of its principal part are stable. This reduces the global stability question to that of a scalar, homogeneous, outflow problem which thereafter becomes the main object of the paper.

Investigating the stability of the reduced problem, our main results are restricted to the case where the boundary conditions are translatory, i.e., determined at all boundary points by the same coefficients. Such boundary conditions are commonly used in practice; and, in particular, when the numerical boundary consists of a single point the boundary conditions are translatory by definition.

The main stability criteria for the translatory case, stated without proof in Section 3, are given essentially in terms of the boundary conditions. Such scheme-independent criteria eliminate the need to analyze the intricate and often complicated interaction between the basic scheme and the boundary conditions; hence

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providing convenient alternatives to the well-known stability criterion of Gustafsson, Kreiss, and Sundström [3], which is the basis for our work.

As in [3], we assume that the basic scheme is stable for the pure Cauchy problem and that the approximation is solvable. Under these basic assumptions–which are obviously necessary for stability–we obtain, for example, in Theorems 3.3 and 3.4, that the reduced problem is stable if the (translatory) boundary conditions are solvable and satisfy the von Neumann condition as well as an additional simple inequality. If the basic scheme is unitary, it is also required that the boundary conditions be dissipative.

Having the new stability criteria, we continue in Section 3 to study several examples. First, we reestablish the known fact that if the basic scheme is two-level and dissipative, then outflow boundary conditions determined by horizontal extrapolation always maintain stability. Surprisingly, we show that this result is false if the basic scheme is of more than two levels. Next, for arbitrary multi-level dissipative basic schemes, we find that if the outflow boundary conditions are generated, for example, by oblique extrapolation, by the Box-Scheme, or by the right-sided Euler scheme, then overall stability is assured. Finally, for basic schemes (dissipative or unitary), we show that overall stability holds if the outflow boundary conditions are determined by the right-sided explicit or implicit Euler schemes. These examples incorporate many special cases discussed in recent literature [1]–[4], [6], [9], [10].

In Sections 4 and 5 we prove the results stated in Section 3.

It should be pointed out that there is no difficulty in extending our stability criteria to cases with two boundaries. In fact, if the corresponding left and right quarter-plane problems are stable, then, by Theorem 5.4 of [3], the original two-boundary problem is stable as well.

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1. The Difference Approximation. Consider the first order hyperbolic system of partial differential equations

$$\partial \mathbf{u}(x,t)/\partial t = A\,\partial \mathbf{u}(x,t)/\partial x + B\mathbf{u}(x,t) + \mathbf{f}(x,t), \qquad x > 0,\ t > 0, \tag{1.1a}$$

where $\mathbf{u}(x, t) = (u^{(1)}(x, t), \ldots, u^{(n)}(x, t))'$ is the vector of unknowns (prime denoting the transpose), $\mathbf{f}(x, t) = (f^{(1)}(x, t), \ldots, f^{(n)}(x, t))'$ is a given vector, and A and B are fixed $n \times n$ matrices so that A is Hermitian and nonsingular. Without restriction we may assume that A is diagonal of the form

$$A = \begin{pmatrix} A^{\mathrm{I\,I}} & 0 \\ 0 & A^{\mathrm{II\,II}} \end{pmatrix}, \qquad A^{\mathrm{I\,I}} < 0, A^{\mathrm{II\,II}} > 0, \tag{1.2}$$

where $A^{\mathrm{I\,I}}$ and $A^{\mathrm{II\,II}}$ are of orders $l \times l$ and $(n - l) \times (n - l)$, respectively.

The solution of (1.1) is uniquely determined if we prescribe initial values

$$\mathbf{u}(x, 0) = \mathring{\mathbf{u}}(x), \qquad x \geq 0, \tag{1.1b}$$

and boundary conditions

$$\mathbf{u}^{\mathrm{I}}(0, t) = S\mathbf{u}^{\mathrm{II}}(0, t) + \mathbf{g}(t), \qquad t > 0, \tag{1.1c}$$

where S is a fixed $l \times (n - l)$ matrix, $\mathbf{g}(t)$ is a given l-vector, and

$$\mathbf{u}^{\mathrm{I}} = (u^{(1)}, \ldots, u^{(l)})', \qquad \mathbf{u}^{\mathrm{II}} = (u^{(l+1)}, \ldots, u^{(n)})' \tag{1.3}$$

is a partition of $\mathbf{u}$ into inflow and outflow unknowns, respectively, corresponding to the partition of A.

In order to solve the initial-boundary value problem (1.1) by difference approximations we introduce a mesh size $h \equiv \Delta x > 0$, $k \equiv \Delta t > 0$, such that $\lambda \equiv h/k =$ constant. Using the notation $\mathbf{v}_\nu(t) = \mathbf{v}(\nu h, t)$, we approximate (1.1a) by a consistent, two-sided, general multi-step basic scheme of the form

$$\begin{aligned} &Q_{-1}\mathbf{v}_\nu(t + k) = \sum_{\sigma=0}^{s} Q_\sigma \mathbf{v}_\nu(t - \sigma k) + k\mathbf{b}_\nu(t), \qquad \nu = r, r+1, \ldots, \\ &Q_\sigma = \sum_{j=-r}^{p} A_{j\sigma}E^j, \qquad E\mathbf{v}_\nu = \mathbf{v}_{\nu+1}, \qquad \sigma = -1, \ldots, s, \end{aligned} \tag{1.4a}$$

where the $n \times n$ matrices $A_{j\sigma}$ are polynomials in A and kB, and the n-vectors $\mathbf{b}_\nu(t)$ depend smoothly on $f(x, t)$ and its derivatives.

To solve (1.4a) uniquely, we provide initial values

$$\mathbf{v}_\nu(\sigma k) = \mathring{\mathbf{v}}_\nu(\sigma k), \qquad \sigma = 0, \ldots, s, \nu = 0, 1, \ldots, \tag{1.4b}$$

where in addition we must specify, at each time step $t = \sigma k \geqslant sk$, boundary values $\mathbf{v}_\mu(t + k)$, $\mu = 0, \ldots, r - 1$. The required boundary values will be determined by two sets of boundary conditions, the first of which is obtained by taking the last $n - l$ components of general boundary conditions of the form

$$\begin{aligned} T_{-1}^{(\mu)}\mathbf{v}_\mu(t + k) &= \sum_{\sigma=0}^{q} T_\sigma^{(\mu)}\mathbf{v}_\mu(t - \sigma k) + k\mathbf{d}_\mu(t), \\ T_\sigma^{(\mu)} &= \sum_{j=0}^{m} C_{j\sigma}^{(\mu)}E^j, \qquad \mu = 0, \ldots, r - 1, \sigma = -1, \ldots, q, \end{aligned}$$

where the matrices $C_{j\sigma}^{(\mu)}$ are polynomials in A and kB, the $C_{j(-1)}^{(\mu)}$ are nonsingular, and the n-vectors $\mathbf{d}_\mu(t)$ are functions of $\mathbf{f}(x, t)$, $\mathbf{g}(t)$ and their derivatives. If we put

$$C_{j\sigma}^{(\mu)} = \begin{bmatrix} C_{j\sigma}^{\mathrm{I\,I}(\mu)} & C_{j\sigma}^{\mathrm{I\,II}(\mu)} \\ C_{j\sigma}^{\mathrm{II\,I}(\mu)} & C_{j\sigma}^{\mathrm{II\,II}(\mu)} \end{bmatrix}, \qquad \mathbf{v}_\mu = \begin{bmatrix} \mathbf{v}_\mu^{\mathrm{I}} \\ \mathbf{v}_\mu^{\mathrm{II}} \end{bmatrix}, \qquad \mathbf{d}_\mu = \begin{bmatrix} d_\mu^{\mathrm{I}} \\ \mathbf{d}_\mu^{\mathrm{II}} \end{bmatrix}$$

in accordance with the partitions of A and U in (1.2), (1.3), this set of conditions takes the form

$$\begin{aligned} &T_{-1}^{\mathrm{II\,I}(\mu)}\mathbf{v}_\mu^{\mathrm{I}}(t + k) + T_{-1}^{\mathrm{II\,II}(\mu)}\mathbf{v}_\mu^{\mathrm{II}}(t + k) \\ &\qquad = \sum_{\sigma=0}^{q} \left[T_\sigma^{\mathrm{II\,I}(\mu)}\mathbf{v}_\mu^{\mathrm{I}}(t - \sigma k) + T_\sigma^{\mathrm{II\,II}(\mu)}\mathbf{v}_\mu^{\mathrm{II}}(t - \sigma k) \right] + k\mathbf{e}_\mu^{\mathrm{II}}(t), \\ &T_\sigma^{\mathrm{II}\,\alpha(\mu)} = \sum_{j=0}^{m} C_{j\sigma}^{\mathrm{II}\,\alpha(\mu)}E^j, \qquad \alpha = \mathrm{I}, \mathrm{II}, \mu = 0, \ldots, r - 1. \end{aligned} \tag{1.4c}$$

For the second set of boundary conditions we use the analytic condition

$$\mathbf{v}_0^{\mathrm{I}}(t + k) = S\mathbf{v}_0^{\mathrm{II}}(t + k) + \mathbf{g}(t + k), \tag{1.4d}$$

together with $r-1$ additional conditions of the form

$$\mathbf{v}_\mu^{\mathrm{I}}(t+k)=\sum_{j=1}^{p}\left[D_j^{\mathrm{I}\,\mathrm{I}(\mu)}\mathbf{v}_j^{\mathrm{I}}(t+k)+D_j^{\mathrm{I}\,\mathrm{II}(\mu)}\mathbf{v}_j^{\mathrm{II}}(t+k)\right]+k\mathbf{e}_\mu^{\mathrm{I}}(t), \tag{1.4e}$$

$$\mu=1,\ldots,r-1,$$

where the matrices $D_j^{\mathrm{I}\,\mathrm{I}(\mu)}$ and $D_j^{\mathrm{I}\,\mathrm{II}(\mu)}$–of orders $l\times l$ and $l\times(n-l)$, respectively–are polynomials in the blocks $(A^{\alpha\alpha})^{-1}$ and $kB^{\alpha\beta}$, $\alpha,\beta=\mathrm{I},\mathrm{II}$, of the matching partitions

$$A^{-1}=\begin{bmatrix}(A^{\mathrm{I}\,\mathrm{I}})^{-1} & 0\\ 0 & (A^{\mathrm{II}\,\mathrm{II}})^{-1}\end{bmatrix},\qquad kB=k\begin{pmatrix}B^{\mathrm{I}\,\mathrm{I}} & B^{\mathrm{I}\,\mathrm{II}}\\ B^{\mathrm{II}\,\mathrm{I}} & B^{\mathrm{II}\,\mathrm{II}}\end{pmatrix},$$

so that $D_j^{\mathrm{I}\,\mathrm{I}(\mu)}$ are homogeneous in $B^{\mathrm{I}\,\mathrm{II}}$ and $B^{\mathrm{II}\,\mathrm{I}}$, and the l-vectors $\mathbf{e}_\mu^{\mathrm{I}}(t)$ are again functions of $\mathbf{f}(x,t)$, $\mathbf{g}(t)$ and their derivatives.

