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Spline Interpolation at Knot Averages on a Two-Sided Geometric Mesh*

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Abstract. For splines of degree $k > 1$ with knots $-t_i = t_{2m+1-i} = 1 + q + q^2 + \cdots + q^{m-i}$, $i = 1, \ldots, m$, where $0 < q < 1$, it is shown that spline interpolation to continuous functions at nodes $\tau_i = \Sigma_1^k w_j t_{i+j}$, $i = 1, \ldots, n = 2m - k - 1$, has operator norm $\|P\|$ which is bounded independently of q and m as q tends to zero if and only if $(1 - w_1)^k < \frac{1}{2}$, $(1 - w_k)^k < \frac{1}{2}$, and $w_j > 0$, $j = 1, \ldots, k$. The choice of nodes $\tau_i = \Sigma_0^{k+1} w_j t_{i+j}$ and the growth rate of $\|P\|$ with k are also discussed.

1. Two-Sided q-Splines. To integers $n > 0$, $k \geqslant 0$, and a nondecreasing sequence $\mathbf{t} = (t_i)_1^{n+k+1}$ with $t_i < t_{i+k+1}$, $i = 1, \ldots, n$, is associated $\mathcal{S}_{k+1,\mathbf{t}}$, the space of polynomial splines of order $k + 1$ with knot sequence $\mathbf{t}$, defined by $\mathcal{S}_{k+1,\mathbf{t}} = \operatorname{span}\{N_1, \ldots, N_n\}$, where each $N_i = N_{i,k+1}$ is an appropriate normalized B-spline. See [1] for specific details.

With $q > 0$, m a positive integer, $n = 2m - k - 1$, and

$$(1.1)\qquad \begin{aligned} t_i &= -(1 + q + \cdots + q^{m-i}), & i &= 1, \ldots, m, \\ &= 1 + q + \cdots + q^{i-m-1}, & i &= m + 1, \ldots, 2m, \end{aligned}$$

$\mathcal{S}_{k+1,\mathbf{t}}$ is the space of *two-sided q-splines.*

Each two-sided q-spline can be represented as

$$(1.2)\qquad \begin{aligned} s(t) = {}& \sum_1^{m-1} A_j\left[q^{j-m}(t_{j+1} - t)_+\right]^k + \sum_0^k A_{m+j} t^j \\ & + \sum_1^{m-1} A_{m+k+j}\left[q^{-j}(t - t_{m+j})_+\right]^k, \end{aligned}$$

where $u_+ = \max\{u, 0\}$, with the endpoint conditions

$$(1.3)\qquad s^{(i)}(t_1) = s^{(i)}(t_{2m}) = 0, \qquad i = 0, \ldots, k - 1.$$

Conversely, each function of the form (1.2) which satisfies (1.3) is a two-sided q-spline.

With the notation

$$[i] = 1 + q + \cdots + q^{i-1}, \qquad i = 0, 1, \ldots,$$

relations such as

$$t_{j+1} - t_i = q^{m-j}[j + 1 - i], \qquad 0 < i \leqslant j < m,$$

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and

$$t_{i+1} - t_j = q^{j-m}[i+1-j], \qquad m < j \leqslant i < 2m,$$

can be stated in a compact form. The notation

$$[i]! = [i][i-1] \cdots [2][1] \quad \text{and} \quad \begin{bmatrix} j \\ i \end{bmatrix} = \frac{[j]!}{[i]![j-i]!}$$

will also be useful.

The clause "as q tends to zero" appears throughout this paper. It will always mean "for all q satisfying $0 < q \leqslant q_0$". The specific choice of q_0 will vary from instance to instance. However, q_0 will never depend on m.

LEMMA 1.1. *With k and m fixed, let $\{s\}$ be a set of two-sided q-splines with $\{(A_1, \ldots, A_{2m+k-1})\}$ the corresponding set of coefficient vectors in* (1.2). *Then $\{s\}$ is uniformly bounded as q tends to zero if and only if $\{(A_j)\}$ is uniformly bounded as q tends to zero. Moreover, if the bound on $\{s\}$ is independent of m, then so is the bound on $\{(A_j)\}$.*

Proof. Let $1 > q_0 > 0$ and C be such that

$$|A_j| \leqslant C, \quad \text{all } j \text{ and } 0 < q \leqslant q_0.$$

Then, for each real t and $0 < q \leqslant q_0$,

$$\begin{aligned} |s(t)| &\leqslant C\left(\sum_1^{m-1} \left[q^{j-m}(t_{j+1} - t_1)\right]^k + \sum_0^k t_{2m}^j + \sum_1^{m-1} \left[q^{-j}(t_{2m} - t_{m+j})\right]^k \right) \\ &= C\left(\sum_1^{m-1} [j]^k + \sum_0^k [m]^j + \sum_1^{m-1} [m-j]^k \leqslant (2m+k-1)C[m]^k \right) \\ &< (2m+k-1)Cm^k. \end{aligned}$$

Conversely, let $1 > q_0 > 0$ and B be such that

$$|s(t)| \leqslant B, \quad \text{all real } t \text{ and } 0 < q \leqslant q_0.$$

