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A Class of Steiner Triple Systems of Order 21 and Associated Kirkman Systems

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Abstract. We examine a class of Steiner triple systems or order 21 with an automorphism consisting of three disjoint cycles of length 7. We exhibit explicitly all members of this class: they number 95 including the 7 cyclic systems. We then examine resolvability of the obtained systems; only 6 of the 95 are resolvable yielding a total of 30 nonisomorphic Kirkman triple systems of order 21. We also list several invariants of the systems and investigate their further properties.

1. Introduction. A *Steiner triple system* (STS) is a pair (V, B) where V is a v-set and B is a family of 3-subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple. The number v is called the *order* of the STS (V, B). It is well known (see, e.g., [12]) that an STS of order v exists if and only if $v \equiv 1$ or 3 (mod 6). A *Kirkman triple system* (KTS) is a Steiner triple system (V, B) together with a partition R of the set of triples B into subsets $R_1, R_2, \ldots, R_u$ called *parallel classes* such that each R_i is a partition of V. R is called a *resolution* of (V, B). It is only relatively recently that Kirkman triple systems have been shown to exist if and only if $v \equiv 3$ (mod 6) [17].

With the existence problem for both STSs and KTSs completely settled, one considers next the enumeration problem. Two STSs, (V_1, B_1), (V_2, B_2), are *isomorphic* if there exists a bijection α: $V_1 \to V_2$ such that $\alpha B_1 = B_2$. Similarly, two KTSs, (V_1, B_1, R_1), (V_2, B_2, R_2), are isomorphic if there exists a bijection α: $V_1 \to V_2$ such that $\alpha B_1 = B_2$ and $\alpha R_1 = R_2$. The numbers $N(v)$ of nonisomorphic STSs and the numbers $K(v)$ of nonisomorphic KTSs are known exactly only for $v \leqslant 15$. One has $N(3) = N(7) = N(9) = 1$, $N(13) = 2$, $N(15) = 80$, but $N(19) \geqslant 284407$ and $N(21) \geqslant 2160980$ (cf. [10]); it is known that $N(v) \sim \exp(v(v-1)/6)$ as $v \to \infty$ [2], [21]. In order to circumvent the "combinatorial explosion" effect encountered as one moves past the order 15, one imposes additional conditions that the STSs should satisfy, so that the enumeration problem for such a suitably restricted class becomes once again feasible, and the obtained numbers stay within "reasonable" bounds. Examples of this approach include: (1) Enumeration of cyclic STSs initiated by Bays [4] and carried out by him for orders $v \leqslant 37$ and $v = 43$, and repeated recently, independently, by Colbourn [6] who verified the values obtained

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by Bays for $v \leqslant 37$ and extended the enumeration of cyclic STSs to orders $v \leqslant 45$ (correcting in the process the value obtained by Bays for $v = 43$); in particular, there are 4 nonisomorphic cyclic STSs of order 19 and 7 nonisomorphic cyclic STSs of order 21. (2) Enumeration of reverse STSs of order 19 [9]; there are exactly 184 of them. (3) Enumeration of 1-rotational STSs of order 27 and of 2-rotational STSs of order 19 [16]; there are exactly 10 of the latter kind.

On the other hand, for Kirkman triple systems, one has $K(3) = K(9) = 1$, $K(15) = 7$ ([8], [14]; see also [18]) but virtually nothing is known about $K(v)$ for $v \geqslant 21$ (except, of course, that $K(v) \geqslant 1$).

In this paper we examine a class of STSs of order 21 possessing an automorphism consisting of three disjoint cycles of length 7. The purpose of this paper is twofold. First, we exhibit explicitly all members of this class; they number 95 and include, of course, the 7 cyclic STSs. Second, we then examine resolvability of the obtained systems; only 6 of the 95 above STSs are resolvable, yielding a total of 30 nonisomorphic KTSs or order 21. As a by-product, this enables us to show easily that the above STSs include all transitive STSs of order 21. We also list several invariants of the systems, both STSs and KTSs, and investigate their further properties such as double resolvability, etc.

2. A Class of Steiner Triple Systems of Order 21. We will take $V = Z_7 \times \{1, 2, 3\}$; and element (x, i) of $V_i = Z_7 \times \{i\}$ will be written for brevity as x_i. Any of the constructed STSs will have $\alpha = (0_1\ 1_1 \cdots 6_1)(0_2\ 1_2 \cdots 6_2)(0_3\ 1_3 \cdots 6_3)$ as its automorphism. The triples of any STS fall into 10 orbits under α containing 7 triples each; only one representative of each orbit needs to be listed.

The STSs are classified according to how many of the sets V_i carry subsystems of order 7: STSs with three subsystems (type A), one subsystem (type B), or no subsystem of order 7 (type C). (It is easily seen that there can be no STS with exactly two disjoint subsystems of order 7.) There are altogether 4, 41, and 50 STSs of types A, B, and C, respectively, for a total of 95 systems. These are listed in Table 1.

A set of STSs, containing representatives of all isomorphism classes of STSs with an automorphism of above type, was generated by hand (with double- and triple-checking). However, the isomorphism testing and analysis was performed on computer. Table 2 surveys some of the characteristics of the analyzed STSs that are listed in Table 1. The column headed G contains the order of the automorphism group, the column headed R gives the number of distinct resolutions, and the column headed K the number of nonisomorphic resolutions of the given design (for more about this last invariant see Section 3 below). Letters c and t, respectively, indicate that the STS is cyclic or that it is transitive (but not cyclic), respectively. The column headed F counts the number of fragments (cf., e.g., [7]), and the columns headed k_6, k_7, and k_7-base vector count the number of 6-cliques, 7-cliques, and 7-cliques through the 10 base blocks in the block-intersection graph, respectively (cf. [7]). These last four invariants distinguish the systems completely; on the other hand, each of F, k_6, and k_7-base vector is essential for distinguishing, as shown by the pairs of designs (C.45, C.46), (C.36, C.42), and (C.5, C.26), respectively.

TABLE 1

STSs with 3 subsystems

Common base blocks: $0_1 1_1 3_1$ $0_2 1_2 3_2$ $0_3 1_3 3_3$ $0_1 0_2 0_3$, and

A.1	$0_1 1_2 2_3$	$0_1 2_2 4_3$	$0_1 3_2 6_3$	$0_1 4_2 1_3$	$0_1 5_2 3_3$	$0_1 6_2 5_3$
A.2	$0_1 1_2 2_3$	$0_1 2_2 5_3$	$0_1 3_2 1_3$	$0_1 4_2 6_3$	$0_1 5_2 4_3$	$0_1 6_2 3_3$
A.3	$0_1 1_2 2_3$	$0_1 2_2 6_3$	$0_1 3_2 5_3$	$0_1 4_2 3_3$	$0_1 5_2 1_3$	$0_1 6_2 4_3$
A.4	$0_1 1_2 3_3$	$0_1 2_2 6_3$	$0_1 3_2 2_3$	$0_1 4_2 5_3$	$0_1 5_2 1_3$	$0_1 6_2 4_3$