We remark that (1.4c, d, e) can be solved uniquely for the required boundary values $\mathbf{v}_\mu(t+k)$, $\mu=0,\ldots,r-1$, in terms of neighboring values of $\mathbf{v}$, at least for sufficiently small k. Indeed, since B introduces an $O(k)$ perturbation of the matrix coefficients in (1.4c, e), it suffices to prove this statement for $B=0$. But then, using the properties of $C_{j\sigma}^{(\mu)}$ and $D_j^{\mathrm{I}\,\mathrm{I}(\mu)}$, it is not hard to see that $C_j^{\mathrm{II}\,\mathrm{I}(\mu)}=D_j^{\mathrm{I}\,\mathrm{I}(\mu)}=0$ and that the $C_{j(-1)}^{\mathrm{II}\,\mathrm{II}(\mu)}$ are nonsingular; hence (1.4c) uniquely determines the vectors $\mathbf{v}_\mu^{\mathrm{II}}(t+k)$, $\mu=r-1,\ldots,0$ (in that order), and, substituting in (1.4d, e), we explicitly obtain $\mathbf{v}_\mu^{\mathrm{I}}(t+k)$, $\mu=0,\ldots,r-1$.

We also remark that, while it is a standard matter to construct boundary conditions of the form (1.4c) to any degree of accuracy, the construction of (1.4e) is less obvious. For example, using (1.1), we find by induction on $j>1$ that

$$\frac{\partial^j}{\partial x^j}\mathbf{u}(x,t)=(L_t)^j\mathbf{u}(x,t)-\mathbf{y}_j(x,t) \tag{1.5}$$

and

$$\frac{\partial^j}{\partial t^j}\mathbf{u}(x,t)=(L_x)^j\mathbf{u}(x,t)+\mathbf{z}_j(x,t), \tag{1.6}$$

where the operators L_t, L_x and the vectors $\mathbf{y}_j(x,t)$, $\mathbf{z}_j(x,t)$ are given by

$$L_t=A^{-1}\left(\frac{\partial}{\partial t}-B\right),\qquad \mathbf{y}_j(x,t)=\sum_{i=0}^{j-1}(L_t)^iA^{-1}\frac{\partial^{j-i-1}}{\partial x^{j-i-1}}\mathbf{f}(x,t),$$

$$L_x=A\frac{\partial}{\partial x}+B,\qquad \mathbf{z}_j(x,t)=\sum_{i=0}^{j-1}(L_x)^i\frac{\partial^{j-i-1}}{\partial t^{j-i-1}}\mathbf{f}(x,t).$$

Now, if conditions of the form (1.4e) are required to ρ order of accuracy, we take a Taylor expansion of $\mathbf{u}_\mu^{\mathrm{I}}(t)$ and use (1.5) and (1.1c) to obtain

$$\begin{aligned}\mathbf{u}_\mu^{\mathrm{I}}(t+k)&=\sum_{j=0}^{p}\frac{(\mu h)^j}{j!}\frac{\partial^j}{\partial x^j}\mathbf{u}^{\mathrm{I}}(0,t+k)+O(h^{p+1})\\ &=\sum_{j=0}^{p}\frac{(\mu h)^j}{j!}\left[(L_t)^j\mathbf{u}(0,t+k)-\mathbf{y}_j(0,t+k)\right]^{\mathrm{I}}+O(h^{p+1})\\ &=\sum_{j=0}^{p}\frac{(\mu h)^j}{j!}\left[(L_t)^j\begin{pmatrix}S\mathbf{u}^{\mathrm{II}}(0,t+k)+\mathbf{g}(t+k)\\ \mathbf{u}^{\mathrm{II}}(0,t+k)\end{pmatrix}-\mathbf{y}_j(0,t+k)\right]^{\mathrm{I}}\\ &\quad+O(h^{p+1}),\end{aligned} \tag{1.7}$$

where $[\cdot]^{\mathrm{I}}$ denotes the first l components of the enclosed vectors. We see that $\mathbf{u}_\mu^{\mathrm{I}}(t+k)$ depends on time derivatives of $\mathbf{u}^{\mathrm{II}}(0, t+k)$ which, using (1.6), may be replaced by space derivatives of $\mathbf{u}^{\mathrm{I}}(0, t+k)$ and $\mathbf{u}^{\mathrm{II}}(0, t+k)$. Approximating these space derivatives by ρ-order accurate linear combinations of $\mathbf{u}_0^{\mathrm{I}}(t+k), \ldots, \mathbf{u}_\rho^{\mathrm{I}}(t+k)$ and $\mathbf{u}_0^{\mathrm{II}}(t+k), \ldots, \mathbf{u}_\rho^{\mathrm{II}}(t+k)$, respectively, we finally obtain (1.4e) if $\mathbf{u}$ is replaced by $\mathbf{v}$ and terms of order $O(h^{\rho+1})$ are dropped.

A concrete example of a second order accurate boundary condition of form (1.4e), for the special case $B = \mathbf{f} = 0$, is given in [2].

2. The Reduced Problem. The difference approximation is completely defined now by (1.4), and we wish to apply to it the stability theory of Gustafsson, Kreiss, and Sundström [3]. Trying to fit our approximation into the form discussed in [3], we realize, however, that while in the present paper the vector $\mathbf{b}$ of the basic scheme (1.4a) is a general combination of $\mathbf{f}$ and its derivatives, in [3] we have $\mathbf{b} = \mathbf{f}$. Indeed, the general $\mathbf{b}$ admitted by us here is necessary if arbitrary high order approximations to (1.1a) are desired.*** Yet, it is not hard to see that this generalization does not affect the results of [3]. We conclude, therefore, that *making the same assumptions about our difference approximation as were made in* [3], *the theory of Gustafsson et al. holds for our case*, and we raise the question of stability in the sense of Definition 3.3 of [3].

In Theorem 2.1 below, we shall reduce the above stability question to that of a scalar outflow approximation with homogeneous boundary conditions. To obtain this theorem, we begin by recalling Lemma 10.3 of [3] which provides a necessary and sufficient determinantal stability criterion given entirely in terms of the *principal part* of the approximation, i.e., the part obtained by neglecting B and eliminating all inhomogeneity vectors. The mere existence of such a criterion implies that for stability purposes we may study (1.4) with $\mathbf{b}_\nu(t) = \mathbf{d}_\mu(t) = \mathbf{e}_\mu(t) = \mathbf{g}(t) = C_{j\sigma}^{\mathrm{II\,I}(\mu)} = D_j^{\mathrm{I\,I}(\mu)} = 0$; hence, instead of (1.4) we may consider a basic scheme of the form

$$\text{(2.1a)}\qquad \begin{aligned} Q_{-1}\mathbf{v}_\nu(t+k) &= \sum_{\sigma=0}^{s} Q_\sigma \mathbf{v}_\nu(t-\sigma k), \qquad \nu = r, r+1, \ldots, \\ Q_\sigma &= \sum_{j=-r}^{p} A_{j\sigma}E^j, \qquad E\mathbf{v}_\nu = \mathbf{v}_{\nu+1}, \qquad \nu = -1, \ldots, s, \end{aligned}$$

with initial values

$$\text{(2.1b)}\qquad \mathbf{v}_\nu(\sigma k) = \mathring{\mathbf{v}}_\nu(\sigma k), \qquad \sigma = 0, \ldots, s, \ \ \nu = 0, 1, \ldots,$$

and boundary conditions

$$\text{(2.1c)}\qquad \begin{aligned} T_{-1}^{\mathrm{II\,II}(\mu)}\mathbf{v}_\mu^{\mathrm{II}}(t+k) &= \sum_{\sigma=0}^{q} T_\sigma^{\mathrm{II\,II}(\mu)}\mathbf{v}^{\mathrm{II}}(t-\sigma k), \\ T_\sigma^{\mathrm{II\,II}(\mu)} &= \sum_{j=0}^{m} C_{j\sigma}^{\mathrm{II\,II}(\mu)}E^j, \qquad \mu = 0, \ldots, r-1, \end{aligned}$$

*** For example, the Lax-Wendroff scheme [7] for (1.1a) is

$$\mathbf{v}_\nu(t+k) = A_{-1}\mathbf{v}_{\nu-1}(t) + A_0\mathbf{v}_\nu(t) + A_1\mathbf{v}_{\nu+1}(t) + k\mathbf{b}_\nu(t), \qquad A_0 = I + kB + \tfrac{1}{2}k^2B^2 - \lambda^2A^2,$$

$$A_{\pm 1} = \tfrac{1}{2}\lambda A + \lambda^2A^2 - \tfrac{1}{4}\lambda k(AB + BA), \qquad \mathbf{b}(x, t) = \left[I + \tfrac{1}{2}k\left(B + A\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\right]\mathbf{f}(x, t).$$

(2.1d) $$\mathbf{v}_0^{\mathrm{I}}(t+k) = S\mathbf{v}_0^{\mathrm{II}}(t+k),$$

(2.1e) $$\mathbf{v}_\mu^{\mathrm{I}}(t+k) = \sum_{j=1}^{p} D^{\mathrm{I\,II}(\mu)}\mathbf{v}_j^{\mathrm{II}}(t+k), \qquad \mu = 1, \dots, r-1,$$

where (2.1a) is now consistent with

(2.2) $$\partial\mathbf{u}/\partial t = A\,\partial\mathbf{u}/\partial x,$$

and the $A_{j\sigma}$ and $C_{j\sigma}^{\mathrm{II\,II}(\mu)}$ are polynomials in A and in $A^{\mathrm{II\,II}}$, respectively.

