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The Finite Element Method in a Family of Improperly Posed Problems

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Abstract. The numerical solution of the Cauchy problem for elliptic equations is considered. We reformulate the original problem as a variational inequality problem, which we solve using the finite element method. Moreover, we prove the convergence of the approximate solution.

Let $\mathfrak{D}$ be a bounded open set in the space $\mathbf{R}^n$ and $\partial\mathfrak{D}$ be the boundary of $\mathfrak{D}$. Then $\Omega_T = \mathfrak{D} \times (0, T)$ is a bounded open set in $\mathbf{R}^{n+1}$. We discuss the following boundary value problem for the elliptic equation:

$$\text{(I)}\qquad \begin{gathered} Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\lambda \frac{\partial u}{\partial t}\right) = 0, \\ u|_{\partial\mathfrak{D}\times[0,T]} = 0, \\ u|_{t=0} = f(x), \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \end{gathered}$$

Here A_{ij}, λ are functions of x and, moreover,

$$\nu(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2) \leqslant \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \leqslant \nu_1(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2),$$

$\nu_1 \geqslant \nu > 0$; $\lambda \geqslant \lambda_0 > 0$. ν, ν_1, λ_0 are constants.

If there are no added restrictions to the solutions of (I), J. Hadamard [1] has pointed out that the solution of (I) is not continuously dependent on the Cauchy data. So problem (I) is an improperly posed problem. As the famous example of J. Hadamard has shown, it is impossible to solve this improperly posed problem by the classical theory of partial differential equations. But these types of problems arise naturally in many kinds of practical problems and therefore have required the attention of many mathematicians. First, M. M. Lavrentiev [2] has discussed bounded solutions of the Laplace equation in a special two-dimensional domain. These solutions are dependent on the Cauchy data continuously. After this L. E. Payne [3], [4] studied solutions of more general second-order elliptic equations, which are dependent on the Cauchy data continuously. Of course, it is necessary to add some restrictions to the domains and the solutions. In 1975 L. E. Payne outlined this problem in [5].

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In this paper we shall first discuss the solutions of problem (I) which are continuously dependent on Cauchy data in a set of solutions with bounded energy. Next we change this problem into a variational inequality problem. We shall give a method for solving this last problem by the finite element method.

1. The Continuous Dependence of the Solutions of Problem (I) on the Cauchy Data. Let us define

$$a(u, v) \equiv \iint_{\Omega_T} \left(\sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \lambda \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \right) dx\, dt, \qquad J(u) = \frac{1}{2} a(u, u).$$

Let us assume that u is the solution of problem (I) and that u satisfies

$$J(u) \leqslant M, \tag{1.1}$$

where $M > 0$ is a constant. Now we consider the continuous dependence of u on f and g.

We define

$$E_1 = \|f\|^2_{H^0(\mathfrak{D})}, \qquad E_2 = \|g\|^2_{H^0(\mathfrak{D})},$$

where $H^0(\mathfrak{D})$ is the Sobolev space on the n-dimensional domain $\mathfrak{D}$, and assume

$$E_1 + E_2 \leqslant M_0, \tag{1.2}$$

where M_0 is a constant.

Consider the following functions,

$$A(t) = \iint_{\Omega_t} \left(\sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \lambda \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} \right) dx\, dt,$$

$$F(t) = \int_0^t A(\tau)\, d\tau + k_1 E_1 + k_2 E_2,$$

where $\Omega_t = \mathfrak{D} \times (0, t)$ and k_1, k_2 are constants to be determined. From $F(t)$ we introduce the function

$$V(t) = \ln F(t) + \frac{t^2}{2}. \tag{1.3}$$

Independently of u, f, and g, we may choose k_1 and k_2 sufficiently large to get

$$F^2 \frac{d^2 V}{dt^2} \geqslant 0, \qquad 0 < t < T. \tag{1.4}$$

The detailed computation is omitted here. A similar computation can be found in [5].

Thus V is a convex function of t on $[0, T]$. For every $t \in [0, T]$, we have

$$V(t) \leqslant V(0) \frac{T - t}{T} + V(T) \frac{t}{T}. \tag{1.5}$$

From (1.5), we obtain the following theorem.