STSs with 1 subsystem

Common base blocks: $0_1 1_1 3_1$ $0_1 0_2 0_3$, and

B.1	$0_1 1_2 2_3$	$0_1 2_2 1_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 5_3$	$0_2 2_3 4_3$
B.2	$0_1 1_2 2_3$	$0_1 2_2 1_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 4_3$	$0_2 3_3 5_3$
B.3	$0_1 1_2 2_3$	$0_1 2_2 5_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 1_3 6_3$	$0_1 3_3 4_3$	$0_2 2_2 6_3$	$0_2 2_3 5_3$
B.4	$0_1 1_2 2_3$	$0_1 2_2 6_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 1_3 3_3$	$0_1 4_3 5_3$	$0_2 2_2 5_3$	$0_2 2_3 6_3$
B.5	$0_1 1_2 2_3$	$0_1 3_2 1_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 3_2 6_3$	$0_2 2_3 4_3$
B.6	$0_1 1_2 2_3$	$0_1 3_2 1_3$	$0_1 2_2 5_2$	$0_1 4_2 6_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 1_2 3_3$	$0_2 4_3 6_3$
B.7	$0_1 1_2 2_3$	$0_1 3_2 6_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 4_3$	$0_1 3_3 5_3$	$0_2 3_2 2_3$	$0_2 4_3 5_3$
B.8	$0_1 1_2 2_3$	$0_1 3_2 6_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 5_3$	$0_1 3_3 4_3$	$0_2 3_2 5_3$	$0_2 4_3 6_3$
B.9	$0_1 1_2 2_3$	$0_1 5_2 3_3$	$0_1 2_2 4_2$	$0_1 3_2 6_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 1_2 4_3$	$0_2 2_3 6_3$
B.10	$0_1 1_2 2_3$	$0_1 5_2 3_3$	$0_1 2_2 6_2$	$0_1 3_2 4_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 4_3$	$0_2 3_3 6_3$
B.11	$0_1 1_2 2_3$	$0_1 6_2 1_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 6_3$	$0_2 3_3 5_3$
B.12	$0_1 1_2 2_3$	$0_1 6_2 1_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 3_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 5_3$	$0_2 4_3 6_3$
B.13	$0_1 1_2 2_3$	$0_1 6_2 3_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 5_3$	$0_2 2_3 6_3$
B.14	$0_1 1_2 2_3$	$0_1 6_2 4_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 1_3 3_3$	$0_1 5_3 6_3$	$0_2 2_2 4_3$	$0_2 3_3 6_3$
B.15	$0_1 1_2 3_3$	$0_1 2_2 6_3$	$0_1 4_2 5_2$	$0_1 3_2 6_2$	$0_1 1_3 5_3$	$0_1 2_3 4_3$	$0_2 2_2 3_3$	$0_2 5_3 6_3$
B.16	$0_1 1_2 3_3$	$0_1 3_2 1_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 2_3 4_3$	$0_1 5_3 6_3$	$0_2 3_2 6_3$	$0_2 1_3 4_3$
B.17	$0_1 1_2 3_3$	$0_1 3_2 1_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 2_3 4_3$	$0_1 5_3 6_3$	$0_2 3_2 4_3$	$0_2 3_3 6_3$
B.18	$0_1 1_2 3_3$	$0_1 3_2 1_3$	$0_1 2_2 6_2$	$0_1 4_2 5_2$	$0_1 2_3 5_3$	$0_1 4_3 6_3$	$0_2 2_2 1_3$	$0_2 3_3 4_3$
B.19	$0_1 1_2 3_3$	$0_1 3_2 1_3$	$0_1 2_2 6_2$	$0_1 4_2 5_2$	$0_1 2_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 6_3$	$0_2 1_3 3_3$
B.20	$0_1 1_2 3_3$	$0_1 3_2 1_3$	$0_1 2_2 6_2$	$0_1 4_2 5_2$	$0_1 2_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 3_3$	$0_2 4_3 6_3$
B.21	$0_1 1_2 3_3$	$0_1 3_2 2_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 4_3$	$0_1 5_3 6_3$	$0_2 3_2 4_3$	$0_2 3_3 5_3$
B.22	$0_1 1_2 3_3$	$0_1 3_2 2_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 5_3$	$0_1 4_3 6_3$	$0_2 3_2 1_3$	$0_2 3_3 4_3$

TABLE 1 (*continued*)

B.23	$0_1 1_2 3_3$	$0_1 3_2 4_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 6_3$	$0_1 2_3 5_3$	$0_2 3_2 6_3$	$0_2 4_3 5_3$
B.24	$0_1 1_2 3_3$	$0_1 3_2 4_3$	$0_1 2_2 5_2$	$0_1 4_2 6_2$	$0_1 1_3 6_3$	$0_1 2_3 5_3$	$0_2 1_2 6_3$	$0_2 3_3 4_3$
B.25	$0_1 1_2 3_3$	$0_1 3_2 4_3$	$0_1 2_2 5_2$	$0_1 4_2 6_2$	$0_1 1_3 6_3$	$0_1 2_3 5_3$	$0_2 1_2 4_3$	$0_2 5_3 6_3$
B.26	$0_1 1_2 3_3$	$0_1 5_2 2_3$	$0_1 2_2 3_2$	$0_1 4_2 6_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 3_2 1_3$	$0_2 3_3 6_3$
B.27	$0_1 1_2 3_3$	$0_1 5_2 2_3$	$0_1 2_2 3_2$	$0_1 4_2 6_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 3_2 6_3$	$0_2 1_3 5_3$
B.28	$0_1 1_2 3_3$	$0_1 5_2 2_3$	$0_1 2_2 6_2$	$0_1 3_2 4_2$	$0_1 1_3 4_3$	$0_1 5_3 6_3$	$0_2 2_2 1_3$	$0_2 3_3 5_3$
B.29	$0_1 1_2 3_3$	$0_1 5_2 2_3$	$0_1 2_2 6_2$	$0_1 3_2 4_2$	$0_1 1_3 4_3$	$0_1 5_3 6_3$	$0_2 2_2 5_3$	$0_2 1_3 6_3$
B.30	$0_1 1_2 3_3$	$0_1 5_2 2_3$	$0_1 2_2 6_2$	$0_1 3_2 4_2$	$0_1 1_3 5_3$	$0_1 4_3 6_3$	$0_2 2_2 3_3$	$0_2 5_3 6_3$
B.31	$0_1 1_2 3_3$	$0_1 6_2 2_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 1_3 5_3$	$0_1 4_3 6_3$	$0_2 2_2 1_3$	$0_2 4_3 5_3$
B.32	$0_1 1_2 3_3$	$0_1 6_2 2_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 1_3 6_3$	$0_1 4_3 5_3$	$0_2 2_2 6_3$	$0_2 1_3 5_3$
B.33	$0_1 1_2 5_3$	$0_1 3_2 1_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 2_3 4_3$	$0_1 3_3 6_3$	$0_2 3_2 6_3$	$0_2 1_3 2_3$
B.34	$0_1 1_2 5_3$	$0_1 3_2 1_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 2_3 6_3$	$0_1 3_3 4_3$	$0_2 3_2 2_3$	$0_2 1_3 3_3$
B.35	$0_1 1_2 5_3$	$0_1 6_2 1_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 2_3 4_3$	$0_1 3_3 6_3$	$0_2 2_2 3_3$	$0_2 5_3 6_3$
B.36	$0_1 1_2 5_3$	$0_1 6_2 1_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 2_3 6_3$	$0_1 3_3 4_3$	$0_2 2_2 1_3$	$0_2 3_3 5_3$
B.37	$0_1 1_2 5_3$	$0_1 6_2 1_3$	$0_1 2_2 5_2$	$0_1 3_2 4_2$	$0_1 2_3 6_3$	$0_1 3_3 4_3$	$0_2 2_2 5_3$	$0_2 1_3 6_3$
B.38	$0_1 1_2 6_3$	$0_1 2_2 5_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 1_3 4_3$	$0_1 2_3 3_3$	$0_2 2_2 1_3$	$0_2 2_3 4_3$
B.39	$0_1 1_2 6_3$	$0_1 2_2 5_3$	$0_1 3_2 6_2$	$0_1 4_2 5_2$	$0_1 1_3 4_3$	$0_1 2_3 3_3$	$0_2 2_2 4_3$	$0_2 1_2 6_3$
B.40	$0_1 1_2 6_3$	$0_1 3_2 4_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 3_3$	$0_1 2_3 5_3$	$0_2 3_2 2_3$	$0_2 3_3 4_3$
B.41	$0_1 1_2 6_3$	$0_1 3_2 4_3$	$0_1 2_2 4_2$	$0_1 5_2 6_2$	$0_1 1_3 5_3$	$0_1 2_3 3_3$	$0_2 3_2 6_3$	$0_2 2_3 4_3$