We thus obtain,

LEMMA 2.1. *Approximation* (1.4) *is stable if and only if its principal part* (2.1) *is stable.*

Setting

$$A_{j\sigma} = \begin{bmatrix} A_{j\sigma}^{\mathrm{I\,I}} & 0 \\ 0 & A_{j\sigma}^{\mathrm{II\,II}} \end{bmatrix}, \qquad j = -r, \dots, p, \sigma = -1, \dots, s,$$

according to the partition of A in (1.2), our next step is to split the basic scheme (2.1a) and the initial values (2.1b) into

(2.3a) $$Q_{-1}^{\mathrm{I\,I}}\mathbf{v}_\nu^{\mathrm{I}}(t+k) = \sum_{\sigma=0}^{s} Q_\sigma^{\mathrm{I\,I}}\mathbf{v}_\nu^{\mathrm{I}}(t-\sigma k), \qquad \nu = r, r+1, \dots,$$
$$Q_\sigma^{\mathrm{I\,I}} = \sum_{j=-r}^{p} A_{j\sigma}^{\mathrm{I\,I}}E^j,$$

(2.3b) $$\mathbf{v}_\nu^{\mathrm{I}}(\sigma k) = \mathring{\mathbf{v}}_\nu^{\mathrm{I}}(\sigma k), \qquad \sigma = 0, \dots, s, \nu = 0, 1, \dots,$$

and

(2.4a) $$Q_{-1}^{\mathrm{II\,II}}\mathbf{v}_\nu^{\mathrm{II}}(t+k) = \sum_{\sigma=0}^{s} Q_\sigma^{\mathrm{II\,II}}\mathbf{v}_\nu^{\mathrm{II}}(t-\sigma k), \qquad \nu = r, r+1, \dots,$$
$$Q_\sigma^{\mathrm{II\,II}} = \sum_{j=-r}^{p} A_{j\sigma}^{\mathrm{II\,II}}E^j,$$

(2.4b) $$\mathbf{v}_\nu^{\mathrm{II}}(\sigma k) = \mathring{\mathbf{v}}_\nu^{\mathrm{II}}(\sigma k), \qquad \sigma = 0, \dots, s, \nu = 0, 1, \dots;$$

thus viewing approximation (2.1) as consisting of inflow and outflow parts given by (2.3) (2.1d, e) and (2.4) (2.1c), respectively. Obviously, (2.1) is stable if and only if both parts are.

We observe that the outflow part (2.4) (2.1c) is self-contained and provides, via (2.1d, e), the boundary values $\mathbf{v}_\mu^{\mathrm{I}}(t+k)$, $\mu = 0, \dots, r-1$. We may therefore consider (2.1d, e) as arbitrary inhomogeneous boundary values for the inflow part. So, by the argument involving Lemma 10.3 of [3] preceding Lemma 2.1, we may replace (2.1d, e)–without affecting stability–by homogeneous boundary values

(2.5) $$\mathbf{v}_\mu^{\mathrm{I}}(t+k) = 0, \qquad \mu = 0, \dots, r-1.$$

This gives us a new self-contained inflow part, (2.3) (2.5), whose stability together with that of (2.4) (2.1c) is equivalent to the overall stability of (2.1).

Since the $A_{j\sigma}$ and $C_{j\sigma}^{\mathrm{II\,II}(\mu)}$ are diagonal, we write

$$A_{j\sigma} = \operatorname{diag}(a_{j\sigma}), \qquad C_{j\sigma}^{\mathrm{II\,II}(\mu)} = \operatorname{diag}\big(c_{j\sigma}^{(\mu)}\big),$$

and split (2.3) (2.5) and (2.4) (2.1c) into n scalar components, each of the form

$$Q_{-1}v_\nu(t+k) = \sum_{\sigma=0}^{s} Q_\sigma v_\nu(t-\sigma k), \qquad \nu = r, r+1, \ldots,$$

(2.6a)

$$Q_\sigma = \sum_{j=-r}^{p} a_{j\sigma}E^j,$$

(2.6b) $$v_\nu(\sigma k) = \mathring{v}_\nu(\sigma k), \qquad \sigma = 0, \ldots, s, \nu = 0, 1, \ldots,$$

(2.6c) $$T_{-1}^{(\mu)}v_\mu(t+k) = \sum_{\sigma=0}^{q} T_\sigma^{(\mu)}v_\mu(t-\sigma k), \qquad \mu = 0, \ldots, r-1,$$

where (2.6a) is consistent with a corresponding component of (2.2),

(2.7) $$\partial u/\partial t = a\partial u/\partial x, \qquad a \neq 0;$$

and the boundary conditions (2.6c) are either homogeneous, i.e.,

(2.8) $$T_{-1}^{(\mu)} = 1, \qquad T_\sigma^{(\mu)} = 0, \qquad \mu = 0, \ldots, r-1, \sigma = 0, \ldots, s,$$

or are given by

(2.9) $$T_\sigma^{(\mu)} = \sum_{j=0}^{m} c_{j\sigma}^{(\mu)}E^j, \qquad c_{0(-1)}^{(\mu)} \neq 0, \mu = 0, \ldots, r-1, \sigma = -1, \ldots, q,$$

depending on whether $a < 0$ or $a > 0$, respectively.

Since (2.1) is stable if and only if (2.3) (2.5) and (2.4) (2.1c) are stable, and since the latter are stable if and only if their scalar components are, we obtain immediately

LEMMA 2.2. *Approximation* (2.1) *is stable if and only if the scalar components of* (2.3) (2.5) *and* (2.4) (2.1c), *given by* (2.6) (2.8) *and* (2.6) (2.9), *are stable.*

In Section 4 we shall prove

LEMMA 2.3. *The inflow approximation* (2.6) (2.8) *is unconditionally stable.*

This lemma–due to Kreiss [4] in the special case when the basic scheme is dissipative, explicit and two-level–combined with the previous two, finally yields the main result of this section:

THEOREM 2.1. *Approximation* (1.4) *is stable if and only if the scalar outflow components of its principal part are stable.*

The above discussion implies that from now on we may reduce our stability study to scalar approximations of form (2.6) with either (2.8) or (2.9). We thus conclude this section by stating the basic assumptions of [3] relating to these approximations which will hereafter hold throughout the paper.

Assumption 2.1 (Assumption 3.1, [3]). Approximation (2.6) is *solvable*, i.e., there exists a constant $K > 0$ such that for each $y \in l_2(x)$ there is a unique solution $w \in l_2(x)$ to

$$Q_{-1}w_\nu = y_\nu, \qquad \nu = r, r+1, \ldots,$$
$$T_{-1}^{(\mu)}w_\mu = y_\mu, \qquad \mu = 0, \ldots, r-1,$$

with $\|w\| \leqslant K\|y\|$. Here, $l_2(x)$ is the space of all grid functions $w = \{w_\nu\}_{\nu=0}^{\infty}$ with $\|w\|^2 \equiv h\sum_{\nu=0}^{\infty}|w_\nu|^2 < \infty$.

Assumption 2.2 (Assumption 5.1 [3]). The basic scheme (2.6a) is stable for the pure Cauchy problem, $-\infty < \nu < \infty$. That is, putting

$$a_j(z) \equiv a_{j(-1)} - \sum_{\sigma=0}^{s} z^{-\sigma-1} a_{j\sigma}, \qquad j = -r, \ldots, p, \tag{2.10}$$

we have

(i) The von Neumann condition; i.e., the solutions $z(\xi)$ of the equation

$$\sum_{j=-r}^{p} a_j(z) e^{ij\xi} = 0 \tag{2.11}$$

satisfy $|z(\xi)| \leq 1$ for all $|\xi| \leq \pi$.

(ii) Those $z(\xi)$ which lie on the unit circle are simple roots of (2.11).

Assumption 2.3 (Assumption 5.4 [3]). The basic scheme (2.6a) is either *dissipative*, i.e., the roots of (2.11) satisfy

$$|z(\xi)| < 1, \qquad 0 < |\xi| \leq \pi, \tag{2.12}$$

or it is *unitary*, namely

$$|z(\xi)| = 1, \qquad |\xi| \leq \pi. \tag{2.13}$$

Finally, for convenience only, we make

Assumption 2.4 (Assumption 5.5 [3]).

$$a_{-r}(z), a_p(z) \neq 0 \quad \text{for } |z| \geq 1. \tag{2.14}$$

3. Statement of Main Results and Examples. The purpose of this section is to provide easily checkable stability criteria for outflow approximations of form (2.6) (2.9). In view of Theorem 2.1, this is the key to the overall stability question of approximation (1.4).

Our results–stated below and proved in Section 5–are essentially independent of the basic scheme (2.6a) and are given solely in terms of the boundary conditions. These results, however, do not apply to general boundary conditions of form (2.6c) (2.9); instead we are concerned in this section with the *translatory* case where (2.6c) (2.9) are of the form

$$\begin{aligned} T_{-1} v_\mu(t+k) &= \sum_{\sigma=0}^{q} T_\sigma v_\mu(t - \sigma k), \\ T_\sigma = \sum_{j=0}^{m} c_{j\sigma} E^j, &\qquad c_{0(-1)} \neq 0, \mu = 0, \ldots, r-1. \end{aligned} \tag{3.1}$$

As mentioned in the introduction, such boundary conditions are widely used in practice since the coefficients $c_{j\sigma}$ are independent of μ and all boundary values are conveniently determined by the same procedure. Especially, when the numerical boundary consists of a single grid point ($r = 1$), the computation at the boundary is translatory by definition.

