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Hydrocode Subcycling Stability*

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Abstract. The method of artificial viscosity was originally designed by von Neumann and Richtmyer for calculating the propagation of waves in materials that were hydrodynamic and rate-independent (e.g., ideal gas law). However, hydrocodes (such as WONDY) based on this method continue to expand their repertoire of material laws even unto material laws that are rate-dependent (e.g., Maxwell's material law). Restrictions on the timestep required for stability with material laws that are rate-dependent can be considerably more severe than restrictions of the Courant-Friedrichs-Lewy (CFL) type that are imposed in these hydrocodes. These very small timesteps can make computations very expensive. An alternative is to go ahead and integrate the conservation laws with the usual CFL timestep while subcycling (integrating with a smaller timestep) the integration of the stress-rate equation. If the subcycling is done with a large enough number of subcycles (i.e., with a small enough subcycle timestep), then the calculation is stable. Specifically, the number of subcycles must be one greater than the ratio of the CFL timestep to the relaxation time of the material.

1. Introduction. A previous paper [2] presented the results of a stability analysis of the WONDY [3] hydrocode with a material law that was rate-dependent. WONDY is a computer program based on the artificial viscosity method of von Neumann and Richtmyer [5]. The timestep in WONDY is called the CFL timestep because it is determined by a constraint that is essentially just a modification of the Courant-Friedrichs-Lewy condition [4]. This CFL timestep restriction arose from an approximate stability analysis of the von Neumann-Richtmyer method for the case when the material law is a rate-independent law such as the ideal gas law or Hooke's law. See [2] for further details of the history of the stability analyses of the von Neumann-Richtmyer scheme.

The previous paper [2] showed that the timestep restriction required for stability with material laws that are rate-dependent can be much more stringent than the CFL timestep restriction, especially when the relaxation time $\tau > 0$ of the material is small compared to the CFL timestep Δt_c.

The present paper presents a proof that the CFL timestep restriction now in WONDY need not be altered if the integration of the stress-rate relation is subcycled with a sufficiently large subcycle number, $m \geqslant 1$. A subcycle timestep is $\Delta t_s = \Delta t_c/m$. To subcycle the integration of the stress-rate relation means to integrate it with the subcycle timestep instead of the CFL timestep. As proved herein, $m > 1 + \Delta t_c/\tau$ suffices for stability in the case of a simple but representative material law that is rate-dependent, namely the Malvern material law. Malvern's material law is a special case of Maxwell's material law. See [1] for further details about these rate-dependent material laws.

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Note that another way of stating this stability result is that

$$\frac{1}{\Delta t_s} > \frac{1}{\Delta t_c} + \frac{1}{\tau}$$

is sufficient for the stability of the subcycling scheme presented herein.

2. Notation and Nomenclature. The conservation laws in one-dimensional, Lagrangean conservative form are expressed by

$$\partial \mathbf{U}/\partial t + \partial \mathbf{F}/\partial \mu = 0, \tag{2.1}$$

where $\mathbf{U} = (V, u, E)^T$ and $\mathbf{F} = (-u, \sigma, u\sigma)^T$. Here t is time; μ is material coordinate; V is specific volume; u is specific momentum; E is specific total energy; $E = \mathcal{E} + u^2/2$, where $\mathcal{E}$ is specific internal energy; and σ is stress. The artificial viscosity is a finite difference analog of

$$q = -\Lambda \Delta\mu \frac{\partial u}{\partial \mu}, \tag{2.2}$$

where $\Lambda \geqslant 0$ is the coefficient of the artificial viscosity and $\Delta\mu$ is the material increment. The von Neumann-Richtmyer scheme is a discrete analog of a system of differential equations derived from (2.1) with σ augmented by q; see [3]–[5]. The material law they originally considered was the ideal gas law

$$\sigma = \Gamma\mathcal{E}/V,$$

where Γ is a positive constant.

In the following analysis the material law is Malvern's [1]:

$$\partial\sigma/\partial t + a^2 \partial V/\partial t + (\sigma - \sigma_{eq})/\tau = 0, \tag{2.3}$$

where $a > 0$ is the acoustic impedance; σ_{eq} is the equilibrium stress; a, σ_{eq}, and τ are assumed constant here.