STSs with 0 subsystems

Common base block: $0_1 0_2 0_3$, and

C.1	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 4_3$	$0_2 1_2 2_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 3_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.2	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 2_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 3_2$	$0_3 2_3 4_1$	$0_3 3_3 6_1$
C.3	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_3 6_3$	$0_2 1_2 2_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 3_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.4	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 1_3$	$0_2 1_2 3_3$	$0_2 2_2 3_1$	$0_2 3_2 1_3$	$0_3 1_3 4_1$	$0_3 2_3 3_2$	$0_3 3_3 1_1$
C.5	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 3_3$	$0_2 2_2 3_1$	$0_2 3_2 1_3$	$0_3 1_3 4_1$	$0_3 2_3 3_2$	$0_3 3_3 2_1$
C.6	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 3_3$	$0_2 2_2 3_1$	$0_2 3_2 4_3$	$0_3 1_3 2_2$	$0_3 2_3 4_1$	$0_3 3_3 6_1$
C.7	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 4_3$	$0_2 1_2 3_3$	$0_2 2_2 3_1$	$0_2 3_2 4_3$	$0_3 1_3 2_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.8	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 6_3$	$0_2 1_2 3_3$	$0_2 3_2 4_3$	$0_2 3_2 4_3$	$0_3 1_3 2_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.9	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 1_3$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 1_2$
C.10	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 5_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 1_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 1_2$

TABLE 1 (*continued*)

C.11	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 1_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 1_3$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 1_2$
C.12	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 2_3$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 2_2$
C.13	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 5_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 2_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 2_2$
C.14	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 1_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 2_3$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 2_2$
C.15	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 1_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 1_2$	$0_3 3_3 1_1$
C.16	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 4_3$	$0_2 2_2 3_1$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 1_2$	$0_3 3_3 2_1$
C.17	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 1_3$	$0_2 1_2 5_3$	$0_2 2_2 3_1$	$0_2 3_2 2_3$	$0_3 1_3 4_1$	$0_3 2_3 6_2$	$0_3 3_3 1_1$
C.18	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 5_3$	$0_2 2_2 3_1$	$0_2 3_2 2_3$	$0_3 1_3 4_1$	$0_3 2_3 6_2$	$0_3 3_3 2_1$
C.19	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 5_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 6_2$	$0_3 2_3 4_1$	$0_3 3_3 6_1$
C.20	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 4_3$	$0_2 1_2 5_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 6_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.21	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 6_3$	$0_2 1_2 5_3$	$0_2 2_2 3_1$	$0_2 3_2 6_3$	$0_3 1_3 6_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.22	$0_1 1_1 2_2$	$0_1 2_1 5_2$	$0_1 3_1 2_3$	$0_2 1_2 6_3$	$0_2 2_2 3_1$	$0_2 3_2 4_3$	$0_3 1_3 5_2$	$0_3 2_3 4_1$	$0_3 3_3 6_1$
C.23	$0_1 1_1 2_2$	$0_1 2_1 6_2$	$0_1 3_1 4_3$	$0_2 1_2 3_3$	$0_2 2_2 4_1$	$0_2 3_2 4_3$	$0_3 1_3 2_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.24	$0_1 1_1 2_2$	$0_1 2_1 6_2$	$0_1 3_1 6_3$	$0_2 1_2 3_3$	$0_2 2_2 4_1$	$0_2 3_2 4_3$	$0_3 1_3 2_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.25	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 4_3$	$0_2 1_2 3_1$	$0_2 2_2 1_3$	$0_2 3_2 5_3$	$0_3 1_3 4_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.26	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 6_3$	$0_2 1_2 3_1$	$0_2 2_2 1_3$	$0_2 3_2 5_3$	$0_3 1_3 4_2$	$0_3 2_3 5_2$	$0_3 3_3 2_1$
C.27	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 4_3$	$0_2 1_2 3_1$	$0_2 2_2 3_3$	$0_2 3_2 2_3$	$0_3 1_3 3_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.28	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 6_3$	$0_2 1_2 3_1$	$0_2 2_2 3_3$	$0_2 3_2 2_3$	$0_3 1_3 3_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.29	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 1_3$	$0_2 1_2 3_1$	$0_2 2_2 3_3$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 3_2$	$0_3 3_3 1_1$
C.30	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 2_3$	$0_2 1_2 3_1$	$0_2 2_2 4_3$	$0_2 3_2 1_3$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 4_2$
C.31	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 5_3$	$0_2 1_2 3_1$	$0_2 2_2 4_3$	$0_2 3_2 1_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 4_2$
C.32	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 1_3$	$0_2 1_2 3_1$	$0_2 2_2 4_3$	$0_2 3_2 1_3$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 4_2$
C.33	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 2_3$	$0_2 1_2 3_1$	$0_2 2_2 5_3$	$0_2 3_2 2_3$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 6_2$
C.34	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 5_3$	$0_2 1_2 3_1$	$0_2 2_2 5_3$	$0_2 3_2 2_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 6_2$
C.35	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 1_3$	$0_2 1_2 3_1$	$0_2 2_2 5_3$	$0_2 3_2 2_3$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 6_2$
C.36	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 4_3$	$0_2 1_2 3_1$	$0_2 2_2 6_3$	$0_2 3_2 1_3$	$0_3 1_3 5_2$	$0_3 2_3 4_1$	$0_3 3_3 1_1$
C.37	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 6_3$	$0_2 1_2 3_1$	$0_2 2_2 6_3$	$0_2 3_2 1_3$	$0_3 1_3 5_2$	$0_3 2_3 5_1$	$0_3 3_3 2_1$
C.38	$0_1 1_1 3_2$	$0_1 2_1 1_2$	$0_1 3_1 1_3$	$0_2 1_2 3_1$	$0_2 2_2 6_3$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 6_2$	$0_3 3_3 1_1$
C.39	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 2_3$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 2_1$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 2_2$
C.40	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 5_3$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 2_1$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 2_2$
C.41	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_3$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 2_1$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 2_2$
C.42	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 5_3$	$0_2 1_2 4_3$	$0_2 2_2 1_3$	$0_2 3_2 2_1$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_2$