We associate now with the boundary conditions (3.1) the *boundary characteristic function*

$$R(z, \kappa) \equiv \sum_{j=0}^{m} c_j(z) \kappa^j, \tag{3.2}$$

where

$$c_j(z) \equiv c_{j(-1)} - \sum_{\sigma=0}^{q} z^{-\sigma-1} c_{j\sigma}, \qquad j = 0, \ldots, n. \tag{3.3}$$

This allows us to state

THEOREM 3.1. *Let the basic scheme* (2.6a) *be two-level and dissipative. Then the outflow approximation* (2.6a, b) (3.1) *is stable if*

$$R(z, \kappa) \neq 0 \quad \forall |z| \geqslant 1, 0 < |\kappa| < 1. \tag{3.4}$$

Example 3.1 (Kreiss [4, Theorem 6]; see also [1] and [2, Example (4.5)]). Let the basic scheme (2.6a) be two-level and dissipative, and let the boundary conditions be determined by horizontal extrapolation of order $\omega - 1$, i.e.,

$$v_\mu(t + k) = \sum_{j=1}^{\omega} \binom{\omega}{j} (-1)^{j+1} v_{\mu+j}(t + k), \qquad \mu = 0, \ldots, r - 1. \tag{3.5a}$$

The boundary characteristic function–which for one-level boundary conditions is always z-independent–satisfies

$$R(\kappa) = 1 - \sum_{j=1}^{\omega} \binom{\omega}{j} (-1)^{j+1} \kappa^j = (1 - \kappa)^\omega \neq 0, \qquad 0 < |\kappa| < 1. \tag{3.5b}$$

Hence, (3.4) holds and, by Theorem 3.1, (2.6a, b) (3.5) is stable.

It should be pointed out that Theorem 3.1 is generally false if the basic scheme is of more than two levels. Surprisingly, even the well-known result in Example 3.1 may fail to hold; namely, *outflow dissipative multi-level basic schemes* ($s > 3$), *with boundary values determined by extrapolation of type* (3.5a), *are not always stable.* For example, consider the 3-level, 5-point basic scheme

$$\begin{aligned} v_\nu(t + k) &= \left[I - \frac{\varepsilon}{16}(E - I)^2 (I - E^{-1})^2 \right] v_\nu(t - k) + \lambda a (E - E^{-1}) v_\nu(t), \\ &\qquad\qquad 0 < \varepsilon < 1, 0 < \lambda a \leqslant 1 - \varepsilon, \nu = 2, 3, \ldots, \end{aligned} \tag{3.6}$$

with boundary values $v_\mu(t + k)$, $\mu = 0, 1$, determined by (3.5a). As shown in Section 9 of [6] the basic scheme is dissipative, and it is not hard to verify that the rest of our basic assumptions are fulfilled as well. Yet, although condition (3.4) of Theorem 3.1 is satisfied as exhibited by (3.5b), we prove in Example 4.1 below that approximation (3.6) (3.5a) is unstable.

Despite the above observation we can strengthen Theorem 3.1 for multi-level dissipative basic schemes as follows.

THEOREM 3.2. *Let the basic scheme* (2.6a) *be dissipative. Then the outflow approximation* (2.6a, b) (3.1) *is stable if* (3.4) *holds and if*

$$R(z, \kappa = 1) \neq 0 \quad \forall |z| = 1, z \neq 1. \tag{3.7}$$

Evidently, Example (3.6) (3.5a) implies that the additional condition (3.7) is essential for Theorem 3.2.

Having stated Theorems 3.1, 3.2, we see that, when the boundary conditions (3.1) are not single-level (as in Examples 3.1), condition (3.4) may become a cumbersome inequality in two variables, z and κ. Seeking a convenient alternative to these theorems, we extend the range of μ in (3.1) to obtain the *boundary scheme*,

$$(3.8)\qquad \begin{aligned} T_{-1}v_\mu(t+k) &= \sum_{\sigma=0}^{q} T_\sigma v_\mu(t-\sigma k), \qquad \mu = 0, 1, 2, \ldots, \\ T_\sigma &= \sum_{j=0}^{m} c_{j\sigma}E^j, \qquad \sigma = -1, \ldots, r, \end{aligned}$$

and in analogy to the definitions in Assumptions 2.1, 2.2, we introduce

Definition 3.1. The boundary scheme (3.8) is said to be *solvable* if there exists a constant $K > 0$ so that for each $y \in l_2(x)$ there is a unique solution $w \in l_2(x)$ to

$$(3.9)\qquad T_{-1}w_\mu = y_\mu, \qquad \mu = 0, 1, 2, \ldots,$$

with $\|w\| \leqslant K\|y\|$.

Definition 3.2. The boundary scheme is said to fulfill the von Neumann condition if the roots $z(\xi)$ of

$$(3.10)\qquad R(z, \kappa = e^{i\xi}) \equiv \sum_{j=0}^{m} c_j(z)e^{ij\xi} = 0$$

satisfy $|z(\xi)| \leqslant 1$ for all $|\xi| \leqslant \pi$.

We can now state

THEOREM 3.3 (1st Main Theorem). *Let the basic scheme* (2.6a) *be dissipative. If* (3.7) *holds, and if the boundary scheme* (3.8) *is solvable and satisfies the von Neumann condition, then the outflow approximation* (2.6a, b) (3.1) *is stable.*

This result is an extended analogue of the main theorem (Theorem 2.2) of [2].

Useful sufficient conditions for (3.7), as well as for the solvability of the boundary scheme (3.8), are given in the next two lemmas.

LEMMA 3.1. *Condition* (3.7) *holds if any of the following is satisfied*:

(i) *The boundary conditions* (3.1) *are two-level* (*i.e.*, $q = 0$) *and accurate of order zero at least.*

(ii) *The boundary conditions are three-level, accurate of order zero at least, and in addition* $R(z = -1, \kappa = 1) \neq 0$.

LEMMA 3.2. (i) *The boundary scheme* (3.8) *is solvable if*

$$T_{-1}(\kappa) \equiv \sum_{j=0}^{q} c_{j(-1)}\kappa^j \neq 0, \qquad 0 < |\kappa| \leqslant 1.$$

(ii) *In particular, explicit boundary schemes are always solvable.*

Example 3.2. (Compare the special cases [3, (6.11)] and [2, Example 1].) Let the basic scheme (2.6a) be dissipative and determine the boundary conditions by oblique extrapolation of order $\omega - 1$:

$$(3.11\text{a})\qquad v_\mu(t+k) = \sum_{j=1}^{\omega} \binom{\omega}{j}(-1)^{j+1}v_{\mu+j}[t-(j-1)k], \qquad \mu = 0, \ldots, r-1.$$

The boundary characteristic function associated with (3.11a) is given by

$$(3.11\text{b})\qquad R(z, \kappa) = 1 - \sum_{j=1}^{\omega} \binom{\omega}{j}(-1)^{j+1}z^{-j}\kappa^j = (1 - z^{-1}\kappa)^\omega,$$

so obviously (3.7) holds. Further, the roots of (3.11b) satisfy

$$|z(\kappa = e^{i\xi})| = |e^{i\xi}| = 1.$$

Thus, the associated boundary scheme–which by Lemma 3.2 (ii) is solvable–fulfills the von Neumann condition, and, by Theorem 3.3, (2.6a, b) (3.11a) is stable.

Example 3.3. (Compare the special cases [3, (6.3c)], [9, (3.4)] and [4, Example 4].) Let the basic scheme (2.6a) be dissipative and let the boundary conditions be generated by the second order accurate Box-Scheme

$$\begin{aligned} &v_\mu(t+k) + v_{\mu+1}(t+k) - \lambda a\big[v_{\mu+1}(t+k) - v_\mu(t+k)\big] \\ &\qquad = v_\mu(t) + v_{\mu+1}(t) + \lambda a\big[v_{\mu+1}(t) - v_\mu(t)\big], \qquad \mu = 0, \ldots, r-1. \end{aligned} \tag{3.12}$$

By Lemma 3.1(i), (3.7) is fulfilled, and, since

$$\operatorname{Re}\big[T_{-1}(\kappa)\big] = 1 + \operatorname{Re}(\kappa) + \lambda a\big[1 - \operatorname{Re}(\kappa)\big] \neq 0, \qquad |\kappa| \leqslant 1,$$

then by Lemma 3.2(i), the boundary scheme is solvable. The boundary characteristic function is

$$R(z, \kappa) = 1 + \kappa - \lambda a(\kappa - 1) - z^{-1}\big[1 + \kappa + \lambda a(\kappa - 1)\big],$$

so its root satisfies

$$|z(\kappa = e^{i\xi})| = \left|\frac{1 + e^{i\xi} + \lambda a(e^{i\xi} - 1)}{1 + e^{i\xi} - \lambda a(e^{i\xi} - 1)}\right| = 1, \qquad |\xi| \leqslant \pi;$$

hence, the von Neumann condition holds as well, and Theorem 3.3 implies stability.

Example 3.4. (Compare the special case [10, (6.26)].) Let the basic scheme (2.6a) be dissipative and define the boundary conditions by the right-sided weighted Euler scheme

$$\begin{aligned} &v_\mu(t+k) = v_\mu(t-k) + \lambda a\big[2v_{\mu+1}(t) - v_\mu(t+k) - v_\mu(t-k)\big], \\ &\qquad\qquad 0 < \lambda a \leqslant 1, \mu = 0, \ldots, r-1. \end{aligned} \tag{3.13}$$

The characteristic function for (3.13) is

$$R(z, \kappa) = 1 - z^{-2} - \lambda a(2\kappa z^{-1} - 1 - z^{-2}),$$

and by equating to zero we find its roots

$$z(\kappa = e^{i\xi}) = e^{i\xi}\frac{\lambda a \pm b(\xi)}{\lambda a + 1}, \qquad b(\xi) \equiv \sqrt{(\lambda a)^2 + e^{-2i\xi}\big[1 - (\lambda a)^2\big]}\ ;$$

hence

$$z(\kappa = 1) = \frac{\lambda a \pm 1}{\lambda a + 1} \neq e^{i\varphi}, \qquad 0 < |\varphi| \leqslant \pi,$$

and (3.7) holds. In addition, since $0 < \lambda a \leqslant 1$, then $|b(\xi)| \leqslant 1$, $|\xi| \leqslant \pi$; so

$$|z(\kappa = e^{i\xi})| \leqslant \frac{\lambda a + |b(\xi)|}{\lambda a + 1} \leqslant 1, \qquad |\xi| \leqslant \pi,$$

and the boundary scheme satisfies the von Neumann condition. Finally, since the boundary scheme is explicit, Lemma 3.2(ii) implies solvability, and by Theorem 3.3 stability follows.