Let $\mu_j = j\Delta\mu$ and $t^n = n\Delta t$, where Δt is the time increment. The approximation to $f(\mu_j, t^n)$ is denoted f_j^n. Differences with respect to μ and t are denoted $\Delta_.$ and $\Delta^{\cdot}$, respectively. For example, $\Delta_. f_{j+1/2}^n \equiv f_{j+1}^n - f_j^n$ and $\Delta^{\cdot} f_j^{n+1/2} \equiv f_j^{n+1} - f_j^n$.

3. Lemmas. These lemmas are used in Section 4. Lemmas 1 and 2 present constraints on B and C to insure that the roots of the quadratic

$$\lambda^2 - 2B\lambda + C = 0$$

lie in the unit circle. Lemma 3 is on the reduction of the quadratic inequality

$$A\alpha^2 + 2B\alpha \leqslant 1$$

to a linear inequality. Lemma 4 presents constraints on the eigenvalues of amplification matrices to insure stability. The proofs of Lemmas 1–3 are left to the reader and a proof of Lemma 4 may be found in [4].

LEMMA 1. *Let B and C be real numbers; $D = B^2 - C$; $\lambda_\pm = B \pm D^{1/2}$; $|\lambda|_{\max} = \max|\lambda_\pm|$.*

Case (a): *If $D \geqslant 0$ and $B^2 > 1$, then*

$$|\lambda|_{\max} > 1.$$

Case (b): *If $D \geqslant 0$ and $B^2 \leqslant 1$, then*

$$\left[\, |\lambda|_{\max} \leqslant 1 \text{ if and only if } 2|B| \leqslant C + 1 \,\right].$$

Case (c): *If* $D < 0$, *then*

$$\left[\,|\lambda|_{\max} \leqslant 1 \text{ if and only if } C \leqslant 1\,\right].$$

Moreover, this also holds when the $\leqslant$ *signs inside the square brackets are replaced by either* $<$ *or* $=$.

LEMMA 2. *Let* $B = 1 - b$, $C = 1 - c$, $2b \geqslant c \geqslant 0$. *Then*

$$\left[\,|\lambda|_{\max} \leqslant 1 \text{ if and only if } 2b + c \leqslant 4\,\right].$$

LEMMA 3. *Assume* A *real,* B *and* α *positive and let* $D = B^2 + A$. *Consider the following inequalities*

$$\left[A\alpha^2 + 2B\alpha \leqslant 1\right], \tag{3.1}$$

and

$$\left[\alpha(B + D^{1/2}) \leqslant 1\right]. \tag{3.2}$$

Case (a): *If* $D \geqslant 0$, *then* (3.1) *if and only if* (3.2).
Case (b): *If* $D < 0$, *then* (3.1) *holds for all* α.

LEMMA 4. *Let* $\mathbf{G}(\Delta t, k)$ *be a* p *by* p *amplification matrix. If there exists a positive number* δ *such that the elements of* $\mathbf{G}(\Delta t, k)$ *are bounded for* $0 < \Delta t < \delta$ *and for all* k *and if all the eigenvalues of* $\mathbf{G}$, *with the possible exception of one, lie in a circle strictly inside the unit circle, then von Neumann's condition is sufficient as well as necessary for stability. That is, if there exists a constant* r *such that for all* k *and all* Δt *in* $(0, \delta)$, $|\lambda_i| \leqslant r < 1$ *for* $i = 2, \ldots, p$, *then* $|\lambda_1| \leqslant 1 + O(\Delta t)$ *is necessary and sufficient for stability.*

4. Results. Results 1–3 are for the case $\Lambda = 0$ (artificial viscosity turned off) and Results 4–6 are for the case $\Lambda > 0$ (artificial viscosity turned on).

When $\Lambda = 0$, the WONDY timestep restriction is given by

$$C_{FL} \leqslant \theta < 1, \tag{4.1}$$

where $C_{FL} = a\Delta t/\Delta\mu$ and $\theta = .9$.