Since

$$\sum_0^k A_{m+j}(i/k)^j = s(i/k), \qquad i = 0, \ldots, k,$$

is a matrix equation with nonsingular coefficient matrix $V = ((i/k)^j)$ depending only on k,

$$|A_{m+j}| \leqslant (k+1)B_k B, \qquad j = 0, \ldots, k,$$

where B_k is a bound on the entries of V^{-1}. Set $C_0 = (k+1)B_k B$ and assume inductively that q_1 is such that $|A_{m-j}| \leqslant C_j$ for $j = 0, 1, \ldots, i-1$ for $q \leqslant q_1$. From (1.2)

$$\begin{aligned} s(t_{m-i}) - s(t_{m-i+1}) = A_{m-i} &+ \sum_1^{i-1} A_{m-j}\left([i-j+1]^k - [i-j]^k\right) \\ &+ \sum_1^k A_{m+j}(-1)^j\left([i+1]^j - [i]^j\right), \end{aligned}$$

so that

$$|A_{m-i}| \leq 2B + \sum_{1}^{i-1} C_j\big([i-j+1]^k - [i-j]^k\big) + C_0\sum_{1}^{k}\big([i+1]^j - [i]^j\big)$$

$$\leq 2B + \sum_{1}^{i-1} C_j q^{i-j} k(1-q_0)^{-k} + C_0 \sum_{1}^{k} q^i j(1-q_0)^{-j}$$

$$\leq 2B + \sum_{0}^{i-1} C_j q^{i-j} R_k \quad \text{with } R_k = k^2(1-q_0)^{-k}.$$

Setting $C_i = 2B + \Sigma_0^{i-1} C_j q_1^{i-j} R_k$ allows the induction to proceed. Then $C_1 = 2B + C_0 q_1 R_k$, and $C_{i+1} = q_1(1+R_k)C_i + 2B(1-q_1)$, $i = 1, \ldots, m-2$. This recurrence solves as

$$C_i = \frac{2B(1-q_1)}{1-q_1-q_1R_k}\left[1-(q_1+q_1R_k)^{i-1}\right] + C_1(q_1+q_1R_k)^{i-1},$$

$$i = 1, \ldots, m-1,$$

if $q_1 + q_1 R_k \neq 1$. Imposing the added restriction $q_1 + q_1 R_k < \frac{1}{2}$ and noting that a symmetric argument will yield $|A_{m+k+j}| \leq C_j, j = 1, \ldots, m-1$, establishes that

$$\max_j |A_j| \leq \max_i C_i \leq 4B + C_1 + C_0.$$

This bound is independent of m if B is independent of m. □

LEMMA 1.2. *Let k and m be fixed. As q tends to zero, the coefficients (A_j) satisfy*

$$A_i + \sum_{i+1}^{m-1} A_j q^{(j-i)(k-i)}\begin{bmatrix} j \\ i \end{bmatrix} + \sum_{k-i}^{k} A_{m+j} O(q^{(m-i)(k-i)}) = 0, \qquad i = 1, \ldots, k-1,$$

and

$$A_k + \sum_{k+1}^{m-1} A_j \begin{bmatrix} j \\ k \end{bmatrix} + \sum_{0}^{k} A_{m+j}\left(\frac{t_1^j}{[k]!} + O(q^{m-k+1})\right) = 0.$$

Proof. This follows from (1.3). Let functionals $\Lambda_{i\nu}$, $1 \leq i \leq \nu \leq k$, be defined by

$$\Lambda_{1\nu}s = q^{(m-1)(k-\nu)}\frac{\nu!}{k!}(-1)^{k-\nu}s^{(k-\nu)}(t_1)$$

and, recursively,

$$\Lambda_{i\nu}s = q^{\nu-k}(\Lambda_{i-1,\nu}s - [i-1]\Lambda_{i-1,\nu-1}s)/[i].$$

From (1.2)

$$s^{(k-\nu)}(t_1) = \sum_{1}^{m-1} A_j q^{(j-m)(k-\nu)}\frac{k!}{\nu!}(-1)^{k-\nu}[j]^\nu + \sum_{k-\nu}^{k} A_{m+j}\frac{j!}{(j-k+\nu)!}t_1^{j-k+\nu},$$

whence

$$\Lambda_{1\nu}s = \sum_{1}^{m-1} A_j q^{(j-1)(k-\nu)}[j]^\nu + \sum_{k-\nu}^{k} A_{m+j} q^{(m-1)(k-\nu)} C_{1j\nu},$$

where

$$C_{1j\nu} = \frac{\nu!j!}{k!(j-k+\nu)!}(-1)^{k-\nu}t_1^{j-k+\nu}.$$

The recursion formula gives

$$\Lambda_{i\nu}s = \sum_{i}^{m-1} A_j q^{(j-i)(k-\nu)}[j]^{\nu-i}\begin{bmatrix} j \\ i \end{bmatrix} + \sum_{k-\nu}^{k} A_{m+j}q^{(m-i)(k-\nu)}C_{ij\nu},$$

where

$$C_{ij\nu} = \left(C_{i-1,j,\nu} - [i-1]q^{m-i+1}C_{i-1,j,\nu-1}\right)/[i].$$

From (1.3) each $\Lambda_{i\nu}s = 0$ and, in particular, $\Lambda_{ii}s = 0$. This fact, along with the observation that $C_{kjk} = C_{1jk}/[k]! + O(q^{m-k+1})$ completes the proof. □

Combining Lemmas 1.1, 1.2, and a symmetric counterpart of Lemma 1.2 yields

LEMMA 1.3. *Let k and m be fixed and let $\{s\}$ be a set of two-sided q-splines which is bounded as q tends to zero. Then the corresponding set of coefficient vectors $\{(A_j)\}$ satisfies*

$$\begin{aligned} A_i &= O(q^{k-i}), && i = 1, \ldots, k-1, \\ A_i &= O(1), && i = k, \ldots, 2m, \\ A_{2m+i} &= O(q^i), && i = 1, \ldots, k-1, \end{aligned}$$

as q tends to zero. If the bound on $\{s\}$ is independent of m, then so are the bounds on the A_j.

The independence of m in the $O(q^{k-i})$ and $O(q^i)$ bounds follows from the exponential decay of the coefficients in the first $k-1$ equations of Lemma 1.2.