TABLE 1 (*continued*)

C.43	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_3$	$0_2 1_2 4_3$	$0_2 2_2 1_3$	$0_2 3_2 2_1$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 5_2$
C.44	$0_1 1_1 4_2$	$0_1 2_1 1_2$	$0_1 3_1 5_3$	$0_2 1_2 4_3$	$0_2 2_2 1_3$	$0_2 3_2 5_1$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_2$
C.45	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_2$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 1_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_1$
C.46	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_2$	$0_2 1_2 5_3$	$0_2 2_2 3_3$	$0_2 3_2 2_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_1$
C.47	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_2$	$0_2 1_2 4_3$	$0_2 2_2 1_3$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_1$
C.48	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_2$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 1_3$	$0_3 1_3 5_1$	$0_3 2_3 3_1$	$0_3 3_3 2_1$
C.49	$0_1 1_1 3_2$	$0_1 2_1 6_2$	$0_1 3_1 1_2$	$0_2 1_2 3_3$	$0_2 2_2 6_3$	$0_2 3_2 1_3$	$0_3 1_3 3_1$	$0_3 2_3 6_1$	$0_3 3_3 1_1$
C.50	$0_1 1_1 4_2$	$0_1 2_1 1_2$	$0_1 3_1 5_2$	$0_2 1_2 4_3$	$0_2 2_2 1_3$	$0_2 3_2 5_3$	$0_3 1_3 4_1$	$0_3 2_3 1_1$	$0_3 3_3 5_1$

TABLE 2

Invariants and properties of the STSs

No.	G	R	K	F	k_6	k_7	k_7-base vector									
A.1	294	28	2	21	4116	182	7	7	7	23	23	23	23	23	23	23
A.2	126 c	44	4	63	4116	126	9	9	9	14	14	14	14	14	14	15
A.3	1008 c	12480	18	63	4858	406	13	41	41	41	41	41	41	49	49	49
A.4	882 c	448	4	21	4263	231	14	14	14	27	27	27	27	27	27	27
B.1	14	0	0	14	4053	59	5	5	5	6	6	6	6	6	7	7
B.2	14	0	0	28	4137	31	2	2	2	3	3	3	3	4	4	5
B.3	7	0	0	21	4018	24	0	1	2	2	2	3	3	3	4	4
B.4	7	0	0	14	4074	52	1	2	3	4	4	6	7	8	8	9
B.5	7	0	0	21	4053	45	2	2	2	3	3	5	5	7	8	8
B.6	7	0	0	42	3976	24	1	2	2	2	2	2	2	3	4	4
B.7	7	0	0	35	4102	10	0	0	0	1	1	1	1	2	2	2
B.8	7	0	0	28	4039	31	0	1	2	3	3	3	4	5	5	5
B.9	7	0	0	28	4074	24	0	0	1	1	2	3	4	4	4	5
B.10	7	0	0	35	4109	31	1	1	1	2	3	3	3	4	5	8
B.11	14	0	0	28	4053	31	2	2	2	2	2	3	3	3	6	6
B.12	14	0	0	14	4081	31	2	2	2	2	2	3	4	4	5	5
B.13	7	0	0	14	4018	38	0	1	2	3	4	4	5	5	6	8
B.14	7	0	0	28	4011	17	0	1	1	1	2	2	2	2	2	4
B.15	7	0	0	42	4081	31	2	2	2	2	3	3	3	4	5	5
B.16	14	0	0	42	3885	31	1	1	2	2	3	4	4	4	5	6
B.17	14	0	0	28	4179	17	0	0	0	0	2	2	2	3	4	4
B.18	7	0	0	35	4018	10	0	0	1	1	1	1	1	1	2	2
B.19	14	0	0	63	4858	150	13	13	13	15	15	15	15	17	17	17
B.20	14	0	0	49	3927	45	2	2	2	4	4	5	5	6	6	9
B.21	7	0	0	14	4074	24	1	1	2	2	2	3	3	3	3	4
B.22	7	1	1	28	3997	45	2	3	3	3	4	4	5	6	7	8
B.23	7	0	0	56	4018	38	2	2	2	3	4	5	5	5	5	5
B.24	7	0	0	21	4081	59	2	3	4	4	5	6	7	8	10	10
B.25	7	0	0	21	3976	52	1	2	3	4	4	5	6	6	7	14
B.26	14	0	0	14	3885	3	0	0	0	0	0	0	0	1	1	1
B.27	14	0	0	28	3983	17	1	1	1	2	2	2	2	2	2	2
B.28	14	0	0	21	4130	52	3	3	3	4	4	4	4	6	6	15
B.29	14	0	0	35	4431	59	4	4	4	4	4	5	5	8	8	13
B.30	7	0	0	21	4088	24	0	1	2	2	2	3	3	3	3	5
B.31	7	0	0	35	4046	10	0	0	0	0	1	1	1	1	2	4
B.32	7	0	0	21	4151	17	0	1	1	1	1	1	2	2	3	5
B.33	7	1	1	14	3997	17	1	1	1	1	2	2	2	2	2	3
B.34	7	0	0	28	4032	24	0	1	1	2	2	3	3	3	3	6
B.35	7	0	0	42	4081	17	0	1	1	1	2	2	2	2	3	3
B.36	7	0	0	14	4039	87	5	5	7	7	7	8	8	8	11	21

TABLE 2 (*continued*)