In the case $m = 1$ (i.e., no subcycling) and $\Lambda = 0$, the WONDY equations for conservation of volume and momentum are

$$(\Delta^{\cdot} V/\Delta t)_{j+1/2}^{n+1/2} = (\Delta_{\cdot} u/\Delta\mu)_{j+1/2}^{n+1/2} \tag{4.2}$$

and

$$(\Delta^{\cdot} u/\Delta t)_j^n = -(\Delta_{\cdot}\sigma/\Delta\mu)_j^n, \tag{4.3}$$

and the stress-rate equation is

$$\sigma_{j+1/2}^{n+1} - \sigma_{eq} = (1 - h)(\sigma_{j+1/2}^n - \sigma_{eq}) - hA, \tag{4.4}$$

where

$$h = \Delta t/\tau \tag{4.5}$$

and

$$A = a^2\tau(\Delta^{\cdot} V/\Delta t)_{j+1/2}^{n+1/2}. \tag{4.6}$$

Note that Eq. (4.4) is just a simple difference analog of the Malvern material law.

Next consider the case $m > 1$. Let the overstress at subcycle ν be denoted

$$S^\nu = \sigma_{j+1/2}^{n+\nu/m} - \sigma_{eq} \tag{4.7}$$

for $0 \leqslant \nu \leqslant m$; the starting value for the subcycling is

$$S^0 = \sigma_{j+1/2}^n - \sigma_{eq}. \tag{4.8}$$

The simple subcycling scheme for (4.4) is given by

$$S^{\nu+1} = (1 - h/m)S^\nu - Ah/m \tag{4.9}$$

for $0 \leqslant \nu \leqslant m - 1$. The solution to (4.9) is

$$S^\nu = (1 - h/m)^\nu S^0 + \left[(1 - h/m)^\nu - 1\right]A \tag{4.10}$$

for $0 \leqslant \nu \leqslant m$. In particular

$$\sigma_{j+1/2}^{n+1} - \sigma_{eq} = S^m \tag{4.11}$$

for $\nu = m$ in (4.10). It follows that

$$\sigma_{j+1/2}^{n+1} - \sigma_{eq} = (1 - hf(m))(\sigma_{j+1/2}^n - \sigma_{eq}) - hf(m)A, \tag{4.12}$$

where

$$f(m) = \left[1 - (1 - h/m)^m\right]/h. \tag{4.13}$$

Note that

$$\lim_{m\to\infty} f(m) = g(h), \tag{4.14}$$

where

$$g(h) = (1 - e^{-h})/h. \tag{4.15}$$

Further, note that for all $h > 0$ and $m \geqslant 1$,

$$0 < g(h) < f(m + 1) < f(m) \leqslant 1.$$

To construct the amplification matrix for the system (4.2), (4.3), and (4.12) let

$$w_{j+1/2}^n = (\sigma_{eq} - \sigma_{j+1/2}^n)/a \tag{4.16}$$

and

$$v_j^{n+1} = u_j^{n+1/2}, \tag{4.17}$$

and replace v_j^n by $v^n\xi^j$ and $w_{j+1/2}^n$ by $w^n\xi^{j+1/2}$, where $\xi = \exp ik\Delta\mu$. Then (4.3) becomes

$$v^{n+1} = v^n + i\beta(k)w^n, \tag{4.18}$$

where

$$\beta(k) = 2C_{FL}\sin(k\Delta\mu/2), \tag{4.19}$$

and (4.12) becomes

$$w^{n+1} - i\beta f(m)v^{n+1} = (1 - hf(m))w^n. \tag{4.20}$$

Let $\mathbf{U}^n = (v^n, w^n)^T$, and it follows that

$$\mathbf{U}^{n+1} = \mathbf{G}\mathbf{U}^n,$$

where the amplification matrix is given by

$$\mathbf{G} = \begin{bmatrix} 1 & i\beta \\ i\beta f(m) & 1 - (\beta^2 + h)f(m) \end{bmatrix}. \tag{4.21}$$

Note that

$$\det(\mathbf{G} - \lambda \mathbf{I}) = \lambda^2 - 2B\lambda + C,$$

where

$$B = 1 - (\beta^2 + h)f(m)/2 \tag{4.22}$$

and

$$C = 1 - hf(m). \tag{4.23}$$

RESULT 1. If $m \geqslant h > 0$, then

$$f(m)\left[C_{FL}^2 + h/2\right] \leqslant 1 \tag{4.24}$$

is a necessary and sufficient condition for the stability of $\mathbf{G}$ in Eq. (4.21).