THEOREM 1.1. *Let us assume that A_{ij}, λ are functions of the variable x only. Then the solutions u of problem* (I), *which satisfy* (1.1) *depend continuously on the Cauchy data, and we have the estimates*

$$\int_{\mathfrak{D}} u^2(x, t)\, dx \leqslant H_1 (k_1 E_1 + k_2 E_2)^{(T-t)/T}, \qquad 0 \leqslant t \leqslant T, \tag{1.6}$$

where H_1 is a constant given in the following proof.

Proof. Substituting (1.3) into (1.5), we obtain

$$\ln F(t) + \frac{t^2}{2} \leqslant \frac{T-t}{T} \ln F(0) + \frac{t}{T}\left(\ln F(T) + \frac{T^2}{2}\right);$$

namely,

$$F(t)e^{t^2/2} \leqslant F(0)^{(T-t)/T} \cdot F(T)^{t/T} e^{tT/2}.$$

Therefore,

$$F(t) \leqslant e^{(T-t)t/2} F(0)^{(T-t)/T} F(T)^{t/T}.$$

As $F(0) = k_1 E_1 + k_2 E_2$,

$$F(T) \leqslant MT + k_1 E_1 + k_2 E_2 \leqslant MT + M_0(k_1 + k_2),$$

then

$$F(t) \leqslant H_1(K_1 E_1 + K_2 E_2)^{(T-t)/T}.$$

Here H_1 is a constant, which depends only on M_0, M, A_{ij}, λ and T. But it may have different numerical values at different places in what follows.

On the other hand, we know

$$F(t) \geqslant \frac{1}{2} \int_{\mathcal{D}} \lambda u^2 \, dx \geqslant \frac{\lambda_0}{2} \int_{\mathcal{D}} u^2 \, dx,$$

so we obtain

$$\int_{\mathcal{D}} u^2 \, dx \leqslant H_1(K_1 E_1 + K_2 E_2)^{(T-t)/T},$$

which is (1.6).

Moreover, by integrating the above expression from 0 to T, we get

$$\iint_{\mathcal{D}_T} u^2 \, dx \leqslant \frac{H_1}{|\ln[(K_1 E_1 + K_2 E_2)]|},$$

where $K_1 E_1 + K_2 E_2 \neq 1$, of course.

2. A Variational Inequality. Since Cauchy data are obtained by measuring, we can only get approximations f_1, g_1 of f, g. How do we find the approximate solution of problem (I) from f_1, g_1? In this section, we give an answer to this question.

We first consider the following problem.

$$\begin{aligned} &u|_{\partial \mathcal{D} \times [0,T]} = 0, \\ &\|u(x,0) - f_1(x)\|_{H^0(\mathcal{D})} \leqslant \alpha_1, \\ &\left\| \frac{\partial u(x,0)}{\partial t} - g_1(x) \right\|_{H^0(\mathcal{D})} \leqslant \alpha_2, \\ &J(u) = \inf J(v), \end{aligned} \tag{2.1}$$

where $\alpha_1, \alpha_2 > 0$, are upper bounds of the errors in the measurements, namely,

$$\begin{aligned} &\|f - f_1\|_{H^0(\mathcal{D})} \leqslant \alpha_1, \\ &\|g - g_1\|_{H^0(\mathcal{D})} \leqslant \alpha_2. \end{aligned} \tag{2.2}$$

Let us assume that $\bar{u}$ is a solution of problem (2.1). If $\bar{u}$ satisfies the differential equation in problem (I), then we know $\bar{u}$ is an approximate solution of problem (I) from Theorem 1.1. (Of course, the solution of problem (I) must exist. Let us denote it by u.) Moreover, we have

$$\int_{\mathfrak{D}} [u(x, t) - \bar{u}(x, t)]^2 \, dx \leqslant H_1(K_1\alpha_1^2 + K_2\alpha_2^2)^{(T-t)/T}, \qquad 0 \leqslant t \leqslant T.$$

In order to find an approximate solution of problem (I), we will solve problem (2.1). Problem (2.1) is a minimum problem of functional $J(v)$ on a convex set. To discuss the solvability, we introduce the space $\overset{\bullet}{H}{}^1(\Omega_T)$.