2. Spline Interpolation. Let $\tau = (\tau_i)_1^n$ be a strictly increasing sequence. It is known [1] that: For each function f defined on τ there is exactly one $s \in \mathcal{S}_{k+1,\mathbf{t}}$ such that $s(\tau_i) = f(\tau_i)$, $i = 1, \ldots, n$, if and only if $N_i(\tau_i) > 0$, $i = 1, \ldots, n$, or, equivalently, if and only if

$$t_i < \tau_i < t_{i+k+1}, \qquad i = 1, \ldots, n. \tag{2.1}$$

When τ satisfies (2.1) a linear map P into $\mathcal{S}_{k+1,\mathbf{t}}$ which reproduces $\mathcal{S}_{k+1,\mathbf{t}}$ may be defined by: *For each function f defined on τ, $Pf \in \mathcal{S}_{k+1,\mathbf{t}}$ and $(Pf)(\tau_i) = f(\tau_i)$, $i = 1, \ldots, n$.* In fact, $Pf = \sum f(\tau_j)L_j$ where $(L_j)_1^n$ is defined by $L_j(\tau_i) = \delta_{ij}$, $i, j = 1, \ldots, n$. The operator norm of P is

$$\|P\| = \sup_f \frac{\|Pf\|}{\|f\|},$$

where the sup is taken over all $f \in C[t_1, t_{n+k+1}]$ and

$$\|f\| = \sup\{|f(t)|: t_1 \leqslant t \leqslant t_{n+k+1}\}.$$

It is well known that

$$\|P\| = \max_t \sum_1^n |L_j(t)| = \max_{0 \leqslant \mu \leqslant n}\Big(\max_{\tau_\mu \leqslant t \leqslant \tau_{\mu+1}} s_\mu(t)\Big),$$

where $\tau_0 = t_1$, $\tau_{n+1} = t_{n+k+1}$ and $(s_\mu)_0^n$ is defined by

$$(2.2)\qquad \begin{aligned} s_\mu(\tau_i) &= (-1)^{i+\mu}, & i &= 1, \dots, \mu, \\ &= -(-1)^{i+\mu}, & i &= \mu + 1, \dots, n. \end{aligned}$$

For each μ, the so-called Lebesgue function $\Sigma\, |L_i(t)|$ coincides with $s_\mu(t)$ on the interval $[\tau_\mu, \tau_{\mu+1}]$.

One way of specifying $\boldsymbol{\tau}$ is to require that the nodes be knot averages, i.e.,

$$(2.3)\qquad \tau_i = \sum_0^{k+1} w_j t_{i+j}, \qquad i = 1, \dots, n,$$

where the w_j are fixed nonnegative numbers which sum to one.

THEOREM 1. *Let $k \geqslant 2$, m, and $(w_i)_0^{k+1}$ be fixed. Let* $\mathbf{t}$ *be given by* (1.1) *and $\boldsymbol{\tau}$ be given by* (2.3). *If $\|P\|$ is bounded as q tends to zero, then*

$$(2.4)\qquad w_i > 0, \qquad i = 1, \dots, k.$$

If the bound on $\|P\|$ is also independent of m, then either

$$(2.5)a \qquad w_0 = 0 \quad and \quad (1 - w_1)^k < \tfrac{1}{2}$$

or

$$(2.5)b \qquad w_0 > 0 \quad and \quad \tfrac{1}{2} < (1 - w_0)^k$$

and, either

$$(2.6)a \qquad w_{k+1} = 0 \quad and \quad (1 - w_k)^k < \tfrac{1}{2}$$

or

$$(2.6)b \qquad w_{k+1} > 0 \quad and \quad \tfrac{1}{2} < (1 - w_{k+1})^k.$$

Conversely, if (2.4), (2.5), (2.6) *hold, then $\|P\|$ is bounded independently of m as q tends to zero.*

Proof. Let w_a be the first positive weight and w_b be the last positive weight, so that $\tau_i = \Sigma_a^b\, w_j t_{i+j}$, and set

$$\theta_1 = (1 - w_a) + (1 - w_a - w_{a+1})q + \cdots + w_b q^{b-a-1},$$

$$\theta_2 = (1 - w_b) + (1 - w_b - w_{b-1})q + \cdots + w_a q^{b-a-1}.$$

If $a = b$, then $\theta_1 = \theta_2 = 0$. If $a < b$, then $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ as q tends to zero. Therefore,

$$(2.7)\qquad \begin{aligned} t_{i+b-1} < \tau_i &= t_{i+b} - \theta_2 q^{m+1-b-i} \leqslant t_{i+b}, & i &= 1, \dots, m - b, \\ t_{i+a} \leqslant \tau_i &= t_{i+a} + \theta_1 q^{i+a-m} < t_{i+a+1}, & i &= m - a + 1, \dots, n, \end{aligned}$$

for all sufficiently small $q > 0$. Since

$$(2.8)\qquad \tau_i = 1 - 2 \sum_a^{m-i} w_j + O(q), \qquad i = m - b + 1, \dots, m - a,$$

as q tends to zero, it follows that also

$$(2.9)\qquad -1 < \tau_{m-b+1} < \tau_{m-b+2} < \cdots < \tau_{m-a} < +1$$

for all sufficiently small $q > 0$.

Henceforth, we require that q be such that the inequalities in (2.7) and (2.9) hold. This requirement is independent of m.

Now let $\|P\|$ be bounded independently of m as q tends to zero. We shall prove that (2.4) and (2.6) must hold. A symmetric argument, which we omit, will give (2.5).