Proof Sketch. In Eqs. (4.22)–(4.23) identify the b of Lemma 2 with $(\beta^2 + h)f(m)/2$ and the c with $hf(m)$. Lemma 2 says $|\lambda_\pm| \leqslant 1$ if and only if

$$2b + c \leqslant 4,$$

and this is the same as

$$f(m)(\beta^2(k) + 2h) \leqslant 4,$$

which is true for all k if and only if (4.24) holds. Therefore (4.24) if and only if $|\lambda_\pm| \leqslant 1$. Thus, necessity is established; by Lemma 4 sufficiency is established. To use Lemma 4, it must be shown that either $|\lambda_+|$ or $|\lambda_-|$ is strictly less than unity. Consider their product:

$$\lambda_+\lambda_- = C = (1 - h/m)^m.$$

If $m \geqslant h > 0$, then $0 \leqslant C < 1$. End of proof sketch.

RESULT 2. If $m \geqslant h > 0$ and $C_{FL} \leqslant \theta < 1$, then there exists a positive integer M such that $m > M$ implies (4.24) holds.

Proof Sketch. Note that (4.24) holds if and only if

$$0 \leqslant h - hf(m)(\theta^2 + h/2) \tag{4.25}$$

holds. If $H(h)$ is the limit as $m \to \infty$ of the RHS of (4.25), then

$$H(h) = h - hg(h)(\theta^2 + h/2).$$

Observe that H' has a minimum at $h_m = 2(1 - \theta^2)$ and $H'(h_m) > 0$, therefore $H'(h) > 0$ for $h > 0$. Since $H(0) = 0$, it follows that $H(h) > 0$ for $h > 0$. Therefore, there exists an M such that if $m > M$, then (4.25) holds. End of proof sketch.

RESULT 3. If $m > h + 1 > 1$, $C_{FL} \leqslant \theta$, and $\theta^2 \leqslant 5/6$, then $\mathbf{G}$ is stable.

Proof Sketch. By Results 1 and 2, there exists an M such that $\mathbf{G}$ is stable for $m > M$. This proof shall show that $M \geqslant h + 1$ suffices when $\theta^2 \leqslant 5/6$. That is, it shall be shown that if $m > h + 1 > 1$ and $\theta^2 \leqslant 5/6$, then

$$\frac{2\theta^2 - h}{2\theta^2 + h} \leqslant (1 - h/m)^m, \tag{4.26}$$

and from (4.26) follows (4.24) when $C_{FL} \leqslant \theta$. The problem of showing (4.26) is split into three cases:

Case (1): $h < 2\theta^2$ and $0 < h < 1$, then $m = 2$.

Case (2): $h < 2\theta^2$ and $1 \leqslant h < 2$, then $m = 3$.

Case (3): $h \geqslant 2\theta^2$.

Case (1). For $m = 2$, (4.26) is equivalent to

$$0 \leqslant hF(h),$$

where

$$F(h) = h^2 - 2h(2 - \theta^2) + 8(1 - \theta^2).$$

The roots of F are

$$h_\pm = 2 - \theta^2 \pm D^{1/2},$$

where

$$D = \theta^4 + 4\theta^2 - 4.$$

Observe that if $D < 0$, then $F(h) > 0$ for all $h \geqslant 0$, and if $D \geqslant 0$, then $F(h) \geqslant 0$ for all $0 \leqslant h \leqslant h_-$. Note that if $2(\sqrt{2} - 1) \leqslant \theta^2 \leqslant 5/6$, then $D \geqslant 0$ and $h_- \geqslant 1$. It follows that if $\theta^2 < 2(\sqrt{2} - 1)$, then $F(h) > 0$ for all $h \geqslant 0$, and if $2(\sqrt{2} - 1) \leqslant \theta^2 \leqslant 5/6$, then $F(h) \geqslant 0$ for all h in $[0, 1]$.