Let us define

$$\overset{\bullet}{C}{}^\infty(\overline{\Omega}_T) = \left\{ v | v \in C^\infty(\overline{\Omega}_T), \text{ support of } v \text{ in } \mathfrak{D} \times [0, T] \right\}.$$

Using the norm

$$\|v\|^2_{\overset{\bullet}{H}{}^1(\Omega_T)} = \|v\|^2_{H^1(\Omega_T)} + \|v(x, 0)\|^2_{H^0(\mathfrak{D})} + \left\| \frac{\partial v}{\partial t}(x, 0) \right\|^2_{H^0(\mathfrak{D})},$$

we complete the space $\overset{\bullet}{C}{}^\infty(\overline{\Omega}_T)$ to obtain a new space denoted by $\overset{\bullet}{H}{}^1(\Omega_T)$. Here $H^1(\Omega_T)$ is the usual Sobolev space on Ω_T.

We consider a subset of $\overset{\bullet}{H}{}^1(\Omega_T)$,

$$K = \left\{ v | v \in \overset{\bullet}{H}{}^1(\Omega_T); \|v(x, 0) - f_1\|^2_{H^0(\mathfrak{D})} \leqslant \alpha_1^2, \left\| \frac{\partial v(x, 0)}{\partial t} - g_1 \right\|^2_{H^0(\mathfrak{D})} \leqslant \alpha_2^2 \right\}.$$

Obviously, K is a closed convex set in $\overset{\bullet}{H}{}^1(\overline{\Omega}_T)$. If the solution of problem (I) exists, then K is not empty. So problem (2.1) is equivalent to a variational problem:

(2.3) $$J(\bar{u}) = \inf_{v \in K} J(v).$$

For the variational problem (2.3), we have

LEMMA 2.1. *The variational problem* (2.3) *and the following variational inequality* (2.4) *are equivalent*,

(2.4) $$a(\bar{u}, v - \bar{u}) \geqslant 0 \quad \forall v \in K.$$

LEMMA 2.2. *The solution of problem* (2.4) *is unique.*

The proof of Lemma 2.1 and Lemma 2.2 can be found in [6].

From Lemmas 2.1, 2.2 we also know that the solution of (2.3) is unique.

LEMMA 2.3. *If K is not empty, the solution of* (2.3) *exists.*

Proof. Since the functional $J(v)$ is nonnegative, we have

$$\alpha = \operatorname*{Inf}_{v \in K} J(v) \geqslant 0.$$

Thus there exists a minimization sequence $u_n \in K$, namely,

$$J(u_n) \to \alpha, \qquad n \to \infty, u_n \in K.$$

We know that the bilinear form $a(u, u)$ is not coercive and that K is not bounded, but $J(u_n)$ is bounded. On the other hand, as $u_n \in K$, then

$$\|u_n(x, 0)\|_{H^0(\mathfrak{D})}, \qquad \left\| \frac{\partial u_n(x, 0)}{\partial t} \right\|_{H^0(\mathfrak{D})}$$

are bounded.

Therefore we know u_n is a bounded sequence in the space $\overset{*}{H}(\Omega_T)$. We can extract a subsequence u_{n_K} so that it will converge to $\bar{u}$ weakly. As K is a closed convex set, K is also a closed set in the weak topology. Thus $\bar{u} \in K$. On the other hand, $J(v)$ is a convex functional and $J(v)$ is lower semicontinuous [6]. We have $\underline{\lim}_{k \to \infty} J(u_{n_K}) \geqslant J(\bar{u})$. From this we obtain $J(\bar{u}) = \alpha$. Hence $\bar{u}$ is a solution of (2.4). From Lemma 2.1 we know $\bar{u}$ is a solution of (2.3).