Let $s = s_\mu$ be defined by (2.2) with $\mu < m - b + 1$ or $\mu > m - a - 1$. There is a constant C which bounds $\|P\|$ so that $\|s\| \leqslant C$ as q tends to zero. Since the restriction of s to $[-1, +1]$ is a polynomial of degree k, it follows from a theorem of A. A. Markov (see [7]) that

$$\max\{|s'(t)|: -1 \leqslant t \leqslant 1\} \leqslant Ck^2.$$

Thus, (2.8), (2.9), and the mean-value theorem imply that

$$2 = |s(\tau_i) - s(\tau_{i+1})| \leqslant Ck^2(\tau_{i+1} - \tau_t) \leqslant 2Ck^2 w_{m-i} + O(q)$$

for $i = m - b + 1, \ldots, m - a - 1$ as q tends to zero. Thus, $w_i \geqslant 1/Ck^2 > 0$, $i = a + 1, \ldots, b - 1$.

Suppose that $b < k$. Then, on the one hand, (1.2) gives

$$\begin{aligned}\pm 1 = s(\tau_1) &= \sum_b^{m-1} A_j([j-b] + \theta_2 q^{j-b})^k + \sum_0^k A_{m+j}(-[m-b] - \theta_2 q^{m-b})^j \\ &= A_b\theta_2^k + \sum_{b+1}^{m-1} A_j([j-b]^k + O(q^{j-b})) \\ &\quad + \sum_0^k A_{m+j}((-[m-b])^j + O(q^{m-b})),\end{aligned}$$

whereas, on the other hand, with $\Lambda_{ii}s$ as in the proof of Lemma 1.2,

$$\begin{aligned}0 &= \theta_2^k \Lambda_{bb}s + \sum_{b+1}^{k-1} [i-b]^k \Lambda_{ii}s + [k]!\Lambda_{kk}s \\ &= A_b\theta_2^k + \sum_{b+1}^{m-1} A_j([j-b]^k + O(q^{j-b})) \\ &\quad + \sum_0^k A_{m+j}((-[m-b])^j + O(q^{m-b})).\end{aligned}$$

Subtraction yields

$$\pm 1 = \sum_{b+1}^{m-1} A_j O(q^{j-b}) + \sum_0^k A_{m+j} O(q^{m-b}),$$

so that (A_j) cannot be bounded as q tends to zero. This contradiction to Lemma 1.3 shows that $b \geqslant k$.

A similar argument with $s(\tau_n)$ shows that $a \leqslant 1$, so that (2.4) is proved.

To prove (2.6), we first suppose that $w_{k+1} = 0$. We must show that $(1 - w_k)^k < \frac{1}{2}$ or, equivalently, that

(2.10) $$r_2 = \theta_2^k/(1 - \theta_2^k) < 1 \quad \text{as } q \text{ tends to zero.}$$

Again, let $s = s_\mu$ be defined by (2.2). Then Lemma 1.2 and (1.2) give

(2.11) $$-s(\tau_1) = [k]!\Lambda_{kk}s - s(\tau_1) = \sum_0^{m-k-1} M_{0j}A_{k+j} + \sum_0^k R_{0j}A_{m+j}$$

and

$$(2.12)\quad s(\tau_i) - s(\tau_{i+1}) = \sum_{i-1}^{m-k-1} M_{ij}A_{k+j} + \sum_{0}^{k} R_{ij}A_{m+j}, \qquad i = 1, \ldots, m-k-1,$$

where

$$M_{0j} = [k+j]!/[j]! - ([j] + \theta_2 q^j)^k, \qquad j = 0, \ldots, m-k-1,$$

$$M_{i,i-1} = \theta_2^k, \qquad i = 1, \ldots, m-k-1,$$

$$M_{ij} = ([j-i+1] + \theta_2 q^{j-i+1})^k - ([j-i] + \theta_2 q^{j-i})^k,$$
$$i = 1, \ldots, m-k-1; j = i, \ldots, m-k-1,$$

$$R_{0j} = [k]!C_{kjk} - \tau_1^j, \qquad j = 0, \ldots, k,$$

$$R_{ij} = \tau_i^j - \tau_{i+1}^j, \qquad i = 1, \ldots, m-k-1; j = 0, \ldots, k,$$

with $C_{kjk} = t_1^k/[k]! + O(q^{m-k+1})$ as in the proof of Lemma 1.2.

Since the A_j are bounded and

$$M_{ii} = 1 - \theta_2^k + O(q), \qquad i = 0, \ldots, m-k-1,$$

$$M_{0j} < [k+j]^k - [j]^k < q^j[k]k[k+j]^{k-1} < q^j k(1-q)^{-k},$$

$$M_{ij} < [j-i+2]^k - [j-i]^k < q^{j-i}k(1-q)^{-k},$$
$$j = i+1, \ldots, m-k-1,$$

$$|R_{0j}| < q^{m-k}(j+1)(1-q)^{-j},$$

$$|R_{ij}| \leqslant q^{m-k-i}j(1-q)^{-j},$$

the system (2.11) and (2.12) has the form

$$(1 - \theta_2^k)A_k = -s(\tau_1) + O(q),$$

$$\theta_2^k A_{k+i-1} + (1 - \theta_2^k)A_{k+i} = s(\tau_i) - s(\tau_{i+1}) + O(q), \qquad i = 1, \ldots, m-k-1,$$

which solves as

$$A_{k+i} = \frac{2(-1)^{i+\mu}}{1 - 2\theta_2^k}\left[1 - \frac{r_2^{i+1}}{2\theta_2^k}\right] + O(q),$$
$$i = 0, \ldots, \min(\mu - 1, m-k-1),$$

$$(2.13)\quad A_{k+\mu+i} = \frac{2(-1)^{i+1}}{1 - 2\theta_2^k}\left[1 - \frac{r_2^{i+1}}{\theta_2^i} + \frac{r_2^{i+\mu+1}}{2\theta_2^k}\right] + O(q),$$
$$i = 0, \ldots, m-k-1-\mu,$$

if $r_2 \neq 1$ and as

$$A_{k+1} = (-1)^{i+1}(2 + 4i) + O(q), \qquad i = 0, \ldots, \min(\mu - 1, m-k-1),$$

$$A_{k+\mu+i} = (-1)^{i+1}(2 + 4i - 4\mu) + O(q), \qquad i = 0, \ldots, m-k-1-\mu,$$

if $r_2 = 1$ provided that the buildup of $O(q)$ terms is bounded independently of m. This will be the case if $qr_2 < 1$ as q tends to zero, a condition that can be met independently of m. By Lemma 1.3 these A_j are bounded independently of m. Therefore, (2.10) must hold and $(1 - w_k)^k < \frac{1}{2}$.