Case (2). For $m = 3$, (4.26) is equivalent to

$$0 \leqslant hF(h),$$

where

$$F(h) = -h^3 + h^2(9 - 2\theta^2) - 9h(3 - 2\theta^2) + 54(1 - \theta^2).$$

Note that

$$-F'(h)/3 = h^2 - 2h(9 - 2\theta^2)/3 + 9 - 6\theta^2,$$

and that the extrema of F occur at

$$h_\pm = (9 - 2\theta^2)/3 \pm \left[2\theta^2(9 + 2\theta^2)/9\right]^{1/2}.$$

The minimum is at $h_m = h_-$ and the maximum is at $h_M = h_+$. Using the relation

$$h_m^2 = 2h_m(9 - 2\theta^2)/3 - (9 - 6\theta^2),$$

the evaluation of F at h_m may be reduced to

$$F(h_m) = h_m(9 + 2\theta^2)4\theta^2/9 + 27 - 30\theta^2 - 4\theta^4.$$

An elementary calculation shows that $F(h_m) \geqslant 0$ for $\theta^2 \leqslant 5/6$. It follows that $F(h) \geqslant 0$ for $h \leqslant h_M$. Observing that $h_M \geqslant 2\theta^2$ for $\theta^2 \leqslant 5/6$ completes Case (2).

Case (3). Note that the LHS of (4.26) is nonpositive in this case while the RHS is nonnegative. Note also that for Case (3) (the large h case) the conditions may be relaxed to $m \geqslant h$ and $\theta < 1$. End of proof sketch.

Remark. Result 3 shows that if Λ (the coefficient of the artificial viscosity) is zero, then the timestep restriction routine in WONDY need not be altered provided that the integration of the stress-rate relation is subcycled with $m > h + 1$. The $\theta^2 \leqslant 5/6$ restriction is satisfied in WONDY because $\theta = .9$ there. The next part of this paper deals with the subcycling stability when $\Lambda > 0$. In this case the restriction on the timestep in WONDY is

$$C'_{FL} \leqslant \theta,$$

where

$$C'_{FL} = C_{FL}\left(\Lambda/a + \left[1 + (\Lambda/a)^2\right]^{1/2}\right). \tag{4.27}$$

The artificial viscosity in WONDY is of the form

$$q_{j+1/2}^{n-1/2} = -\Lambda(\Delta_x u)_{j+1/2}^{n-1/2}, \tag{4.28}$$

where Λ depends on a, V, and $\Delta_x u$. For simplicity the Λ is taken to be a positive constant here. The addition of this artificial viscous stress to σ in the conservation of momentum equation results in

$$(\Delta_t u/\Delta t)_j^n = -(\Delta_x \tilde{\sigma}/\Delta\mu)_j^n,$$

where

$$\tilde{\sigma}_{j+1/2}^n = \sigma_{j+1/2}^n + q_{j+1/2}^{n-1/2}$$

for all j. Therefore Eq. (4.18) becomes

$$v^{n+1} = v^n(1-\delta) + i\beta w^n,$$

where

$$\delta = (4\Lambda\Delta t/\Delta\mu)\sin^2(k\Delta\mu/2). \tag{4.29}$$

Then the amplification matrix becomes

$$\mathbf{G} = \begin{bmatrix} 1-\delta & i\beta \\ (1-\delta)i\beta f(m) & 1-(\beta^2+h)f(m) \end{bmatrix}, \tag{4.30}$$

and B and C become

$$B = 1 - \left[\delta + (\beta^2+h)f(m)\right]/2 \tag{4.31}$$

and

$$C = (1-\delta)(1-hf(m)), \tag{4.32}$$

where

$$\det(\mathbf{G} - \lambda\mathbf{I}) = \lambda^2 - 2B\lambda + C.$$

RESULT 4. If $2\Lambda\Delta t/\Delta\mu \leqslant 1$ and $m \geqslant h > 0$, then a necessary and sufficient condition for $\mathbf{G}$ to be stable is