No.	G	R	K	F	k_6	k_7	k_7-base vector									
B.37	7	0	0	14	4074	52	1	2	3	4	5	6	6	6	7	12
B.38	14	0	0	7	4165	101	4	4	4	12	12	12	12	13	13	15
B.39	14	0	0	7	4046	80	1	1	1	10	10	10	10	11	11	15
B.40	7	0	0	49	4004	10	0	0	0	1	1	1	1	1	2	3
B.41	7	0	0	49	3969	31	2	2	3	3	3	3	3	3	4	5
C.1	7	0	0	0	4102	36	2	2	2	2	3	4	4	4	5	8
C.2	7	0	0	0	4039	43	1	3	4	4	5	5	5	5	5	6
C.3	7	0	0	21	4018	8	0	0	0	0	1	1	1	1	2	2
C.4	7	0	0	0	4102	22	1	1	1	1	2	3	3	3	3	4
C.5	7	0	0	7	4116	22	0	1	1	2	2	2	3	3	3	5
C.6	7	0	0	7	4193	29	1	2	2	2	2	3	3	4	5	5
C.7	7	0	0	7	4158	50	2	3	5	5	5	5	6	6	6	7
C.8	7	0	0	14	4039	29	1	2	2	2	2	3	3	4	5	5
C.9	7	0	0	7	3955	29	0	1	1	1	2	3	4	5	6	6
C.10	7	0	0	7	4137	29	0	0	2	2	3	3	4	4	4	7
C.11	7	0	0	14	3976	36	0	2	3	3	3	4	4	4	6	7
C.12	7	0	0	7	4102	8	0	0	1	1	1	1	1	1	1	1
C.13	7	0	0	14	4193	15	1	1	1	1	1	1	2	2	2	3
C.14	7	0	0	14	4074	50	4	4	4	4	4	5	5	6	7	7
C.15	7	0	0	21	4088	50	3	3	3	5	5	5	6	6	7	7
C.16	7	0	0	21	4186	22	0	2	2	2	2	2	2	2	4	4
C.17	7	0	0	14	4032	36	1	2	2	2	3	4	5	5	6	6
C.18	7	0	0	28	4039	29	1	1	1	2	3	3	4	4	5	5
C.19	7	0	0	14	4109	29	1	1	2	2	3	3	3	3	5	6
C.20	7	0	0	7	4060	22	1	1	1	2	2	2	2	3	4	4
C.21	7	0	0	7	4214	50	2	3	3	3	6	6	6	6	7	8
C.22	21 t	0	0	0	4165	29	1	1	1	2	3	3	3	5	5	5
C.23	21 t	0	0	7	4109	15	1	1	1	1	1	1	2	2	2	3
C.24	7	0	0	14	3934	22	1	1	1	1	2	2	2	3	4	5
C.25	7	0	0	14	4172	50	1	3	3	4	5	6	6	7	7	8
C.26	7	0	0	7	4116	22	0	1	1	1	3	3	3	3	3	4
C.27	7	0	0	0	4067	15	0	0	0	1	1	2	2	3	3	3
C.28	7	0	0	0	4053	1	0	0	0	0	0	0	0	0	0	1
C.29	7	0	0	0	4130	36	1	2	2	3	3	3	3	5	6	8
C.30	7	0	0	0	4165	15	0	0	0	1	1	1	2	2	3	5
C.31	7	0	0	14	4004	22	0	1	1	2	2	2	3	3	4	4
C.32	7	0	0	7	4137	29	0	2	2	3	3	3	3	4	4	5
C.33	7	0	0	7	4004	22	0	0	1	1	2	3	3	3	4	5
C.34	7	0	0	21	4151	43	1	3	3	3	4	4	6	6	6	7
C.35	7	0	0	0	4067	29	1	2	2	2	2	3	3	4	5	5
C.36	7	0	0	7	4074	8	0	0	0	0	1	1	1	1	2	2
C.37	7	0	0	7	4074	22	0	0	1	1	2	3	3	3	4	5
C.38	21 t	0	0	0	4193	29	2	3	3	3	3	3	3	3	3	3
C.39	21 c	0	0	0	4179	57	3	3	3	5	5	5	6	6	6	15
C.40	7	0	0	7	4095	15	0	0	0	1	2	2	2	2	3	3
C.41	7	0	0	21	4060	22	0	1	1	1	2	2	2	4	4	5
C.42	7	0	0	7	4046	8	0	0	0	0	1	1	1	1	2	2
C.43	7	0	0	21	4172	36	2	3	3	3	4	4	4	4	4	5
C.44	42 c	0	0	42	4060	106	7	7	7	10	10	10	11	11	11	22
C.45	21	0	0	42	3955	43	4	4	4	4	4	4	4	4	4	7
C.46	21	0	0	63	3955	43	4	4	4	4	4	4	4	4	4	7
C.47	21	0	0	21	3976	22	2	2	2	2	2	2	2	2	2	4
C.48	21	0	0	63	4039	1	0	0	0	0	0	0	0	0	0	1
C.49	504 c	0	0	63	4858	22	1	1	1	1	1	1	1	1	1	13
C.50	126 c	0	0	42	4396	64	5	5	5	5	5	5	5	5	5	19

3. A Class of Kirkman Triple Systems of Order 21. As seen from Table 2, only six of the 95 STSs admitting α as an automorphism can be resolved, i.e. can serve as an underlying STS of a Kirkman triple system. However, these six resolvable STSs account for 30 nonisomorphic KTSs. Surprisingly, very few Kirkman triple systems of order 21 seem to have appeared in the literature. Ahrens [1] mentions that there exists a system of this order due to Dudeney [11] but does not display it; the paper [11] is inaccessible to us. The KTS of order 21 in [5] and the KTS in [12] are both isomorphic to the KTS No. A.4.b (see Table 3). Four distinct KTSs of order 21 appear in older editions of Rouse Ball's book on mathematical recreations [3]; the first three are isomorphic to the KTS No. A.4.b while the remaining one is isomorphic to the KTS No. A.3.r (cf. also [15] where the nonisomorphism of these two KTSs is established by using an invariant based on types of interlacing of two parallel classes).

The listing of all 30 KTSs appears in Table 3. Each KTS is listed in a compact form as a 10×7 array of ordered pairs; each of the 10 rows of the array represents a parallel class. If a pair ij occurs in row k, this means that the triple belonging to the orbit determined by the jth base block and obtained from the representative of this orbit by adding i (mod 7) to each its element belongs to the kth parallel class. For easy reference, the representatives of the orbits of the underlying STSs appear again in Table 3 under the heading "base-block classes". Since the orbits are numbered 0 through 9 and each orbit contains 7 triples, each ordered pair ij, with $0 \leqslant i \leqslant 6$, $0 \leqslant j \leqslant 9$, occurs exactly once in the array representing a KTS. The numbering of the KTSs is such that the letter and the digit in the first two places indicate the type of the underlying STS.