We remark that solvability of the boundary scheme is necessary for Theorem 3.3. To see this, consider any dissipative basic scheme with zero-order accurate boundary conditions of the form

$$\text{(3.14a)}\qquad \begin{aligned} &v_\mu(t+k) - \theta v_{\mu+1}(t+k) = v_\mu(t) - \theta v_{\mu+1}(t), \\ &\qquad\qquad\qquad\qquad \theta > 1,\ \mu = 0, \ldots, r-1. \end{aligned}$$

By Lemma 3.1(i), (3.7) is fulfilled. Also, the boundary characteristic function is

$$\text{(3.14b)}\qquad R(z, \kappa) = (1 - z^{-1})(1 - \theta\kappa),$$

hence its single root, $z = 1$, satisfies the von Neumann condition. As shown in Example 4.2, however, the approximation is unstable, which is explained by the failure of the associated boundary scheme to be solvable. Indeed, taking $y = 0$ in (3.9), we find that the grid function $w = \{\theta^{-\mu} w_0\}_{\mu=0}^{\infty}$, with arbitrary w_0, belongs to $l_2(x)$ and satisfies (3.9); thus we have neither the uniqueness nor the boundedness of w required by Definition 3.1.

Condition (3.7) is also necessary for Theorem 3.3 as can be shown by taking (3.6) with $0 < \lambda a < \frac{1}{2}$ and consistent boundary conditions of the form

$$\text{(3.15a)}\qquad v_\mu(t+k) = v_\mu(t-k) + 2\lambda a\left[v_{\mu+1}(t-k) - v_\mu(t-k)\right], \qquad \mu = 0, 1.$$

As mentioned before, the basic scheme is dissipative, and by Lemma 3.2(ii) the boundary scheme is solvable. The boundary characteristic function is

$$\text{(3.15b)}\qquad R(z, k) = 1 - z^{-2}\left[1 + 2\lambda a(\kappa - 1)\right],$$

so it is not hard to verify that the boundary scheme satisfies the von Neumann condition (and is, in fact, even dissipative). Yet, as demonstrated by Example 4.3 below, (3.6) (3.15a) is unstable. The reason Theorem 3.3 does not apply in this case is that $R(z = -1, \kappa = 1) = 0$, i.e., (3.7) is violated.

So far we have treated, in this section, the case where the basic scheme is dissipative. For the general case, where the basic scheme might also be unitary, we need

Definition 3.3. The boundary scheme (3.8) is said to be *dissipative* if the roots of Eq. (3.10) satisfy $|z(\xi)| < 1$ for $0 < |\xi| \leqslant \pi$.

This enables us to state

THEOREM 3.4 (2nd Main Theorem). *Let the basic scheme* (2.6a) *be dissipative or unitary, let* (3.7) *hold, and let the boundary scheme* (3.8) *be solvable and dissipative. Then the outflow approximation* (2.6a, b) (3.1) *is stable.*

Example 3.5. (Compare the special cases [3, (6.3a)], [8, (3.2)] and [2, Example 2].) Let the basic scheme (2.6a) be dissipative or unitary and let the boundary conditions be generated by the right-sided explicit Euler scheme

$$\text{(3.16)}\qquad \begin{aligned} &v_\mu(t+k) = v_\mu(t) + \lambda a\left[v_{\mu+1}(t) - v_\mu(t)\right], \\ &\qquad\qquad\qquad\qquad 0 < \lambda a < 1,\ \mu = 0, \ldots, r-1. \end{aligned}$$

The boundary characteristic function is now

$$R(z, \kappa) = 1 - z^{-1}\left[1 + \lambda a(\kappa - 1)\right],$$

and, since $0 < \lambda a < 1$, its root satisfies

$$|z(\kappa = e^{i\xi})|^2 = (\lambda a)^2 + (1 - \lambda a)^2 + 2\lambda a(1 - \lambda a)\cos\xi$$
$$< (\lambda a)^2 + (1 - \lambda a)^2 + 2\lambda a(1 - \lambda a) = 1, \qquad 0 < |\xi| \leqslant \pi;$$

hence the corresponding boundary scheme is dissipative. Moreover, since (3.16) is two-level, first order accurate and explicit, Lemmas 3.1(i) and (3.2)(ii) imply that (3.7) holds and that the boundary scheme is solvable. The hypotheses of Theorem 3.4 are fulfilled therefore, and approximation (2.6a, b) (3.16) is stable.

Example 3.6. (Compare the special cases [8, (3.3)] and [2, Example 3].) Let the basic scheme (2.6a) be dissipative or unitary, and define the boundary conditions by the right-sided, first order accurate, implicit Euler scheme:

$$\begin{aligned} &v_\mu(t + k) - \lambda a\left[v_{\mu+1}(t + k) - v_\mu(t + k)\right] = v_\mu(t), \\ &\qquad\qquad\qquad\qquad \lambda a > 0,\ \mu = 0, \ldots, r - 1. \end{aligned} \tag{3.17}$$

The characteristic function associated with (3.17) is given by

$$R(z, \kappa) = 1 - \lambda a(\kappa - 1) - z^{-1},$$

so its root satisfies

$$|z(\kappa = e^{i\xi})|^2 = \left[(\lambda a)^2 + (1 + \lambda a)^2 - 2\lambda a(1 + \lambda a)\cos\xi\right]^{-1}$$
$$< \left[(\lambda a)^2 + (1 + \lambda a)^2 - 2\lambda a(1 + \lambda a)\right]^{-1} = 1, \qquad 0 < |\xi| \leqslant \pi,$$

and the boundary scheme is dissipative. Also, Lemma 3.1(i) implies (3.7), and, since

$$\operatorname{Re}\left[T_{-1}(\kappa)\right] = 1 + \lambda a\left[1 - \operatorname{Re}(\kappa)\right] \neq 0, \qquad |\kappa| \leqslant 1,$$

then by Lemma 3.2(i), the boundary scheme is solvable. Thus, Theorem 3.4 applies and stability is assured.

In concluding this section, we claim that condition (3.7) is necessary for Theorem 3.4 (as well as for Theorem 3.3). For example, consider the nondissipative, 3-level Leap-Frog scheme

$$\begin{aligned} &v_\nu(t + k) = v_\nu(t - k) + \lambda a\left[v_{\nu+1}(t) - v_{\nu-1}(t)\right], \\ &\qquad\qquad\qquad\qquad 0 < \lambda a < \tfrac{1}{2},\ \nu = 1, 2, \ldots, \end{aligned} \tag{3.18}$$

where for the boundary condition we take (3.15a) with $\mu = 0$. As mentioned earlier, the corresponding boundary scheme is solvable and dissipative whereas (3.7) is violated. In Example 4.4 we show that approximation (3.18) (3.15a) is unstable, thus proving our claim.

We conjecture that the solvability and dissipitativity of the boundary scheme are essential for Theorem 3.4.

4. A Preliminary Stability Criterion. In this section we use the theory of Gustafsson et al. [3] to obtain in Theorem 4.2 below, a preliminary stability criterion for approximation (2.6). This criterion will be a major tool in proving Lemma 2.3 and Theorems 3.1–3.4.

Following [3], we associate with approximation (2.6) the *resolvent equation*

$$\left(Q_{-1} - \sum_{\sigma=0}^{s} z^{-\sigma-1}Q_\sigma\right)w_\nu = 0, \qquad \nu = r, r + 1, \ldots, \tag{4.1a}$$

with boundary conditions

(4.1b) $$\left(T_{-1}^{(\mu)} - \sum_{\sigma=0}^{q} z^{-\sigma-1} T_{\sigma}^{(\mu)}\right) w_{\mu} = 0, \qquad \mu = 0, \ldots, r-1,$$

where $z \neq 0$ is a fixed arbitrary complex number. This can be written as

(4.2) $$G(z)w = 0,$$

where $G(z)$ is a linear bounded operator on $l_2(x)$ defined by the left-hand sides of (4.1). We say that z is an eigenvalue of approximation (2.6) if (4.2) has a nontrivial solution $w = w(z) \in l_2(x)$. If z is not an eigenvalue but there exists a sequence $\{w(z)^{(j)}\}_{j=1}^{\infty} \subseteq l_2(x)$ with $\|w(z)^{(j)}\| = 1$ such that

$$G(z)w(z)^{(j)} \underset{j\to\infty}{\to} 0,$$

we call z a generalized eigenvalue of approximation (2.6).

Having these definitions, we restate the main result of [3] in the language of [5]:

THEOREM 4.1 (Gustafsson et al. [3]). *Approximation* (2.6) *is stable if and only if it has no eigenvalues nor generalized eigenvalues z with $|z| \geq 1$.*

Seeking a practical version of Theorem 4.1, we first need the following characterization of solvability.

LEMMA 4.1 (essentially Osher, [8]). *Approximation* (2.6) *is solvable if and only if*

(i) *The difference equations*

$$Q_{-1}w_{\nu} = 0, \quad \nu = r, r+1, \ldots,$$
$$T_{-1}^{(\mu)}w_{\mu} = 0, \quad \mu = 0, \ldots, r-1,$$

have no nontrivial solution $w \in l_2(x)$.

(ii) *The equation*

(4.3a) $$Q_{-1}(\kappa) \equiv \sum_{j=-r}^{p} a_{j(-1)}\kappa^{j} = 0$$

has precisely r_0 solutions κ_j with $0 < |\kappa_j| < 1$ where

(4.3b) $$r_0 = \max\{-j\colon a_{j(-1)} \neq 0\}.$$

(iii) *$Q_{-1}(\kappa)$ does not vanish on the unit circle, i.e.,*

$$Q_{-1}(\kappa) \neq 0, \qquad |\kappa| = 1.$$

Proof. Conditions (i) and (iii) coincide with Osher's conditions (d) and (g) in [8]. Regarding (ii), we note that $r_0 \geq 0$ (or else the basic scheme is unnaturally shifted to the right). Hence,

$$Q_{-1}(\kappa) = \kappa^{-r_0} \sum_{j=-r_0}^{p} a_{j(-1)}\kappa^{j+r_0}$$

has a single pole of order r_0 at the origin. Since, by (iii), $Q_{-1}(\kappa)$ does not vanish on the unit circle, we use the Argument Principle to find that there is no change in $\arg[Q_{-1}(\kappa = e^{i\xi})]$ as ξ varies from $-\pi$ to π if and only if (ii) holds. Thus, conditions (i)–(iii) are equivalent to (d), (e), (g) in [8], and Theorem I of [8] completes the proof.