$$a(m)C_{FL}^2 + 2b(m)C_{FL} \leqslant 1, \tag{4.33}$$

where

$$a(m) = f(m) - \Lambda\Delta\mu/(a^2\tau)$$

and

$$b(m) = \Lambda/a + \Delta\mu f(m)/(4a\tau).$$

Proof Sketch. Lemma 2 implies that (4.33) holds if and only if $|\lambda_\pm| \leqslant 1$. To use Lemma 4, it must be shown that either $|\lambda_+|$ or $|\lambda_-|$ is strictly less than unity. Consider their product

$$\lambda_+\lambda_- = C = (1-\delta)(1-hf(m)).$$

By hypothesis $2\Lambda\Delta t/\Delta\mu \leqslant 1$ and therefore $|1-\delta| \leqslant 1$. Thus, $|\lambda_+\lambda_-| \leqslant |1-hf(m)|$, and, since $1-hf(m) = (1-h/m)^m$ and $m \geqslant h > 0$, the result follows. End of proof sketch.

Remark. The inequality $2\Lambda\Delta t/\Delta\mu \leqslant 1$ in the hypothesis of Result 4 is satisfied in WONDY because

$$\frac{a\Delta t}{\Delta\mu}\left[\frac{\Lambda}{a}+\sqrt{1+\left(\frac{\Lambda}{a}\right)^2}\,\right] \leqslant \theta < 1$$

is enforced there. It follows that

$$\frac{2\Lambda\Delta t}{\Delta\mu} \leqslant \theta < 1.$$

Also note that Result 4 is an improvement over Results 3 and 4 of [2]. This is easily seen by noting that (4.33) is equivalent to

$$f(m)\left[C_{FL}^2 + h/2\right] + C_{FL}\frac{\Lambda}{a}\left[2 - h\right] \leqslant 1,$$

and recalling that $f(1) = 1$.

RESULT 5. Let

$$z = \Lambda/a + \left[1 + (\Lambda/a)^2\right]^{1/2},$$

and

$$C'_{FL} = zC_{FL}.$$

If

$$C'_{FL} \leqslant \theta < 1,$$

then there exists an M such that $m > M$ implies (4.33) holds.

Proof Sketch. Let

$$H(h) \equiv \lim_{m\to\infty} h\left[1 - a(m)C_{FL}^2 - 2b(m)C_{FL}\right],$$

and note that

$$H(h) = h\left[1 + (h-2)C_{FL}\Lambda/a\right] - (1 - e^{-h})\left(C_{FL}^2 + h/2\right).$$

Use the fact that

$$\Lambda/a = (z^2 - 1)/2z$$

to get

$$H(h) = h\left[1 + C'_{FL}(h-2)(1 - 1/z^2)/2\right] - (1 - e^{-h})\left[(C'_{FL})^2/z^2 + h/2\right].$$

Let $x = 1 - 1/z^2$ to get

$$H(h) = h\left[1 + C'_{FL}(h-2)x/2\right] - (1 - e^{-h})\left[(C'_{FL})^2(1 - x) + h/2\right],$$

$$H(h) = h - (1 - e^{-h})\left[(C'_{FL})^2 + h/2\right] + x\left\{C'_{FL}h(h-2)/2 + (1 - e^{-h})(C'_{FL})^2\right\}.$$

Note that if

$$0 < H(h) \quad \text{for } h > 0 \tag{4.34}$$

and for all x in [0, 1], then the desired result is established. Since H is of the form $H = Ax + B$, all that is required is to establish (4.34) for $x = 0$ and $x = 1$. In the proof of Result 2 the case $x = 0$ was established. That leaves the case $x = 1$ to establish. If $x = 1$, then

$$H(h) = \frac{h}{2}(1 + e^{-h}) + \frac{h(h-2)}{2}C'_{FL}.$$

It is easy to see that (4.34) holds if $h \geqslant 2$. If $h < 2$, then the worst value for C'_{FL} is unity, but then $H(h) = h[e^{-h} - (1 - h)]/2$ and since $e^{-h} > 1 - h$ for $h > 0$, the desired result follows. End of proof sketch.