In Table 4 we display some invariants of the KTSs given in Table 3: the order of the automorphism group (G), the number of 6-sets ($k_6^{\perp}$) and 7-sets ($k_7^{\perp}$) of disjoint triples (= parallel classes) orthogonal to the given resolution (i.e. to the given KTS), and the r-dimension of the KTS (RD_K). The last three invariants require some explanation. A collection of triples C is orthogonal to a resolution R if, for each $R_i \in R$, $|C \cap R_i| \leqslant 1$. (The invariants $k_6^{\perp}$ and $k_7^{\perp}$ distinguish the KTSs completely.) Similarly, two resolutions $R = \{R_1, \dots, R_u\}$, $R' = \{R'_1, \dots, R'_u\}$ of an STS (V, B) are orthogonal if $|R_i \cap R'_j| \leqslant 1$ for all $i, j = 1, \dots, u$. More generally, a set $\{R^{(i)} | R^{(i)} = \{R_1^{(i)}, \dots, R_u^{(i)}\}, i = 1, \dots, t\}$ of t ($t \geqslant 2$) resolutions of (V, B) is a d-orthogonal set ($d \geqslant 2$) of resolutions if

$$|R_{i_1}^{(j_1)} \cap R_{i_2}^{(j_2)} \cap \cdots \cap R_{i_d}^{(j_d)}| \leqslant 1$$

for any $\{j_1, j_2, \dots, j_d\} \subset \{1, 2, \dots, t\}$, and any $i_k \in \{1, \dots, u\}$, $k = 1, \dots, d$. The smallest number d (if it exists) such that there exists a d-orthogonal set of d resolutions $(V, B, R^{(i)})$ of an STS (V, B) is called the r-dimension (RD_S) of the STS; otherwise $RD_S = \infty$. The smallest number d (if it exists) such that there exists a d-orthogonal set of d resolutions $(V, B, R^{(i)})$ of the KTS (V, B, R) with all $R^{(i)} \sim R$ is called the r-dimension (RD_K) of the KTS; otherwise $RD_K = \infty$. If $RD_S = 2$, the corresponding STS is said to be doubly resolvable [13]; doubly resolvable STSs provide perhaps the nicest examples of generalized Room squares; cf. [13]. Unfortunately, none of the 30 KTSs in Table 3 are doubly resolvable; it follows then, from Table 4, that RD_S equals 3 for the STSs A.1, A.2, and A.4 (and so one can construct from them Kirkman cubes, cf. [20]), and equals ∞ for A.3, B.22, and B.33. However, one of the KTSs is "almost" doubly resolvable. If one takes the KTS No. A.3.r and omits the orbit No. 8 (triples $0_1 5_2 1_3$ mod 7) from the set of triples, the remaining set of triples can be partitioned into two orthogonal "resolutions". The resulting square of side 9 below is a special example of a generalized Howell design of degree 3 (cf. [19]).

$0_{1}1_{1}3_{1}$		$6_{1}6_{2}6_{3}$	$0_{3}1_{3}3_{3}$		$2_{1}2_{2}2_{3}$	$0_{2}1_{2}3_{2}$	$4_{1}4_{2}4_{3}$	$5_{1}5_{2}5_{3}$
$5_{1}0_{2}4_{3}$	$1_{1}2_{1}4_{1}$	$0_{1}4_{2}3_{3}$		$5_{2}6_{2}1_{2}$	$5_{3}6_{3}1_{3}$		$6_{1}3_{2}2_{3}$	$3_{1}2_{2}0_{3}$
	$3_{2}4_{2}6_{2}$	$2_{1}3_{1}5_{1}$	$1_{1}5_{2}4_{3}$	$2_{3}3_{3}5_{3}$	$4_{1}1_{2}0_{3}$		$0_{1}2_{2}6_{3}$	$6_{1}0_{2}1_{3}$
	$3_{3}4_{3}6_{3}$	$2_{2}3_{2}5_{2}$	$3_{1}4_{1}6_{1}$	$2_{1}4_{2}1_{3}$		$5_{1}6_{2}0_{3}$	$1_{1}0_{2}5_{3}$	$0_{1}1_{2}2_{3}$
$6_{3}0_{3}2_{3}$	$6_{1}1_{2}5_{3}$			$4_{1}5_{1}0_{1}$	$4_{2}5_{2}0_{2}$	$1_{1}2_{2}3_{3}$	$3_{1}6_{2}1_{3}$	$2_{1}3_{2}4_{3}$
$1_{2}2_{2}4_{2}$			$0_{1}3_{2}5_{3}$	$3_{1}0_{2}6_{3}$	$5_{1}6_{1}1_{1}$	$1_{3}2_{3}4_{3}$	$2_{1}5_{2}0_{3}$	$4_{1}6_{2}3_{3}$
$4_{1}3_{2}1_{3}$	$3_{1}5_{2}2_{3}$	$4_{3}5_{3}0_{3}$	$6_{2}0_{2}2_{2}$			$6_{1}0_{1}2_{1}$	$5_{1}1_{2}3_{3}$	$1_{1}4_{2}6_{3}$
$2_{1}6_{2}5_{3}$	$0_{1}0_{2}0_{3}$	$1_{1}1_{2}1_{3}$	$5_{1}4_{2}2_{3}$	$6_{1}2_{2}4_{3}$	$3_{1}3_{2}3_{3}$	$4_{1}5_{2}6_{3}$		
$6_{1}5_{2}3_{3}$	$5_{1}2_{2}1_{3}$	$4_{1}0_{2}2_{3}$	$2_{1}1_{2}6_{3}$	$1_{1}3_{2}0_{3}$	$0_{1}6_{2}4_{3}$	$3_{1}4_{2}5_{3}$		

TABLE 3

Kirkman triple systems

Base-block classes:

	A.1	A.2	A.3	A.4	B.22	B.33
0	$0_{1}1_{1}3_{1}$	$0_{1}1_{1}3_{1}$	$0_{1}1_{1}3_{1}$	$0_{1}1_{1}3_{1}$	$0_{1}1_{1}3_{1}$	$0_{1}1_{1}3_{1}$
1	$0_{2}1_{2}3_{2}$	$0_{2}1_{2}3_{2}$	$0_{2}1_{2}3_{2}$	$0_{2}1_{2}3_{2}$	$0_{1}0_{2}0_{3}$	$0_{1}0_{2}0_{3}$
2	$0_{3}1_{3}3_{3}$	$0_{3}1_{3}3_{3}$	$0_{3}1_{3}3_{3}$	$0_{3}1_{3}3_{3}$	$0_{1}1_{2}3_{3}$	$0_{1}1_{2}5_{3}$
3	$0_{1}0_{2}0_{3}$	$0_{1}0_{2}0_{3}$	$0_{1}0_{2}0_{3}$	$0_{1}0_{2}0_{3}$	$0_{1}3_{2}2_{3}$	$0_{1}3_{2}1_{3}$
4	$0_{1}1_{2}2_{3}$	$0_{1}1_{2}2_{3}$	$0_{1}1_{2}2_{3}$	$0_{1}1_{2}3_{3}$	$0_{1}2_{2}4_{2}$	$0_{1}2_{2}4_{2}$
5	$0_{1}2_{2}4_{3}$	$0_{1}2_{2}5_{3}$	$0_{1}2_{2}6_{3}$	$0_{1}2_{2}6_{3}$	$0_{1}5_{2}6_{2}$	$0_{1}5_{2}6_{2}$
6	$0_{1}3_{2}6_{3}$	$0_{1}3_{2}1_{3}$	$0_{1}3_{2}5_{3}$	$0_{1}3_{2}2_{3}$	$0_{1}1_{3}5_{3}$	$0_{1}2_{3}4_{3}$
7	$0_{1}4_{2}1_{3}$	$0_{1}4_{2}6_{3}$	$0_{1}4_{2}3_{3}$	$0_{1}4_{2}5_{3}$	$0_{1}4_{3}6_{3}$	$0_{1}3_{3}6_{3}$
8	$0_{1}5_{2}3_{3}$	$0_{1}5_{2}4_{3}$	$0_{1}5_{2}1_{3}$	$0_{1}5_{2}1_{3}$	$0_{2}3_{2}1_{3}$	$0_{2}3_{2}6_{3}$
9	$0_{1}6_{2}5_{3}$	$0_{1}6_{2}3_{3}$	$0_{1}6_{2}4_{3}$	$0_{1}6_{2}4_{3}$	$0_{2}3_{3}4_{3}$	$0_{2}1_{3}2_{3}$