LEMMA 2.4. *Let us assume A_{ij}, $\lambda \in C^{1,\beta}(\Omega_T)$. Then the solution of problem* (2.3) *$\bar{u} \in C^2(\Omega_T)$ satisfies the differential equation*

$$L\bar{u} = 0.$$

Proof. Let us denote an arbitrary function in the space $C_0^\infty(\Omega_T)$ by w (namely, $w \in C^\infty(\Omega_T)$ and the support of w is in Ω_T).

Let us define

$$v_+ \equiv \bar{u} + w, \qquad v_- \equiv \bar{u} - w,$$

then $v_+, v_- \in K$. As $\bar{u}$ is the solution of (2.4), we obtain

$$a(\bar{u}, v_+ - \bar{u}) \geqslant 0, \qquad a(\bar{u}, v_- - \bar{u}) \geqslant 0.$$

Namely

$$a(\bar{u}, w) \geqslant 0, \qquad -a(\bar{u}, w) \geqslant 0.$$

Thus we have

$$a(\bar{u}, w) = 0.$$

Let us denote an arbitrary element of the Sobolev space $\overset{\circ}{H}{}^1(\Omega_T)$ by $\mathring{w}$. Then for $\mathring{w}$ we have a sequence $w_n \in C_0^\infty(\Omega_T)$ $(n = 1, 2, \ldots)$ and $\{w_n\}$ converges to $\mathring{w}$ in $\overset{\circ}{H}{}^1(\Omega_T)$. For every w_n, we have

$$a(\bar{u}, w_n) = 0.$$

As $n \to +\infty$, we obtain

$$a(\bar{u}, \mathring{w}) = 0, \qquad \mathring{w} \in \overset{\circ}{H}{}^1(\Omega_T).$$

From this we know that $\bar{u}$ is a generalized solution of the following Dirichlet problem

$$(2.5) \qquad \begin{cases} L\bar{u} = 0, \\ \bar{u}|_{\partial\Omega_T} = \bar{u}|_{\partial\Omega_T}. \end{cases}$$

As the generalized solution of (2.5) is unique and smooth in Ω_T, we obtain $\bar{u} \in C^2(\Omega_T)$. From this, we have

THEOREM 2.1. *$\bar{u}$, the solution of* (2.3), *is an approximate solution of problem* (I) *and we have the estimate*

$$\int_{\mathfrak{D}} [u(x, t) - \bar{u}(x, t)]^2 \, dx \leqslant H_1(K_1\alpha_1^2 + K_2\alpha_2^2)^{(T-t)/T},$$

where $u(x, t)$ is the solution of problem (I).

3. The Approximate Solution of (2.3) in the Finite Element Method. In this section we consider the approximate solution of (2.3) in the finite element method. We divide $\bar{\Omega}_T$ as follows: at first $[0, T]$ is divided into N equal parts by

$$t_0 = 0, \quad t_1 = \frac{T}{N}, \ldots, \quad t_i = \frac{iT}{N}, \ldots, \quad t_N = T.$$

So $\bar{\Omega}_T$ is divided into N layers; the shape of every layer being identical. Next every layer is divided into many elements and the way of dividing every layer is completely identical. When choosing proper displacement functions, we can obtain a finite-dimensional subspace of $\overset{\bullet}{H}{}^1(\Omega_T)$. For instance, every layer is divided into simplices (when $n = 1$, a simplex is a triangle), and we choose the displacement functions so that they are continuous on $\bar{\Omega}_T$ and linear functions of (x, t) on every element. From this we obtain a subspace of $\overset{\bullet}{H}{}^1(\Omega_T)$. Let us denote it by S^h (where h is the longest length of the sides of simplices). Of course, the angles of the simplices require some restrictions for convergence of the finite element method.

Now we simplify the problem (2.3) on the finite-dimensional space S^h.