To complete the proof of (2.6) we now suppose that $w_{k+1} > 0$. Since $\theta_2 = 1 - w_{k+1} + O(q)$, we must now show that $\theta_2^k > \frac{1}{2}$ as q tends to zero, that is

$$r_2 = \theta_2^k / (1 - \theta_2^k) > 1 \quad \text{as } q \text{ tends to zero.} \tag{2.14}$$

Since the τ_i have "moved over one interval", Eqs. (2.11) and (2.12) are replaced by

$$\begin{aligned} -s(\tau_1) = [k]!A_k &+ \sum_1^{m-k-1} \left([k+j]!/[j]! - \left([j-1] + \theta_2 q^{j-1}\right)^k\right) A_{k+j} \\ &+ \sum_0^k R_{0j} A_{m+j} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} s(\tau_i) - s(\tau_{i+1}) = \sum_i^{m-k-1} M_{i,j-1} A_{k+j} + \sum_0^k R_{ij} A_{m+j}, \\ i = 1, \ldots, m-k-2, \end{aligned} \tag{2.16}$$

and the bounds on M_{0j} and R_{ij} are replaced by

$$|[k+j]!/[j]! - ([j-1] + \theta_2 q^{j-1})^k| < q^{j-1}k(1-q)^{-k},$$

$$|R_{0j}| < q^{m-k-1}(j+1)(1-q)^{-j},$$

$$|R_{ij}| \leqslant q^{m-k-1-i}j(1-q)^{-j}.$$

This incomplete system now has the form

$$A_k + (1 - \theta_2^k)A_{k+1} = -s(\tau_1) + O(q),$$

$$\theta_2^k A_{k+i} + (1 - \theta_2^k)A_{k+i+1} = s(\tau_i) - s(\tau_{i+1}) + O(q), \qquad i = 1, \ldots, m-k-2.$$

Adding the equation

$$s(\tau_{m-k-1}) = A_{m-1}\theta_2^k + \sum_0^k A_{m+j}\tau_{m-k-1}^j = A_{m-1}\theta_2^k + s(-1) + O(q) \tag{2.17}$$

and imposing the restriction $qr_2^{-1} < 1$ permits us to solve this system backwards in terms of $s(-1)$ as

$$\begin{aligned} A_{m-i} &= \frac{(-1)^{m-1-k-\mu-i}}{2\theta_2^k - 1}\left[2 - (1+r_2)r_2^{-i}\right] + (1+r_2)(-r_2)^{-i}s(-1) \\ &\quad + O(q), \qquad i = 1, \ldots, m-k-1-\mu, \\ A_{k+\mu+1-i} &= \frac{(-1)^{i-1}}{2\theta_2^k - 1}\left[2 - (1+r_2)r_2^{-i}(2 - r_2^{-m+k+\mu+1})\right] \\ &\quad + (1+r_2)(-r_2)^{-m+k+\mu+1-i}s(-1) + O(q), \qquad i = 1, \ldots, \mu, \end{aligned} \tag{2.18}$$

if $0 \leqslant \mu \leqslant m-k-2$ and as

$$\begin{aligned} A_{m-i} &= \frac{(-1)^{\mu-m+k-i}}{2\theta_2^k - 1}\left[2 - (1+r_2)r_2^{-i}\right] \\ &\quad + (1+r_2)(-r_2)^{-i}s(-1) + O(q), \qquad i = 1, \ldots, m-k-1, \end{aligned} \tag{2.19}$$

if $\mu \geqslant m-k-1$. Since the A_j are bounded independently of m, (2.14) must hold and $(1 - w_{k+1})^k > \frac{1}{2}$.

The proof that (2.4), (2.5), (2.6) are necessary conditions for $\|P\|$ to be bounded independently of m as q tends to zero is complete.

To prove that (2.4), (2.5), (2.6) are sufficient that $\|P\|$ be bounded independently of m as q tends to zero, we will use the approach outlined in the proof of Lemma 1.1. That is, we will first show that, for each $s = s_\mu$, the block $A_m, \ldots, A_{m+k}$ is bounded and then argue recursively from bounds on $s(\tau_i)$ (replacing $s(t_i)$ in the proof of Lemma 1.1) that A_{m-i} (and A_{m+k+i}), $i = 1, \ldots, m-1$, are bounded independently of m. Finally, we will use (1.2) and (2.13) or (2.18) or (2.19) to bound $s_\mu(t)$ for all t and all μ.

If $a = 0$ and $b = k + 1$, the first step, bounding the block $A_m, \ldots, A_{m+k}$ is easy since (2.9) implies that

$$\sum_0^k A_{m+j}\tau_{m-k+i}^j = \pm 1, \qquad i = k+1-b, \ldots, k-a, \tag{2.20}$$

and (2.4), (2.8) give a bounded inverse for the Vandermonde matrix (τ_{m-k+i}^j). However, if $b = k$ then the $i = 0$ equation of (2.20) is replaced by

$$\theta_2^k A_{m-1} + \sum_0^k A_{m+j}\tau_{m-k}^j = \pm 1. \tag{2.21}$$

If $a = 1$, there is a similar replacement of

$$\sum_0^k A_{m+j}\tau_m^j + \theta_1^k A_{m+k+1} = \pm 1 \tag{2.22}$$

for the $i = k$ equation of (2.20).