RESULT 6. Let $\theta^2 \leqslant 5/6$, $z = \Lambda/a + [1 + (\Lambda/a)^2]^{1/2}$, and $C'_{FL} = zC_{FL}$. If $C'_{FL} \leqslant \theta$ and $1 < h + 1 < m$, then **G** is stable.

Proof Sketch. The plan of the proof is to show that (4.33) holds and then use Result 4. Note that (4.33) holds if and only if

$$f(m)\left[C_{FL}^2 + h/2\right] + C_{FL}(2 - h)\Lambda/a \leqslant 1. \tag{4.35}$$

By the proof of Result 3, it is seen that (4.35) holds for $h \geqslant 2$. The remainder of the proof is broken into three cases:

Case (1): $0 < h < 2\theta^2$, $h < 1$, $m = 2$.

Case (2): $0 < h < 2\theta^2$, $1 \leqslant h < 2\theta^2$, $m = 3$.

Case (3): $2\theta^2 \leqslant h < 2$.

Let

$$H(h) = (1 - h/m)^m\left[\theta^2(1 - x) + h/2\right] - \theta^2(1 - x) + \frac{h}{2} - \theta h(1 - h/2)x,$$

and note that (4.35) holds for $z \geqslant 1$ if

$$0 \leqslant H(h) \tag{4.36}$$

holds for $0 \leqslant x \leqslant 1$.

In Result 3, (4.36) was shown for $x = 0$. Since H is the form $H = Ax + B$, if (4.36) is also shown for $x = 1$, then it holds for all x in [0, 1]. If $x = 1$, then

$$H(h) = \frac{h}{2}J(h),$$

where

$$J(h) = (1 - h/m)^m + 1 - \theta(2 - h).$$

The problem is reduced to showing $0 \leqslant J(h)$.

Case (1). If $m = 2$, then

$$J(h) = 2 - h + h^2/4 - \theta(2 - h),$$

and for $h < 2$ the worst case for θ is $\theta = 5/6$. If $\theta = 5/6$, then $J(h) = h^2/4 + (2 - h)/6$ and $J(h) > 0$ for $h > 0$.

Case (2). If $m = 3$, then

$$J(h) = 2 - h + h^2/3 - h^3/27 - \theta(2 - h),$$

and for $h < 2$ the worst case for θ is $\theta = 5/6$. If $\theta = 5/6$, then $J = h^2/3(1 - h/9) + (2 - h)/6$ which is positive for $h < 2$.

Case (3). $2\theta^2 \leqslant h < 2$,

$$J(h) = (1 - h/m)^m + 1 - \theta(2 - h).$$

The first term $(1 - h/m)^m$ is positive and considering the second term, namely

$$1 - \theta(2 - h)$$

in the interval $2\theta^2 \leqslant h < 2$, the worst case is seen to be $h = 2\theta^2$. Let

$$P(\theta) \equiv 1 - \theta(2 - 2\theta^2),$$

and note that the minimum of $P(\theta)$ occurs at $\theta^2 = 1/3$ and is positive there. End of proof sketch.

5. Concluding Remarks. If the material law with rate-dependence that is considered in this paper is used in WONDY (or related hydrocodes) without subcycling, then Result 4 says that the restrictions on the timestep for stability are $2\Lambda\Delta t/\Delta\mu \leq 1$, $\Delta t \leq \tau$, and $C_{FL}^2 + 2C_{FL}\Lambda/a + \Delta t(1 - 2\Lambda\Delta t/\Delta\mu)/(2\tau) \leq 1$. These constraints can be rather severe, particularly for relaxation times τ that are small compared to the WONDY timestep Δt_c which is determined from

$$C_{FL}^2 + 2C_{FL}\Lambda/a \leq \theta < 1 \tag{5.1}$$

with $\theta = .9$.

Result 6 says that, if (5.1) is satisfied with $\theta^2 \leq 5/6$ and if m (the subcycle number) satisfies $m > h + 1$ (where $h = \Delta t_c/\tau$), then the WONDY subcycling scheme results in a stable calculation. In other words, the existing timestep calculating routine need not be altered provided subcycling is used with subcycle timestep Δt_s satisfying

$$\Delta t_s < \frac{\tau\Delta t_c}{\tau + \Delta t_c}. \tag{5.2}$$

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