Recalling the functions $a_j(z)$ in (2.10), we introduce now the *characteristic equation* of the basic scheme (2.6a)

$$P(z, \kappa) \equiv \sum_{j=-r}^{p} a_j(z)\kappa^j = 0, \tag{4.4}$$

whose $r + p$ roots $\kappa_j(z)$ play a central role in determining the eigenvalues of approximation (2.6).

LEMMA 4.2 (compare Lemmas 5.1 and 5.2, [3]). *For* $|z| > 1$, *the characteristic equation* (4.4) *has precisely* r *roots with* $0 < |\kappa_j(z)| < 1$, p *roots with* $|\kappa_j(z)| > 1$, *and no roots with* $|\kappa_j(z)| = 1$.

Proof. By Assumption 2.4, the leading coefficients of $P(z, \kappa)$ do not vanish for $|z| > 1$; hence (4.4) has $r + p$ roots, all satisfying $|\kappa_j(z)| > 0$. Since these roots are the solutions of the polynomial equation

$$P_r(z, \kappa) \equiv \kappa^r P(z, \kappa) = 0, \tag{4.5}$$

we may study (4.5) rather than (4.4).

By Assumption 2.2(i),

$$P_r(z, \kappa = e^{i\xi}) \neq 0, \qquad |z| > 1, |\xi| \leqslant \pi,$$

i.e., for $|z| > 1$, $P_r(z, \kappa)$ does not vanish on the unit circle $|\kappa| = 1$. Since the roots of $P_r(z, \kappa)$ are continuous functions of z, it follows that for $|z| > 1$ the number of roots satisfying $0 < |\kappa_j(z)| < 1$ is independent of z. In particular, consider the limit case

$$P_r(z \to \infty, \kappa) = \sum_{j=-r}^{p} a_{j(-1)}\kappa^{j+r}.$$

By Lemma 4.1(iii),

$$P_r(z \to \infty, \kappa = e^{i\xi}) \neq 0, \qquad |\xi| \leqslant \pi,$$

so, by continuity again, the number of roots of $P_r(z, \kappa)$ satisfying $0 < |\kappa_j(z)| < 1$ may be determined by counting the roots κ_j, $|\kappa_j| < 1$, of $P_r(z \to \infty, \kappa)$. We have

$$P_r(z \to \infty, \kappa) = \kappa^{r-r_0} \sum_{j=-r_0}^{p} a_{j(-1)}\kappa^{j+r_0},$$

where r_0 is defined in (4.3b). Moreover, by Lemma 4.1(ii), $\sum_{j=-r_0}^{p} a_{j(-1)}\kappa^j$, hence $\sum_{j=-r_0}^{p} a_{j(-1)}\kappa^{j+r_0}$ have precisely r_0 roots with $|\kappa| < 1$. Thus, with its additional $r - r_0$ zeros, $P_r(z \to \infty, \kappa)$ has r roots with $|\kappa| < 1$ and the lemma follows.

According to the above lemma, the roots of the characteristic equation (4.4) split for $|z| > 1$ into two groups: r *inner roots* satisfying $0 < |\kappa_j(z)| < 1$ and p *outer roots* with $|\kappa_j(z)| > 1$. By continuity, therefore, these groups of inner and outer roots remain well defined for $|z| \geqslant 1$ as well, where milder inequalities, $|\kappa_j(z)| \leqslant 1$ and $|\kappa_j(z)| \geqslant 1$, hold, respectively. Since, by Assumption 2.4, $\kappa = 0$ is never a root of $P(z, \kappa)$ for $|z| \geqslant 1$, we finally obtain

LEMMA 4.3. *For* $|z| \geqslant 1$, *the* $r + p$ *roots* $\kappa_j(z)$ *of the characteristic equation* (4.4) *split into* r *inner roots with* $0 < |\kappa_j(z)| \leqslant 1$ *and* p *outer roots with* $|\kappa_j(z)| \geqslant 1$.

Now, let z be given. It is well known (e.g. [5]) that z is an eigenvalue or a generalized eigenvalue of approximation (2.6) if and only if Eqs. (4.1) have a nontrivial solution of the form

$$w_\nu = \sum_{\alpha=1}^{N} \sum_{\beta=0}^{M_\alpha - 1} \tau_{\alpha\beta}\Phi_{\alpha\beta}(\nu)\kappa_\alpha(z)^\nu, \qquad \nu = 0, 1, 2, \ldots, \tag{4.6}$$

where $\kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant N$, are the distinct inner roots of the characteristic equation (4.4) each with multiplicity $M_\alpha = M_\alpha(z)$. Here, $\Phi_{\alpha\beta}(\nu)$ are arbitrary polynomials in ν with $\deg[\Phi_{\alpha\beta}(\nu)] = \beta$, and $\tau_{\alpha\beta}$ are coefficients whose number, by Lemma 4.3, is precisely

$$\sum_{\alpha=1}^{N} M_\alpha = r.$$

To find the $\tau_{\alpha\beta}$, we substitute (4.6) in (4.1b) and obtain a linear homogeneous system of r equations with r unknowns,

$$J(z)\boldsymbol{\tau}' = 0, \tag{4.7}$$

where $J(z)$ is an $r \times r$ coefficient matrix and $\boldsymbol{\tau} = (\tau_{\alpha\beta})$ is the unknown vector. Obviously, $w = w(z)$ in (4.6) does not vanish if and only if (4.7) has a nontrivial solution $\boldsymbol{\tau}$; hence, z is an eigenvalue or a generalized eigenvalue of approximation (2.6) if and only if $\det J(z) \neq 0$.

This observation combined with Theorem 4.1 gives the following equivalent of Lemma 10.3 of [3]:

LEMMA 4.4 (Gustafsson et al. [3]). *Approximation* (2.6) *is stable if and only if*

$$\det J(z) \neq 0, \qquad |z| \geqslant 1.$$

Now for z, $|z| \geqslant 1$, with corresponding distinct inner roots $\kappa_\alpha = \kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant N$, each with multiplicity M_α, we make a specific choice for the polynomials $\Phi_{\alpha\beta}(\nu)$ in (4.6):

$$\Phi_{\alpha\beta}(\nu) = \kappa_\alpha(z)^{-\beta}\beta!\binom{\nu}{\beta}.$$

Thus, (4.6) becomes

$$w_\nu = \sum_{\alpha=1}^{N} \sum_{\beta=0}^{M_\alpha - 1} \tau_{\alpha\beta}\beta!\binom{\nu}{\beta}\kappa_\alpha^{\nu-\beta}, \qquad \nu = 0, 1, 2, \ldots,$$

and substituting in (4.1b)–with $T_\sigma^{(\mu)}$ given by either (2.8) or (2.9)–, we obtain explicit expressions for the system $J(z)\boldsymbol{\tau}' = 0$, namely

$$\sum_{\alpha=1}^{N} \sum_{\beta=0}^{M_\alpha - 1} \beta!\binom{\mu}{\beta}\kappa_\alpha^{\mu-\beta}\tau_{\alpha\beta} = 0, \qquad \mu = 0, \ldots, r-1, \tag{4.8}$$

or

$$\sum_{\alpha=1}^{N} \sum_{\beta=0}^{M_\alpha - 1} \sum_{j=0}^{m} \left(c_{j(-1)}^{(\mu)} - \sum_{\sigma=0}^{q} z^{-\sigma-1}c_{j\sigma}^{(\mu)}\right)\beta!\binom{\mu+j}{\beta}\kappa_\alpha(z)^{\mu+j-\beta}\tau_{\alpha\beta} = 0, \tag{4.9}$$

$$\mu = 0, \ldots, r-1,$$

according to the inflow and outflow cases ($a < 0$ or $a > 0$), respectively. To further simplify these expressions, we use the boundary coefficients to introduce r *boundary functions*

$$
(4.10)\qquad R_\mu(z,\kappa) \equiv \begin{cases} \kappa^\mu, & a < 0, \\ \displaystyle\sum_{j=0}^{m}\left(c_{j(-1)}^{(\mu)} - \sum_{\sigma=0}^{q} z^{-\sigma-1} c_{j\sigma}^{(\mu)}\right)\kappa^{\mu+j}, & a > 0, \end{cases}
$$

$$\mu = 0, \ldots, r-1.$$

Since

$$
\frac{\partial^\beta R_\mu(z,\kappa)}{\partial\kappa^\beta} = \begin{cases} \beta!\dbinom{\mu}{\beta}\kappa^{\mu-\beta}, & a < 0, \\ \displaystyle\sum_{j=0}^{m}\left(c_{j(-1)}^{(\mu)} - \sum_{\sigma=0}^{q} z^{-\sigma-1} c_{j\sigma}^{(\mu)}\right)\beta!\dbinom{\mu+j}{\beta}\kappa^{\mu+j-\beta}, & a > 0, \end{cases}
$$

systems (4.8) and (4.9) both take the unified compact form

$$
\sum_{\alpha=1}^{N}\sum_{\beta=0}^{M_\alpha-1}\left[\frac{\partial^\beta R_\mu(z,\kappa)}{\partial\kappa^\beta}\right]_{\kappa=\kappa_\alpha(z)} \cdot \tau_{\alpha\beta} = 0, \qquad \mu = 0, \ldots, r-1,
$$

so the coefficient matrix $J(z)$ can be conveniently written as

$$
(4.11a)\qquad J(z) = \left[H(z,\kappa_1,M_1), \ldots, H(z,\kappa_N,M_N)\right],
$$

where $H(z, \kappa_\alpha, M_\alpha)$, $1 \leqslant \alpha \leqslant N$, are the $r \times M_\alpha$ blocks

$$
(4.11b)\qquad H(z,\kappa_\alpha,M_\alpha) = \left[\begin{bmatrix} R_0(z,\kappa) \\ R_1(z,\kappa) \\ \vdots \\ R_{r-1}(z,\kappa)\end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} R_0(z,\kappa) \\ R_1(z,\kappa) \\ \vdots \\ R_{r-1}(z,\kappa)\end{bmatrix}, \ldots, \frac{\partial^{M_\alpha-1}}{\partial\kappa^{M_\alpha-1}}\begin{bmatrix} R_0(z,\kappa) \\ R_1(z,\kappa) \\ \vdots \\ R_{r-1}(z,\kappa)\end{bmatrix}\right]_{\kappa=\kappa_\alpha(z)}.
$$

This expression for $J(z)$ gives us a concrete analogue of Lemma 4.4:

LEMMA 4.5. *Approximation* (2.6) *is stable if and only if for every* z, $|z| > 1$,

$$
(4.12)\qquad \det J(z) \equiv \det\left[H(z,\kappa_1,M_1), \ldots, H(z,\kappa_N,M_N)\right] \neq 0,
$$

where $\kappa_\alpha = \kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant N$, *are the distinct inner roots of* (4.4), *each with multiplicity* $M_\alpha = M_\alpha(z)$.