Therefore, if $b = k$ (and/or $a = 1$), a preliminary step to eliminate $\theta_2^k A_{m-1}$ from (2.21), at the expense of adding a bounded quantity to the right member, is necessary. While eliminating $\theta_2^k A_{m-1}$ through a sequence of upper triangulation steps on (2.11), (2.12), (2.21) is straightforward, there must be an argument that $\theta_2^k A_{m-1}$ is bounded independently of m as q tends to zero independently of m. The following lines supply this argument.

Let $b = k$ and let s be any of the s_μ given by (2.2). Using the bounds on $M = (M_{ij})$, we see that this matrix is diagonally dominant if q is such that $1 - \theta_2^k > \theta_2^k + kq(1-q)^{-k-1}$. But (2.5)$a$ is equivalent to $1 - \theta_2^k > \theta_2^k$ for sufficiently small q, so that this condition can be met by imposing a further restriction on q.

Let $q_0 > 0$ and $\delta > 0$ be such that $\delta = 1 - 2\theta_2^k - kq_0(1-q_0)^{-k-1}$. Then the solutions of a system

$$M\mathbf{x} = \mathbf{b}$$

satisfy $\max_i|x_i| \leqslant \delta^{-1}\max_i|b_i|$ by the usual diagonal dominance argument. Applying this fact with

$$b_0 = [k]!\Lambda_{kk}s - s(\tau_1) = -s(\tau_1),$$
$$b_i = s(\tau_i) - s(\tau_{i+1}), \qquad i = 1, \ldots, m-k-1,$$

as well as with

$$b_i = -R_{ij}, \qquad i = 0, \ldots, m-k-1,$$

for each $j = 0, \ldots, k$, yields

$$A_{m-1} = C + \sum_0^k C_j A_{m+j}$$

with

$$|C| \leqslant \delta^{-1} \max\{|s(\tau_1)|, |s(\tau_i) - s(\tau_{i+1})|: i = 1, \ldots, m-k-1\} = 2/\delta,$$
$$|C_0| \leqslant \delta^{-1} \max_i |R_{i0}| = |R_{00}|/\delta = O(q^{m-k}),$$
$$|C_j| < \delta^{-1} \max_i |R_{ij}| = |\tau^j_{m-k-1} - \tau^j_{m-k}|/\delta$$
$$= |(1 + q + \theta_2 q^2)^j - (1 + \theta_2 q)^j|/\delta < q[2]j[3]^{j-1}/\delta$$
$$< qj[3]^j/\delta = O(q), \qquad j = 1, \ldots, k.$$

Combining these deductions with (2.21) gives the equation

$$\sum_0^k A_{m+j}\left(\tau^j_{m-k} + C_j\theta_2^k\right) = s(\tau_{m-k}) - C\theta_2^k, \tag{2.23}$$

which can be adjoined to (2.20). Since $C_j = O(q)$ and $\tau_{m-k+1} - \tau_{m-k} = 2w_k + O(q)$, the resulting system has a bounded solution as q tends to zero. We have assumed that $a = 0$. If $a = 1$, a similar argument at τ_m is needed.

We have completed the first step in the proof of sufficiency, i.e., we have shown that the set $A_m, \ldots, A_{m+k}$ is bounded. But now (2.12) or (2.16) and their symmetric counterparts imply immediately that the set $A_k, \ldots, A_{2m}$ is bounded. An argument similar to the proof of Lemma 1.2 gives $O(q^i)$ bounds on A_{k-i} and A_{2m+i}, $i = 1, \ldots, k-1$. The second step in the proof is completed.

Now we must bound $s_\mu(t)$ for all t and all μ. For $-1 \leqslant t \leqslant +1$, the boundedness of $A_m, \ldots, A_{m+k}$ and (2.4) give a uniform bound on $s_\mu(t)$. If $t_1 \leqslant t \leqslant t_m$, there is a θ_t in [0, 1] and an $i > 0$ such that $t_{m-i} \leqslant t = t_{m-i+1} - \theta_t q^i = -[i] - \theta_t q^i \leqslant t_{m-i+1}$. Then

$$s(t) = \sum_{m-i}^{m-1} A_j\left([i + j - m] + \theta_t q^{i+j-m}\right)^k + \sum_0^k A_{m+j} t^j.$$

If $i \leqslant m - b$, then $\tau_{m+1-b-i} = -[i] - \theta_2 q^i$ and

$$|s(t)| \leqslant |s(\tau_{m+1-b-i})| + |A_{m-i}| + O(q) = 1 + |A_{m-i}| + O(q)$$

can be easily shown. If $i > m - b$, a modified argument gives

$$|s(t)| \leqslant |s(\tau_1)| + \sum_1^k |A_j| + O(q) = 1 + |A_k| + O(q).$$

Thus, the $s_\mu(t)$ are uniformly bounded for all μ and all t so that $\|P\|$ is bounded independently of m as q tends to zero. □

3. Two Special Cases. Theorem 1 provides counterexamples when (2.4), (2.5), (2.6) are not satisfied, e.g., interpolation at the knots with $k \geqslant 2$ or interpolation at weighted two-knot averages with $k \geqslant 3$. The condition that q tend to zero compares (contrasts?) with the often-used condition that the local mesh ratios $(t_{j+1} - t_j)/(t_{i+1} - t_i)$, $|i - j| = 1$ be bounded.

For $k \geqslant 3$ and $q = 1$, it is easy to select weights w_j satisfying (2.4), (2.5), (2.6) which still produce unbounded spline interpolation. Thus, even for two-sided q-splines, these conditions are not sufficient to guarantee bounded interpolation. Indeed, the method of their derivation suggests that they are linked quite closely to the tendency of q to zero.