Let us define the set

$$K^h = \left\{ v^h | v^h \in S^h, \|v^h(x, 0) - f_1\|_{H^0(\mathcal{D})} \leqslant \alpha_1, \right.$$
$$\left. \left\| \frac{\partial v^h(x, 0)}{\partial t} - g_1(x) \right\|_{H^0(\mathcal{D})} \leqslant \alpha_2 \right\}.$$

Obviously, K^h is a closed convex set in S^h. If K^h is not empty, instead of problem (2.3) we consider the minimum of the functional $J(v^h)$ on the closed convex set K^h.

$$J(\bar{u}^h) = \inf_{v^h \in K^h} J(v^h). \tag{3.1}$$

Let us assume that the number of internal nodes of every node layer is m. For arbitrary $v^h \in K^h$, we denote the displacements of v^h on the internal nodes of the ith node layer by $v_i^1, v_i^2, \ldots, v_i^m$; $i = 0, 1, \ldots, N$. (As $v^h|_{\partial\mathcal{D} \times [0,T]} = 0$, the values of v^h on the boundary nodes are zero.)

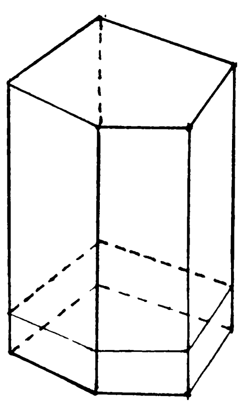

FIGURE 1

Let us define $Y_i = (v_i^1, v_i^2, \ldots, v_i^m)^T$, which is an m-dimensional column vector. Because of the way we divided $\bar{\Omega}_T$ and because A_{ij}, λ are independent of t, the stiffness matrix of every element-layer is the same. Let us denote it by

$$\begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix},$$

which is a $2m$-order nonnegative symmetry matrix. From this we have

$$J(v^h) = \sum_{i=1}^{n} (Y_{i-1}^T\ Y_i^T)\begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix}\begin{pmatrix} Y_{i-1} \\ Y_i \end{pmatrix}.$$

In this case, $J(v^h)$ is a quadratic form of $Y_0, \ldots, Y_N$. Because $v^h(x, 0)$ is dependent only on the vector Y_0 and $\partial v^h(x, 0)/\partial t$ is dependent only on the vector Y_0, Y_1, let us define

$$\|v^h(x, 0) - f_1\|^2_{H^0(\mathfrak{D})} = F(Y_0), \qquad \left\| \frac{\partial v^h(x, 0)}{\partial t} - g_1 \right\|^2_{H^0(\mathfrak{D})} = G(Y_0, Y_1),$$

where $F(Y_0)$ is a quadratic function of the node displacements, $Y_0^1, Y_0^2, \ldots, Y_0^m$, and $G(Y_0, Y_1)$ is a quadratic function of the node displacement, $Y_0^1, Y_0^2, \ldots, Y_0^m$, $Y_1^1, Y^2, \ldots, Y_1^m$.

So problem (3.1) becomes the minimum problem of the quadratic form $J(Y_0, \ldots, Y_N)$ under the following restrictions:

$$F(Y_0) \leqslant \alpha_1^2, \qquad G(Y_0, Y_1) \leqslant \alpha_2^2.$$

Namely,

$$J(\bar{Y}_0, \bar{Y}_1, \ldots, \bar{Y}_N) = \min_{\substack{F(Y_0) < \alpha_1^2 \\ G(Y_0, Y_1) < \alpha_2^2}} J(Y_0, Y_1, \ldots, Y_N). \tag{3.2}$$

This is a convex programming problem. The solution of (3.2) exists and is unique. There is a great number of variables in this problem. Now we shall eliminate all the variables except the first two and the last one.

If $Y_0, Y_1, \ldots, Y_N$ is the solution of (3.2), then we know that $Y_2, Y_3, \ldots, Y_{N-1}$ satisfies the following equations:

$$-A_1 Y_{i-1} + (K_1 + K_1')Y_i - A_1^T Y_{i+1} = 0, \qquad i = 2, 3, \ldots, N-1. \tag{3.3}$$