TABLE 3 (*continued*)

	A.1a	A.1b	A.2a
1	00 01 02 23 43 53 63	00 01 02 23 43 53 63	00 01 02 23 43 53 63
2	10 21 32 04 34 54 64	10 51 22 07 37 57 67	10 41 22 06 36 56 65
3	20 41 62 05 15 45 65	20 31 42 04 14 44 64	20 51 32 03 16 67 48
4	30 61 22 06 16 26 56	30 11 62 08 18 28 58	30 61 42 13 26 07 58
5	40 11 52 17 27 37 67	40 61 12 15 25 35 65	40 21 12 18 28 38 68
6	50 31 12 08 28 38 48	50 41 32 09 29 39 49	50 11 62 33 46 27 08
7	60 51 42 19 39 49 59	60 21 52 16 36 46 56	60 31 52 17 37 47 57
8	03 14 25 36 47 58 69	03 24 45 66 17 38 59	04 14 24 34 44 54 64
9	13 24 35 46 57 68 09	13 34 55 06 27 48 69	05 15 25 35 45 55 65
10	33 44 55 66 07 18 29	33 54 05 26 47 68 19	09 19 29 39 49 59 69

	A.2b	A.2c	A.2d
1	00 01 02 23 43 53 63	00 31 12 53 66 47 28	00 31 12 53 66 47 28
2	10 41 22 06 36 56 65	10 41 22 63 06 57 38	10 41 22 63 06 57 38
3	20 51 32 64 05 16 49	20 51 32 03 16 67 48	20 51 32 03 16 67 48
4	30 61 42 04 15 26 59	30 61 42 13 26 07 58	30 61 42 04 15 26 59
5	40 21 12 18 28 38 68	40 01 52 23 36 17 68	40 01 52 14 25 36 69
6	50 11 62 24 35 46 09	50 11 62 33 46 27 08	50 11 62 33 46 27 08
7	60 31 52 17 37 47 57	60 21 02 43 56 37 18	60 21 02 34 45 56 19
8	03 14 34 44 65 27 58	04 14 24 34 44 54 64	13 24 44 54 05 37 68
9	13 25 45 55 67 08 39	05 15 25 35 45 55 65	23 35 55 65 07 18 49
10	33 54 07 48 19 29 69	09 19 29 39 49 59 69	43 64 17 58 09 29 39

	A.3a	A.3b	A.3c
1	00 01 02 23 43 53 63	00 01 02 23 43 53 63	00 01 02 23 43 53 63
2	10 11 12 54 06 37 69	10 11 12 54 06 37 69	10 11 12 54 06 37 69
3	20 21 22 64 16 47 09	20 21 22 64 16 47 09	20 21 22 64 16 47 09
4	30 31 32 04 26 57 19	30 41 32 04 15 27 57	30 51 62 24 07 57 19
5	40 41 42 14 36 67 29	40 51 62 33 25 66 19	40 41 42 14 36 67 29
6	50 51 52 24 46 07 39	50 61 42 35 07 29 49	50 61 32 04 34 26 49
7	60 61 62 34 56 17 49	60 31 52 46 56 17 39	60 31 52 46 56 17 39
8	03 13 33 44 66 27 59	03 14 24 44 65 36 59	03 13 33 44 66 27 59
9	05 15 25 35 45 55 65	13 34 05 45 55 26 67	05 15 25 35 45 55 65
10	08 18 28 38 48 58 68	08 18 28 38 48 58 68	08 18 28 38 48 58 68

	A.3d	A.3e	A.3f
1	00 01 02 23 43 53 63	00 01 02 23 43 53 63	00 01 02 23 43 53 63
2	10 11 12 54 06 37 69	10 11 12 54 06 37 69	10 11 12 54 06 37 69
3	20 31 42 04 14 44 64	20 41 52 14 47 67 09	20 41 52 14 47 67 09
4	30 41 22 15 57 09 29	30 31 32 04 26 57 19	30 51 62 24 07 57 19
5	40 61 62 33 25 65 17	40 51 22 24 64 16 39	40 31 22 64 17 29 39
6	50 51 52 24 46 07 39	50 21 42 36 46 07 29	50 61 32 04 34 26 49
7	60 21 32 36 47 19 59	60 61 62 34 56 17 49	60 21 42 16 36 46 56
8	03 35 16 56 66 27 49	03 13 33 44 66 27 59	03 13 33 44 66 27 59
9	13 34 05 45 55 26 67	05 15 25 35 45 55 65	05 15 25 35 45 55 65
10	08 18 28 38 48 58 68	08 18 28 38 48 58 68	08 18 28 38 48 58 68

TABLE 3 (*continued*)

1	00	01	02	23	43	53	63	00	01	02	23	43	53	63	00	01	02	23	43	53	63
2	10	11	12	54	06	37	69	10	11	32	54	64	35	06	10	11	32	54	64	35	06
3	20	61	52	07	17	47	67	20	31	12	05	47	19	69	20	51	22	03	16	66	49
4	30	31	32	04	26	57	19	30	51	52	03	14	24	59	30	31	12	05	26	56	19
5	40	51	62	14	24	34	64	40	41	22	15	36	66	29	40	41	62	14	24	65	36
6	50	41	22	09	29	39	49	50	21	42	04	25	45	37	50	21	42	04	25	45	37
7	60	21	42	16	36	46	56	60	61	62	34	56	17	49	60	61	52	33	17	47	59
8	03	13	33	44	66	27	59	13	44	55	07	27	67	39	13	44	55	07	27	67	39
9	05	15	25	35	45	55	65	33	65	16	26	46	57	09	34	15	46	57	09	29	69
10	08	18	28	38	48	58	68	08	18	28	38	48	58	68	08	18	28	38	48	58	68

A.3g A.3h A.3i

1	00	01	02	23	43	53	63	00	01	02	23	43	53	63	00	01	02	23	43	53	63
2	10	11	32	54	64	35	06	10	11	32	54	64	35	06	10	11	32	54	64	35	06
3	20	51	52	15	46	66	07	20	61	42	04	16	49	69	20	61	52	07	17	47	67
4	30	21	62	25	56	09	19	30	41	22	15	57	09	29	30	31	12	05	26	56	19
5	40	31	12	14	65	26	37	40	51	62	33	25	66	19	40	41	62	14	24	65	36
6	50	41	22	24	05	36	47	50	21	12	07	27	37	47	50	21	42	04	25	45	37
7	60	61	42	13	33	44	59	60	31	52	46	56	17	39	60	51	22	55	16	39	49
8	03	34	45	17	57	67	29	03	14	24	44	65	36	59	03	13	33	44	66	27	59
9	04	55	16	27	39	49	69	13	34	05	45	55	26	67	34	15	46	57	09	29	69
10	08	18	28	38	48	58	68	08	18	28	38	48	58	68	08	18	28	38	48	58	68