For the two special cases which follow it is not clear that we need q to tend to zero. Computational evidence with small k suggests, in fact, that q tending to zero gives "worst-case" results. Thus, Theorems 2 and 3 are imperfect in that the condition that q tend to zero may be superfluous.

THEOREM 2. *Let* $\mathbf{t}$ *be given by* (1.1) *and, for each* $k \geqslant 1$ *and* $m > k$, *let* $\boldsymbol{\tau}$ *be given by*

$$\tau_i = (t_{i+1} + t_{i+2} + \cdots + t_{i+k})/k, \qquad i = 1, \ldots, n. \tag{3.1}$$

Then, $\|P\|$ *is bounded as* q *tends to zero. Moreover, there exist absolute constants* $1 < C_1 < C_2$ *such that, for each* $k \geqslant 2$,

$$C_1^k < \|P\| < C_2^k \quad \text{as } q \text{ tends to zero.}$$

THEOREM 3. *Let* $\mathbf{t}$ *be given by* (1.1) *and, for each* $k \geqslant 1$ *and* $m > k$, *let* $\boldsymbol{\tau}$ *be given by* (2.3) *with*

$$\begin{aligned} w_0 &= w_{k+1} = \sin^2(\alpha_k/2), \\ w_j &= \sin(\alpha_k)\sin(2j\alpha_k), \qquad j = 1, \ldots, k, \end{aligned} \tag{3.2}$$

where $\alpha_k = \pi/(2k+2)$. *Then,* $\|P\|$ *is bounded as* q *tends to zero. Moreover, there exist absolute constants* $0 < C_3 < C_4$ *such that, for each* $k \geqslant 2$,

$$C_3 \log k < \|P\| < C_4 \log k \quad \text{as } q \text{ tends to zero.}$$

Proof of Theorems 2 *and* 3. The assertions that $\|P\|$ is bounded as q tends to zero are proved by showing that (2.4), (2.5), (2.6) hold. These follow readily, since, in Theorem 2,

$$(1 - w_k)^k = (1 - w_1)^k = (k-1)^k/k^k < 1/e < 3/8$$

while, in Theorem 3,

$$\begin{aligned} (1 - w_{k+1})^k = (1 - w_0)^k &= \cos^{2k}(\alpha_k/2) > \left(1 - \alpha_k^2/8\right)^{2k} \\ &> 1 - \pi\alpha_k/8 > (8k+3)/(8k+8) > 3/4. \end{aligned}$$

In Theorem 2, the lower bound on $\|P\|$ follows from the fact that, as q tends to zero, the nodes

$$\tau_{m-k+1}, \tau_{m-k+2}, \ldots, \tau_{m-k+j}, \ldots, \tau_{m-1}$$

tend to

$$(2-k)/k, (4-k)/k, \ldots, (2j-k)/k, \ldots, (k-2)/k,$$

and that, for $s = s_\mu$ with $m - k \leqslant \mu \leqslant m - 1$,

$$|s(\pm 1)| = \left(1 - r_2^{m-k}\right)/\left(1 - 2\theta_2^k\right) + O(q) \geqslant 1/\left(1 - \theta_2^k\right) + O(q) > 1,$$

so that $\|P\|$ is bounded below by any lower bound for polynomial interpolation on $[-1, +1]$ at the equally-spaced nodes

$$-1, (2-k)/k, (4-k)/k, \ldots, (2j-k)/k, \ldots, (k-2)/k, +1.$$

See Rivlin [7, pp. 96–99] for a proof that such polynomial interpolation grows exponentially.

Similarly, in Theorem 3, the lower bound on $\|P\|$ follows from the fact that $\tau_{m-k}, \ldots, \tau_m$ approach the Chebyshev nodes $-\cos(2j\alpha_k - \alpha_k)$, $j = 1, \ldots, k+1$, as q tends to zero and the fact that polynomial interpolation on these nodes has logarithmic growth. See [7, pp. 93–96].

To complete the proof that $\|P\|$ grows exponentially or logarithmically in Theorem 2 or Theorem 3, respectively, it is necessary only to show that, for each μ, $s_\mu(t)$ is "controlled" outside $(-1, +1)$. This fact follows from the closing lines of the proof of Theorem 1, where it was noted that, for $t_1 \leqslant t \leqslant t_m$, there is a $j < m$ such that $|s(t)| \leqslant 1 + |A_j| + O(q)$. For Theorem 2, (3.1) and (2.13) imply that

$$\max_{j<m} |A_j| < \frac{2}{1 - 2\theta_2^k} + O(q) < \frac{2e}{e-2} + O(q) < 8,$$

so that $|s(t)| < 10$ for $t \leqslant -1$ as q tends to zero. For Theorem 3, (3.2) and (2.18), (2.19) imply that

$$\begin{aligned} \max_{j<m} |A_j| &< \frac{2}{2\theta_2^k - 1} + 2|s(-1)| + O(q) \\ &< \frac{2}{2\cos^{2k}(\alpha_k/2) - 1} + 2|s(-1)| + O(q) \\ &< 4 + 2|s(-1)| + O(q), \end{aligned}$$

so that $|s(t)| < 6 + 2|s(-1)|$ for $t \leqslant -1$ as q tends to zero. Symmetry considerations give like bounds for $|s(t)|$ on $+1 = t_{m+1} \leqslant t \leqslant t_{2m}$.

The proof of Theorem 2 and Theorem 3 is complete. □

If $q = 1$ (not covered by these theorems), two-sided q-spline interpolation is essentially the same as cardinal spline interpolation, for which logarithmic growth of $\|P\|$ with k has been demonstrated; see [6]. This fact supports the conjecture that q tending to zero gives "worst-case" results for the nodes (3.1).

For cubic spline interpolation with arbitrary knot spacing and the nodes (3.1), de Boor [2] has shown that $\|P\| < 27$. He conjectures that $\|P\| < 3$ or 4 may be true. The following supplies a lower bound on $\limsup\|P\|$, where the lim sup is taken over all ordered knot spacings.