From (3.3) we obtain Y_i ($i = 2, 3, \ldots, N-1$) represented by the vectors Y_1, Y_N, and we can get the stiffness matrix of the $N-1$ element-layers (from 2nd element-layer to Nth element-layer). Namely,

$$\frac{1}{2}\sum_{i=2}^{N} (Y_{i-1}^T\ Y_i^T)\begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix}\begin{pmatrix} Y_{i-1} \\ Y_i \end{pmatrix} = \frac{1}{2}(Y_1^T\ Y_N^T)\begin{pmatrix} K_{N-1} & -A_{N-1}^T \\ -A_{N-1} & K_{N-1}' \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_N^T \end{pmatrix}.$$

Here K_{N-1}, K_{N-1}', A_{N-1} are given by the following iterative formulas (see [7]):

$$\left.\begin{aligned} K_i &= K_{i-1} - A_{i-1}^T(K_1 + K_{i-1}')^{-1}A_{i-1} \\ K_i' &= K_1' - A_1(K_1 + K_{i-1}')^{-1}A_1^T \\ A_i &= -A_1(K_1 + K_{i-1}')^{-1}A_{i-1} \end{aligned}\right\}, \qquad i = 2, 3, \ldots, N-1.$$

If $N - 1 = 2^k$, where k is a positive integer, then we can use the iterative formulas (5) in [7]. We actually need to iterate k times to get K_{N-1}, K'_{N-1} and A_{N-1}. Therefore, instead of (3.3) we obtain a convex programming problem,

$$J^*(\bar{Y}_0, \bar{Y}_1, \bar{Y}_N) = \min_{\substack{F(Y_0) < \alpha_1^2 \\ G(Y_0, Y_1) < \alpha_2^2}} J^*(Y_0, Y_1, Y_N), \tag{3.4}$$

where

$$J^*(Y_0, Y_1, Y_N) = \frac{1}{2}(Y_0^T\ Y_1^T)\begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix}\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}$$
$$+ \frac{1}{2}(Y_1^T\ Y_N^T)\begin{pmatrix} K_{N-1} & -A_{N-1}^T \\ -A_{N-1} & K_{N-1}' \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_N \end{pmatrix}.$$

To solve problem (3.1) we may now proceed as follows: First, we solve a convex programming problem (3.4) with fewer variables to obtain the vectors $\bar{Y}_0$, $\bar{Y}_1$ and $\bar{Y}_n$. Then we get $Y_2, Y_3, \ldots, Y_{N-1}$ from the linear equations (3.3). Finally, from these we obtain the solution $\bar{u}$ of our problem (3.1).

4. Convergence of the Approximate Solution in the Finite Element Method. In the last section, the given subdivision satisfies the following conditions:

(1) S^h is a finite-dimensional subspace of $\dot{H}^1(\Omega_T)$.

(2) K^h is a closed convex subset of K, and K^h is not empty.

If a subdivision satisfies the above conditions (1) and (2), we call it conforming. When the subdivision is made finer by taking smaller h_i's, we get a space sequence S^{h_i} $(i = 1, 2, \ldots)$ and a set sequence K^{h_i}. If for the arbitrary function $v \in K$, we can find a functional sequence

$$\{v^{h_i} | v^{h_i} \in K^{h_i}\} \quad \text{and} \quad \|v - v^{h_i}\|_{\dot{H}^1(\Omega_T)} \to 0 \qquad (i \to \infty),$$

then we say this subdivision sequence satisfies the basic condition A. If there is a function $v_0 \in K$ and

$$\|v_0 - f_1\|_{H^0(\mathfrak{D})} < \alpha_1, \qquad \left\| \frac{\partial v_0}{\partial t} - g_1 \right\|_{H^0(\mathfrak{D})} < \alpha_2,$$

it is easy to prove that the subdivision sequence in the last section satisfies the basic condition A. As before it is clear that the angles of the simplices require some restrictions.

For every S^{h_i} and K^{h_i}, we get the solution u^{h_i} of (3.1), respectively, because both S^{h_i} and K^{h_i} are conforming. To prove the convergence of u^{h_i}, we first prove the following lemmas.

LEMMA 4.1. *If* $\{S^{h_i}\}$, $\{K^{h_i}\}$ *satisfy the basic condition* A, *then the* $\{u^{h_i}\}$ *is a bounded sequence in the space* $\dot{H}^1(\Omega_T)$.