A.3j A.3k A.3ℓ

1	00	01	02	23	43	53	63	00	01	02	23	43	53	63	00	01	02	23	43	53	63
2	10	11	32	54	64	35	06	10	21	12	54	65	07	37	10	21	12	54	65	07	37
3	20	61	62	65	07	17	49	20	11	62	64	45	06	17	20	31	22	64	05	17	47
4	30	31	12	05	26	56	19	30	31	52	04	14	55	26	30	51	62	14	25	55	06
5	40	51	42	14	25	37	67	40	51	32	25	67	19	39	40	11	32	26	36	67	19
6	50	41	52	04	24	45	39	50	41	22	24	05	36	47	50	41	52	04	24	45	39
7	60	21	22	55	16	36	47	60	61	42	13	33	44	59	60	61	42	35	16	56	49
8	03	13	33	44	66	27	59	03	35	16	56	66	27	49	03	13	33	44	66	27	59
9	34	15	46	57	09	29	69	34	15	46	57	09	29	69	34	15	46	57	09	29	69
10	08	18	28	38	48	58	68	08	18	28	38	48	58	68	08	18	28	38	48	58	68

A.3m A.3n A.3o

1	00	01	02	23	43	53	63	00	01	02	23	43	53	63	00	01	02	23	43	53	63
2	10	21	32	04	34	54	64	10	21	62	55	65	36	07	10	51	52	55	07	67	39
3	20	41	62	13	45	06	66	20	31	22	64	05	17	47	20	31	22	64	05	17	47
4	30	51	22	06	15	25	55	30	41	52	04	14	24	54	30	21	32	04	54	25	19
5	40	31	12	19	29	39	69	40	51	32	25	67	19	39	40	41	62	14	24	65	36
6	50	11	52	33	26	46	09	50	11	12	45	06	26	37	50	11	12	45	06	26	37
7	60	61	42	35	16	56	49	60	61	42	35	16	56	49	60	61	42	35	16	56	49
8	03	14	24	44	65	36	59	03	13	33	44	66	27	59	03	13	33	44	66	27	59
9	07	17	27	37	47	57	67	34	15	46	57	09	29	69	34	15	46	57	09	29	69
10	08	18	28	38	48	58	68	08	18	28	38	48	58	68	08	18	28	38	48	58	68

A.3p A.3q A.3r

TABLE 3 (*continued*)

	A.4a	A.4b	A.4c
1	00 01 02 23 43 53 63	00 01 02 23 43 53 63	00 01 02 23 43 53 63
2	10 11 12 03 36 58 69	10 21 42 04 34 54 64	10 41 32 54 65 06 39
3	20 21 22 13 46 68 09	20 41 12 05 15 45 65	20 11 62 45 67 08 19
4	30 61 52 06 16 26 56	30 61 52 06 16 26 56	30 51 22 03 14 25 58
5	40 41 42 33 66 18 29	40 11 22 17 27 37 67	40 21 52 18 28 38 68
6	50 31 62 08 28 38 48	50 31 62 08 28 38 48	50 61 12 04 24 34 44
7	60 51 32 19 39 49 59	60 51 32 19 39 49 59	60 31 42 13 35 46 57
8	04 14 24 34 44 54 64	03 14 25 36 47 58 69	33 55 66 07 17 27 47
9	05 15 25 35 45 55 65	13 24 35 46 57 68 09	64 05 16 26 36 56 49
10	07 17 27 37 47 57 67	33 44 55 66 07 18 29	15 37 48 09 29 59 69

	A.4d	B.22	B.33
1	00 01 02 23 43 53 63	00 44 65 26 57 08 29	00 44 65 56 27 08 29
2	10 41 32 06 36 56 66	10 54 05 36 67 18 39	10 54 05 66 37 18 39
3	20 11 62 09 19 49 69	20 64 15 46 07 28 49	20 64 15 06 47 28 49
4	30 51 22 05 15 25 55	30 04 25 56 17 38 59	30 04 25 16 57 38 59
5	40 21 52 18 28 38 68	40 14 35 66 27 48 69	40 14 35 26 67 48 69
6	50 61 12 04 24 34 44	50 24 45 06 37 58 09	50 24 45 36 07 58 09
7	60 31 42 17 37 47 57	60 34 55 16 47 68 19	60 34 55 46 17 68 19
8	03 54 35 16 67 48 29	01 11 21 31 41 51 61	01 11 21 31 41 51 61
9	13 64 45 26 07 58 39	02 12 22 32 42 52 62	02 12 22 32 42 52 62
10	33 14 65 46 27 08 59	03 13 23 33 43 53 63	03 13 23 33 43 53 63

TABLE 4

Some invariants and properties of Kirkman triple systems

No.	G	$k_6^{\perp}$	$k_7^{\perp}$	d
A.1a	21	448	25	3
A.1b	21	581	32	3
A.2a	9	320	14	6
A.2b	9	380	23	3
A.2c	63	245	32	∞
A.2d	9	332	32	∞
A.3a	3	671	23	∞
A.3b	1	479	4	∞
A.3c	3	569	5	∞
A.3d	1	578	15	∞
A.3e	1	542	3	∞
A.3f	1	517	3	∞
A.3g	1	515	3	∞
A.3h	3	475	1	∞
A.3i	1	517	9	∞
A.3j	1	501	2	∞
A.3k	1	496	4	∞
A.3ℓ	1	533	6	∞
A.3m	3	579	27	∞
A.3n	1	504	5	∞
A.3o	3	586	13	∞
A.3p	3	558	15	∞
A.3q	3	617	11	∞
A.3r	21	875	53	∞
A.4a	9	533	32	4
A.4b	63	560	32	3
A.4c	3	348	3	3
A.4d	21	427	25	3
B.22	7	308	0	∞
B.33	7	322	0	∞

4. Concluding Remarks. Although we obtained 30 KTSs of order 21, almost all of which appear to have been unknown earlier, there are no doubt many other KTSs of this order. Complete enumeration (which, apart from being of independent interest, would enable one to settle the question of whether there exists a doubly resolvable STS of order 21, an important step enroute to determine the spectrum of doubly resolvable STSs) does not appear to be feasible. One could attempt either an enumeration of a class of STSs with some additional property, in a manner similar to that of this paper, and then test the obtained STSs for resolvability, or one could try to derive a set of transformations of (the known) KTSs leading to discovery of new KTSs. One possibility for an approach of the first kind would be to examine the set of STSs admitting an automorphism consisting of one fixed element and four 5-cycles; STSs of this kind do exist; cf. [16]. (On the other hand, an STS of order 21 with an automorphism having one fixed element and one 20-cycle or one fixed element and two 10-cycles cannot exist [16].) As for the latter possibility, we ask at this point only whether there exist automorphism-free STSs of order 21 that are resolvable.

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