A milder version of this theorem is given in Theorem 3.2, [2].

We return now to the case where the inflow boundary conditions are, as before, homogeneous and given by (2.6c) (2.8), whereas the outflow conditions (2.6c) (2.9) are translatory as described in (3.1). In this case–where both the inflow and outflow conditions are of translatory type–the boundary functions in (4.10) become

$$
(4.13)\qquad R_\mu(z,\kappa) = \begin{cases} \kappa^\mu, & a < 0, \\ \displaystyle\sum_{j=0}^{m}\left(c_{j(-1)} - \sum_{\sigma=0}^{q} z^{-\sigma-1} c_{j\sigma}\right)\kappa^{\mu+j}, & a > 0, \end{cases}
$$

$$\mu = 0, \ldots, r-1.$$

Hence, denoting

$$R(z, \kappa) \equiv R_0(z, \kappa)$$

(note: for $a > 0$ this definition coincides with the one in (3.2)), we find that

$$R_\mu(z, \kappa) = \kappa^\mu R(z, \kappa), \qquad \mu = 0, \ldots, r-1,$$

so the blocks of $J(z)$ in (4.11b) become

$$H(z, \kappa_\alpha, M_\alpha)$$

$$= \left[\begin{bmatrix} R(z,\kappa) \\ \kappa R(z,\kappa) \\ \vdots \\ \kappa^{r-1} R(z,\kappa) \end{bmatrix}, \frac{\partial}{\partial \kappa} \begin{bmatrix} R(z,\kappa) \\ \kappa R(z,\kappa) \\ \vdots \\ \kappa^{r-1} R(z,\kappa) \end{bmatrix}, \ldots, \frac{\partial^{M_\alpha - 1}}{\partial \kappa^{M_\alpha - 1}} \begin{bmatrix} R(z,\kappa) \\ \kappa R(z,\kappa) \\ \vdots \\ \kappa^{r-1} R(z,\kappa) \end{bmatrix} \right]_{\kappa = \kappa_\alpha}. \tag{4.14}$$

This allows us to simplify Lemma 4.5 as follows.

THEOREM 4.2 (compare Theorem 4.1, [2]). *Approximation* (2.6), *with translatory boundary conditions given by either* (3.1) *for $a > 0$ or* (2.8) *for $a < 0$, is stable if and only if for every z, $|z| \geqslant 1$, with corresponding inner roots $\kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant N$, we have*

$$R(z, \kappa_\alpha) \neq 0, \qquad \alpha = 1, \ldots, N. \tag{4.15}$$

Proof. Suppose $R(z, \kappa_\alpha) = 0$ for some z, $|z| \geqslant 1$, with a corresponding inner root κ_α. Then, clearly, the first column of $H(z, \kappa_\alpha, M_\alpha)$ in (4.14) vanishes; thus $\det J(z) = 0$, and by Lemma 4.5 we have instability.

Conversely, let (4.15) hold and take an arbitrary z, $|z| \geqslant 1$, with distinct inner roots $\kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant N$. To prove stability it suffices, by Lemma 4.5, to verify that the rows of

$$J(z) = \left[H(z, \kappa_1, M_1), \ldots, H(z, \kappa_N, M_N) \right] \tag{4.16}$$

are linearly independent, where the $H(z, \kappa_\alpha, M_\alpha)$ are given by (4.14). For that purpose, let

$$\sum_{\mu=0}^{r-1} \gamma_\mu \begin{bmatrix} \kappa_1^\mu R(z, \kappa_1) \\ \vdots \\ \partial^{M_N - 1}\left[\kappa_N^\mu R(z, \kappa_N) \right] / \partial \kappa_N^{M_N - 1} \end{bmatrix}' = 0 \tag{4.17}$$

be a vanishing linear combination of the rows of (4.16), and let us rewrite (4.17) as r scalar equations

$$\frac{\partial^j}{\partial \kappa^j} \left\{ R(z, \kappa) \left[\sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu \right] \right\}_{\kappa = \kappa_\alpha} = 0, \qquad 1 \leqslant \alpha \leqslant N, 0 \leqslant j \leqslant M_\alpha - 1. \tag{4.18}$$

Since, by hypothesis,

$$R(z, \kappa)|_{\kappa = \kappa_\alpha} \neq 0, \qquad 1 \leqslant \alpha \leqslant N,$$

we expand the partial derivatives in (4.18) by Leibnitz' rule and use induction on $j \geqslant 0$ to find that the sum in (4.18) must have vanishing derivatives, i.e.,

$$\frac{d^j}{d\kappa^j}\left[\sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu\right]_{\kappa=\kappa_\alpha} = 0, \qquad 1 \leqslant \alpha \leqslant N, 0 \leqslant j \leqslant M_{\alpha-1}.$$

Consequently, the polynomial

$$\Psi(\kappa) \equiv \sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu,$$

which is of degree $r - 1$ at most, has r roots (κ_α, $1 \leqslant \alpha \leqslant N$, each with multiplicity M_α), so $\Psi(\kappa) \equiv 0$ and the coefficients γ_μ must vanish. By (4.17), therefore, the rows of (4.16) are linearly independent and stability follows.

The proof of Lemma 2.3 and the counterexamples of Section 3 are almost at hand now.

Proof of Lemma 2.3. By (4.13), the boundary function associated with the homogeneous boundary conditions (2.6c) (2.8) is

$$R(z, \kappa) \equiv R_0(z, \kappa) = 1.$$

Thus, (4.15) holds trivially, and by the last theorem approximation (2.6) (2.8) is stable.

Example 4.1. Consider the dissipative basic scheme (3.6) with the boundary conditions in (3.5a). The boundary function is given by (3.5b) and for $z = -1$ it can be shown (as in Lemma 6.2, [3]) that the characteristic equation has exactly one inner root satisfying $\kappa(z = -1) = 1$. Hence, $R(z = 1, \kappa = 1) = 0$, and by Theorem 4.2 we have instability.

Example 4.2. Take the zero-order accurate boundary conditions (3.14a) in combination with any basic scheme (dissipative or even unitary). By (3.14b), $R(z = 1, \kappa) = 0$ for all κ; so at $z = 1$ the characteristic boundary function vanishes for all inner roots, and Theorem 4.2 implies instability.

Example 4.3. Take the same basic scheme as in Example 4.1 with the boundary conditions in (3.15a). As in Example 4.1, we have an inner root $\kappa(z = -1) = 1$ for which, by (3.15b), $R(z, \kappa) = 0$. Hence, (4.15) is violated and approximation (3.6) (3.15) is unstable.

Example 4.4. Consider the Leap-Frog scheme (3.18) with a boundary condition as in Example 4.3. In Lemma 6.2 [3] it is shown that the characteristic equation of (3.18) has a single inner root $\kappa(z = -1) = 1$. So as in the previous example, $R(z = -1, \kappa = 1) = 0$, and by Theorem 4.2 instability follows.

5. Proof of Main Results. We turn now to prove the results stated in Section 3, beginning with the following lemma.

LEMMA 5.1. *For* $z = 1$, *the characteristic equation* (4.4) *has exactly one root satisfying* $\kappa(z = 1) = 1$. *In the outflow case* ($a > 0$) *this is always an outer root.*

Proof. Since the basic scheme (2.6a) is consistent with

$$\partial u/\partial t = a\partial u/\partial x, \qquad a \neq 0,$$

the coefficients $a_{j\sigma}$ must satisfy the ordinary consistency conditions

$$\sum_{j=-r}^{p} a_{j(-1)} = \sum_{\sigma=0}^{s} \sum_{j=-r}^{p} a_{j\sigma}$$

and

$$\sum_{j=-r}^{p} j a_{j(-1)} = \sum_{\sigma=0}^{s} \sum_{j=-r}^{p} j a_{j\sigma} - \lambda a \sum_{\sigma=0}^{s} (\sigma + 1) \sum_{j=-r}^{p} a_{j\sigma},$$

which can be written as

$$\sum_{j=-r}^{p} a_j(z)|_{z=1} = 0, \tag{5.1a}$$

$$\sum_{j=-r}^{p} j a_j(z)|_{z=1} = -\lambda a \sum_{j=-r}^{p} \frac{d}{dz} a_j(z)|_{z=1} \tag{5.1b}$$

or, equivalently, as

$$P(z, \kappa)|_{z=\kappa=1} = 0, \tag{5.2a}$$

$$\frac{\partial}{\partial \kappa} P(z, \kappa)|_{z=\kappa=1} = -\lambda a \frac{\partial}{\partial z} P(z, \kappa)|_{z=\kappa=1}. \tag{5.2b}$$

Here, $a_j(z)$ and $P(z, \kappa)$ are defined in (2.10) and (4.4), respectively.