Proof. As K is not empty, we have a $v_0 \in K$. From the basic condition A, there is a functional sequence $\{v_0^{h_i}\}$, $v_0^{h_i} \in K^{h_i}$, and $\|v_0 - v_0^{h_i}\|_{\dot{H}^1(\Omega_T)} \to 0$, when $i \to \infty$.

Therefore,

$$\|v_0 - v_0^{h_i}\|_{H^1(\Omega_T)} \leqslant \|v_0 - v_0^{h_i}\|_{\dot{H}^1(\Omega_T)} < C_1.$$

As u^{h_i} is the solution of (3.1), then

$$J(u^{h_i}) \leqslant J(v_0^{h_i}) \leqslant C_2\|v_0^{h_i}\|^2_{\dot{H}^1(\Omega_0)}$$
$$\leqslant C_2\big(\|v_0\|_{\dot{H}^1(\Omega_T)} + \|v_0 - v^{h_i}\|_{\dot{H}^1(\Omega_T)}\big)^2 \leqslant C_2\big(\|v_0\|_{\dot{H}^1(\Omega_T)} + C_1\big)^2,$$

where C_1, C_2 are two constants. On the other hand, we know that

$$\|u^{h_i}\|^2_{\dot{H}^1(\Omega_T)} \leqslant C_3 J(u^{h_i})$$

and $u^{h_i} \in K^{h_i}$, $C_3 > 0$ is a constant. It follows that $\|u^{h_i}\|_{\dot{H}^1(\Omega_T)}$ is bounded.

LEMMA 4.2. *We can extract a subsequence $\{u^{h_{ij}}\}$ from $\{u^{h_i}\}$. This subsequence converges weakly to $\bar{u}$, the solution of* (2.4).

Proof. From the result of Lemma 4.1, we know that $\{u^{h_i}\}$ is a bounded sequence in $\dot{H}^1(\Omega_T)$. Thus we can extract a subsequence $\{u^{h_{ij}}\}$, which converges in $\dot{H}^1(\Omega_T)$ weakly. Let us denote the limit of this subsequence of $\bar{u}$. Now we prove that $\bar{u}$ is the solution of (2.4). We know that K is a closed convex set, thus it is weakly closed, namely the limit $\bar{u} \in K$. On the other hand, as the basic condition A is satisfied, there is a sequence $\{v^{h_{ij}}\}$, $v^{h_{ij}} \in K^{h_{ij}}$ for arbitrary $v \in K$ and

$$\|v - v^{h_{ij}}\|_{\dot{H}^1(\Omega_T)} \to 0 \qquad (j \to \infty).$$

For $u^{h_{ij}}$, $v^{h_{ij}}$ we have

$$a(u^{h_{ij}}, v^{h_{ij}} - u^{h_{ij}}) > 0,$$

namely,

$$a(u^{h_{ij}}, v^{h_{ij}}) > a(u^{h_{ij}}, u^{h_{ij}}).$$

From

$$\lim_{j\to\infty} a(u^{h_{ij}}, v^{h_{ij}}) = a(\bar{u}, v), \qquad \underline{\lim}_{j\to\infty} a(u^{h_{ij}}, u^{h_{ij}}) > a(\bar{u}, \bar{u}),$$

we get

$$a(\bar{u}, v) > a(\bar{u}, \bar{u}) \quad \forall v \in K.$$

Namely,

$$a(\bar{u}, v - \bar{u}) \geqslant 0 \quad \forall v \in K.$$

From this we know that $\bar{u}$ is the solution of (2.4).

LEMMA 4.3. *$u^{h_{ij}}$ is convergent to $\bar{u}$ in the strong topology in the Sobolev space* $H^1(\Omega_T)$.