By (5.1a), $z = 1$ is a solution of

$$\sum_{j=-r}^{p} a_j(z) = 0,$$

and by Assumption 2.2(ii) this solution is simple. Hence,

$$\frac{d}{dz} \sum_{j=-r}^{p} a_j(z)|_{z=1} \neq 0,$$

so by (5.2b)

$$\begin{aligned} \frac{\partial}{\partial \kappa} P(z, \kappa)|_{z=\kappa=1} &= -\lambda a \frac{\partial}{\partial z} P(z, \kappa)|_{z=\kappa=1} \\ &\equiv -\lambda a \frac{d}{dz} \sum_{j=-r}^{p} a_j(z)|_{z=1} \neq 0. \end{aligned} \tag{5.2c}$$

Having (5.2a) and (5.2c), we employ the implicit function theorem to find that in the neighborhood of $z = 1$ the characteristic equation (4.4) can be uniquely solved for κ as a differentiable function of z such that

$$\kappa(z = 1) = 1. \tag{5.3a}$$

This is the first part of the lemma.

To complete the proof, consider the outflow case $a > 0$. Then (5.2b) yields

$$\left.\frac{d\kappa(z)}{dz}\right|_{z=1} = \left[-\frac{\partial P}{\partial z} \Big/ \frac{\partial P}{\partial \kappa} \right]_{z=\kappa=1} = \frac{1}{\lambda a} > 0; \tag{5.3b}$$

so by (5.3a) (5.3b), for $z = 1 + \varepsilon$ with sufficiently small $\varepsilon > 0$,

$$\kappa(z) = 1 + (\lambda a)^{-1}\varepsilon + O(\varepsilon^2) > 1.$$

That is, for z in the right real neighborhood of $z = 1$,

$$|\kappa(z)| > 1,$$

and by Lemma 4.2 this inequality is valid for all z, $|z| > 1$. By definition, therefore, $\kappa(z)$ of (5.3a) is an outer root of (4.4) and the lemma follows.

Proof of Theorem 3.1. Take an arbitrary z, $|z| \geqslant 1$, and let $\kappa_\alpha = \kappa_\alpha(z)$ be a corresponding inner root. In order to prove stability it suffices, by Theorem 4.2, to show that

$$R(z, \kappa_\alpha) \neq 0. \tag{5.4}$$

By Lemma 4.3, we have $0 < |\kappa_\alpha(z)| \leqslant 1$, where for $0 < |\kappa_\alpha(z)| < 1$, (5.4) is implied by (3.4). Hence, we may restrict attention to inner roots on the unit circle, i.e., $\kappa_\alpha(z) = e^{i\xi}$, $|\xi| \leqslant \pi$. Since the basic scheme is dissipative, then by (2.12) the solutions $z = z(\kappa)$ of (4.4) satisfy

$$|z(\kappa = e^{i\xi})| < 1, \qquad 0 < |\xi| \leqslant \pi. \tag{5.5}$$

Thus, for $|z| \geqslant 1$, (4.4) has no roots $\kappa = e^{i\xi}$, $0 < |\xi| \leqslant \pi$, and our discussion is further reduced to $|z| \geqslant 1$, $\kappa_\alpha(z) = 1$. Next, by continuity, (5.5) yields

$$|z(\kappa = 1)| \leqslant 1;$$

so $\kappa = 1$ is ruled out as an inner root for $|z| > 1$ and it remains to consider $|z| = 1$, $\kappa_\alpha(z) = 1$. Finally, by Lemma 5.1, $\kappa = 1$ is excluded as an inner root for $z = 1$, and we are left with

$$|z| = 1, \quad z \neq 1, \quad \kappa_\alpha(z) = 1. \tag{5.6}$$

Since the basic scheme is consistent, then by (5.2a)

$$P(z = 1, \kappa = 1) = 0. \tag{5.7}$$

Moreover, since the basic scheme is two-level, $P(z, \kappa)$ is a polynomial of first degree in z^{-1} where, by (5.7), its only root is $z^{-1} = 1$. Thus,

$$P(z, \kappa = 1) \neq 0, \qquad |z| = 1, z \neq 1,$$

and the proof is complete.

For Theorem 3.2 we repeat the previous proof to the point where the remaining values of z and $\kappa_\alpha(z)$ to be studied are given in (5.6). At this point, (3.7) implies (5.4) and stability is assured.

To prove Theorems 3.3 and 3.4 we need yet another result.

LEMMA 5.2. *Let the boundary scheme* (3.8) *be solvable and satisfy the von Neumann condition. Then,*

$$R(z, \kappa) \neq 0, \qquad |\kappa| \leqslant 1, |z| > 1, \tag{5.8}$$

and

$$R(z, \kappa) \neq 0, \qquad |\kappa| < 1, |z| \geqslant 1. \tag{5.9}$$

Proof. Apply Lemma 4.1 to the solvable boundary scheme, rather than to the basic scheme, with Eq. (4.3a) replaced by its boundary counterpart

$$T_{-1}(\kappa) \equiv \sum_{j=0}^{m} c_{j(-1)}\kappa^j = 0.$$

Since the boundary scheme is one-sided where by (3.1) $c_{j(-1)} \neq 0$, then parts (ii)–(iii) of the lemma imply that

$$T_{-1}(\kappa) \neq 0, \qquad 0 < |\kappa| \leqslant 1.$$

We also have

$$T_{-1}(0) = c_{0(-1)} \neq 0;$$

so all told

$$(5.10) \qquad T_{-1}(\kappa) \neq 0, \qquad |\kappa| \leqslant 1.$$

Since the boundary scheme satisfies the von Neumann condition then by (3.10), for $|z| > 1$, $R(z, \kappa)$ does not vanish on the unit circle $|\kappa| = 1$. Hence–as in the proof of Lemma 4.2–the number of roots of $R(z, \kappa)$ satisfying $|\kappa(z)| \leqslant 1$ is independent of z, $|z| > 1$, and, by continuity, equals that of the roots κ, $|\kappa| \leqslant 1$, of

$$R(z \to \infty, \kappa) \equiv T_{-1}(\kappa) = 0.$$

By (5.10), therefore, $R(z, \kappa)$ has no roots $|\kappa(z)| \leqslant 1$ for $|z| > 1$, and (5.8) holds.

To obtain (5.9), we merely note that, by (5.8), the roots $\kappa(z)$ of $R(z, \kappa)$ satisfy $|\kappa(z)| > 1$ if $|z| > 1$. Thus, for $|z| \geqslant 1$, those continuous roots satisfy $|\kappa(z)| \geqslant 1$ and the lemma follows.

Proof of Theorem 3.3. Since the boundary scheme is solvable and satisfies the von Neumann condition, Lemma 5.2 implies (3.4), and by Theorem 3.2, approximation (2.6a, b) (3.1) is stable.

Proof of Theorem 3.4. As in the proof of Theorem 3.1, let z satisfy $|z| \geqslant 1$, let $\kappa_\alpha(z)$, $|\kappa_\alpha(z)| \leqslant 1$, be a corresponding inner root, and let us prove that

$$(5.11) \qquad R(z, \kappa_\alpha) \neq 0.$$

By Theorem 4.2, this will imply stability.

Comparing Definitions 3.2 and 3.3, we immediately see that since the boundary scheme is dissipative, it satisfies the von Neumann condition; so Lemma 5.2 applies, and it remains to verify (5.11) for z and $\kappa_\alpha(z)$ with

$$|z| = 1, \quad |\kappa_\alpha(z)| = 1.$$

Indeed, for

$$|z| = 1, \quad |\kappa_\alpha(z)| = 1, \quad \kappa_\alpha(z) \neq 1,$$

(5.11) follows from the dissipativity of the boundary scheme as described in Definition 3.3; for

$$|z| = 1, \quad z \neq 1, \quad \kappa_\alpha(z) = 1,$$

(5.11) is implied by (3.7); and finally, by Lemma 5.1, $\kappa = 1$ is never an inner root for $z = 1$. Thus, (5.11) is verified and the theorem is proven.

We conclude the paper by proving Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. (i) Let the boundary scheme (3.1) be two-level and accurate of order zero at least. By zero-order accuracy, the boundary coefficients satisfy

$$\sum_{j=0}^{m} c_{j(-1)} = \sum_{\sigma=0}^{q} \sum_{j=0}^{m} c_{j\sigma}.$$

Hence,

$$(5.12) \qquad R(z, \kappa)|_{z=\kappa=1} = 0.$$

Since the boundary scheme is two-level, $R(z, \kappa = 1)$ is a first degree polynomial in z^{-1} whose single root, by (5.12), is $z^{-1} = 1$. Consequently,

$$R(z, \kappa = 1) \neq 0, \quad |z| = 1, |z| \neq 1,$$

and (3.7) holds.

(ii) In the three-level case, $R(z, \kappa = 1)$ is a 2nd degree polynomial in z^{-1} with real coefficients. By (5.12) again, $z^{-1} = 1$ is one of the roots, so the other is real as well, namely

$$R(z, \kappa = 1) \neq 0, \quad |z| = 1, z \neq \pm 1. \tag{5.13}$$

Combining (5.13) with our hypothesis

$$R(z = -1, \kappa = 1) \neq 0,$$

(3.7) follows and the proof is complete.

Proof of Lemma 3.2. As in the proof of Lemma 5.2, apply Lemma 4.1 to the boundary scheme (3.8). Since the boundary scheme is right-sided with $c_{0(-1)} \neq 0$, we find that it is solvable if

(a) the difference equations

$$T_{-1}w_\nu \equiv \sum_{j=0}^{m} c_{j(-1)}w_{\nu+j} = 0, \qquad \nu = 0, 1, 2, \ldots, \tag{5.14}$$

have no nontrivial solution $w \in l_2(x)$, and

(b) the hypothesis of the present lemma is fulfilled, i.e.,

$$T_{-1}(\kappa) \equiv \sum_{j=0}^{m} c_{j(-1)}\kappa^j \neq 0, \qquad 0 < |\kappa| \leqslant 1. \tag{5.15}$$

Now, it is well known that the most general solution of (5.14) in $l_2(x)$ is a combination of powers of the roots of $T_{-1}(\kappa)$ which lie inside the unit disc. Thus, if (5.15) holds, then the only solution of (5.14) in $l_2(x)$ is the trivial one, namely (b) implies (a), and the lemma follows.

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