Proof. First we point out that $a(u, u)$ satisfies

$$\frac{1}{2C_3}\|u\|^2_{H^1(\Omega_T)} \leqslant a(u, u) \leqslant 2C_2\|u\|^2_{H^1(\Omega_T)}, \qquad u \in \dot{H}^1(\Omega_T).$$

On the other hand, as $\bar{u} \in K$, there is a sequence $\{v^{h_i} | v^{h_i} \in K^{h_i}\}$ and

$$\|\bar{u} - v^{h_i}\|_{\dot{H}^1(\Omega_T)} \to 0 \qquad (i \to \infty).$$

Therefore, we have

$$\begin{aligned}
a(\bar{u}-u^{h_{j}}, \bar{u}-u^{h_{j}}) &= a(\bar{u}-u^{h_{j}}, \bar{u}-v^{h_{j}}) + a(\bar{u}-u^{h_{j}}, v^{h_{j}}-u^{h_{j}}) \\
&= a(\bar{u}-u^{h_{j}}, \bar{u}-v^{h_{j}}) + a(\bar{u}, v^{h_{j}}-u^{h_{j}}) - a(u^{h_{j}}, v^{h_{j}}-u^{h_{j}}) \\
&\leqslant a(\bar{u}-u^{h_{j}}, \bar{u}-v^{h_{j}}) + a(\bar{u}, v^{h_{j}}-u^{h_{j}}) \\
&= a(\bar{u}-u^{h_{j}}, \bar{u}-v^{h_{j}}) + a(\bar{u}, v^{h_{j}}-\bar{u}) = a(\bar{u}, u^{h_{j}}-\bar{u}) \\
&\leqslant a(\bar{u}-u^{h_{j}}, \bar{u}-v^{h_{j}}) + a(\bar{u}, v^{h_{j}}-\bar{u}) \\
&\leqslant \gamma\|\bar{u}-u^{h_{j}}\|_{H^1(\Omega_T)}\|\bar{u}-v^{h_{j}}\|_{H^1(\Omega_T)} + \gamma_1\|\bar{u}-v^{h_{j}}\|_{H^1(\Omega_T)} \\
&\qquad\qquad (\text{where } \gamma \text{ is a positive constant, } \gamma_1 = \gamma\|\bar{u}\|_{H^1(\Omega_T)}) \\
&\leqslant \frac{1}{4C_3}\|\bar{u}-u^{h_{j}}\|^2_{H^1(\Omega_T)} + C_3\gamma^2\|\bar{u}-v^{h_{j}}\|^2_{H^1(\Omega_T)} + \gamma_1\|\bar{u}-v^{h_{j}}\|_{H^1(\Omega_T)}.
\end{aligned}$$

Thus we obtain

$$\|\bar{u}-u^{h_{j}}\|^2_{H^1(\Omega_T)} \leqslant 4C_3\left\{C_3\gamma^2\|\bar{u}-v^{h_{j}}\|^2_{H^1(\Omega_T)} + \gamma_1\|\bar{u}-v^{h_{j}}\|_{H^1(\Omega_T)}\right\}.$$

Namely, we have

$$\lim_{j\to\infty}\|\bar{u}-u^{h_{j}}\|^2_{H^1(\Omega_T)} = 0.$$

Now we have the following convergence theorem.

THEOREM 4.1. *If the sequences* $\{S^{h_i}\}$ *and* $\{K^{h_i}\}$ *are conforming and satisfy the basic condition* A, *then the complete sequence* u^{h_i} *converges to* $\bar{u}$ *in* $H^1(\Omega_T)$.

Proof. We will prove this by contradiction. If $\{u^{h_i}\}$ does not converge to $\bar{u}$, then there exists a positive constant $\varepsilon_0 > 0$ and a subsequence $u^{h_{i\kappa}}$, which satisfies

$$\|u^{h_{i\kappa}} - \bar{u}\|_{H^1(\Omega_1)} > \varepsilon_0.$$

Now we consider the sequence $\{u^{h_{i\kappa}}\}$. From Lemmas 4.2, 4.3 we know that we can extract a new subsequence from $\{u^{h_{i\kappa}}\}$. This new subsequence converges to the solution of (2.4) in the Sobolev space $H^1(\Omega_T)$. As the uniqueness of (2.4), we obtain a contradiction. So Theorem 4.1 is proved.

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