THEOREM 4. *Let* $k = 3$ *and let* **t** *and* τ *be given by* (1.1) *and* (3.1), *respectively. Then*

$$\lim\|P\| = (222\sqrt{111} + 999)/1331 = 2.507825\ldots,$$

where $\lim\|P\|$ *denotes the limiting value of* $\|P\|$ *as* q *tends to zero and* m *tends to infinity.*

Proof. Let $s = s_\mu$ with $\mu = m - 1$. From (2.21) and (2.13)

$$s(-1) = \frac{(-1)^{\mu-m+k}}{1 - 2\theta_2^k}(1 - r_2^{m-k}) + O(q) \tag{3.3}$$

for each $k > 2$ and $\mu \geq m - k$. Similarly,

$$s(+1) = \frac{(-1)^{m-1-\mu}}{1 - 2\theta_1^k}(1 - r_1^{m-k}) + O(q) \tag{3.4}$$

for $k > 2$ and $\mu \leq m - 1$. Thus, for the case presently under consideration, $s(t)$ tends, on $[-1, +1]$, to the cubic $p(t)$ satisfying $p(\pm 1) = 27/11$ and $p(\pm 1/3) = \pm 1$. This cubic is

$$p(t) = (-297t^3 + 243t^2 + 297t - 27)/88.$$

It has a maximum on $[-1, +1]$ of $(222\sqrt{111} + 999)/1331$ at $t = (9 + 2\sqrt{111})/33$. Showing that $\lim\|P\|$ exists and is equal to this maximum requires a discussion (which we omit) similar to the last paragraph in the proof of Theorem 1 above. □

For arbitrary k it is easy to find $p(t)$, the polynomial which $s_{m-1}(t)$ approaches as q tends to zero and m tends to infinity. From (3.1) and (3.4)

$$\lim s(+1) = z_k = \frac{1}{1 - 2((k-1)/k)^k}.$$

From (3.3), $\lim s(-1) = (-1)^{k-1} z_k$. Then standard combinatorial formulas give (see Gould [4, p. 59])

$$p(t) = -(-1)^l \sum_0^l \frac{(-4)^j T}{T+j}\binom{T+j}{2j} + \frac{2T + kz_k}{T+l}\binom{T+l}{2l},$$

if k is even with $l = k/2$ and $T = lt$, and

$$p(t) = -(-1)^l \sum_0^l \frac{(-4)^j 2T}{2j+1}\binom{T+j-1/2}{2j} + \frac{2T + kz_k}{k}\binom{T+l-1/2}{2l},$$

if k is odd with $l = (k-1)/2$ and $T = kt/2$. The maximum of $p(t)$ on $(k-2)/k \leq t \leq +1$ is a good lower bound on $\|P\|$ as q tends to zero and m tends to infinity.

The following table was computed via double-precision arithmetic in FORTRAN on an Amdahl 470/V7 computer. All entries are rounded down.

Lower bounds on lim sup$\|P\|$

k	max $p(t)$	k	max $p(t)$	k	max $p(t)$	k	max $p(t)$
2	2.0000	7	7.7939	12	9.02×10	27	9.45×10^5
3	2.5078	8	11.8194	15	5.13×10^2	30	6.60×10^6
4	3.0814	9	18.7344	18	3.17×10^3	33	4.67×10^7
5	3.9686	10	30.7986	21	2.05×10^4	36	3.34×10^8
6	5.4087	11	52.1254	24	1.37×10^5	39	2.42×10^9

This table, in which the exponential growth is clear, is associated with Theorem 2 above. A corresponding table of lower bounds on lim sup$\|P\|$ for the node assignment of Theorem 3 can be computed from the fact that the Lebesgue function for polynomial interpolation on the Chebyshev nodes attains its maximum at $t = 1$; see [7, Eq. (4.2.19)]. The first few entries of such a table are:

$$(1,1.414)\quad(2,1.666)\quad(3,1.847)\quad(4,1.988)\quad(5,2.104)\quad(6,2.202).$$

A later entry is (35,3.243). The logarithmic growth is clear. For $k = 1$ with arbitrary knots it can be shown that $\|P\| \leqslant \sqrt{2}$ when nodes are specified by (3.2) above. Whether the other bounds are "good" bounds for the arbitrary knot case is problematical.

4. Remarks. For one-sided q-splines with spline knots $t_i = (1 - q^i)/(1 - q)$, $i = \ldots, -1, 0, 1, 2, \ldots,$ and interpolation nodes $\tau_i = t_i + \theta q^i$, where θ is fixed, $0 \leqslant \theta \leqslant 1$, S. L. Lee [5] has considered eigensplines, i.e., nontrivial splines $s(t)$ satisfying $s(t) = \lambda s(1 + qt)$ for some fixed eigenvalue λ. Setting $\lambda = -1$ yields, for each $k \geqslant 2$, a certain equation $F_k(q, \theta) = 0$. If q and either θ_1 or θ_2 defined above satisfy this equation, then two-sided q-spline interpolation is unbounded. Lee [5] has shown that $F_k(0+, \theta) = C[2\theta^k - 1][2(1 - \theta)^k - 1]$.

For quadratic splines with arbitrary knots t_i, Demko [3] has shown that interpolation is bounded independently of t_i and τ_i if the nodes τ_i satisfy $\tau_i = t_{i+2} - \lambda_i(t_{i+2} - t_{i+1})$ with $\lambda_i^2 \leqslant \gamma < \frac{1}{2}$ and $(1 - \lambda_i)^2 \leqslant \gamma < \frac{1}{2}$. Consequently, for $k = 2$, the results of Theorem 1 above with (2.5)a and (2.6)a are valid for all q and not just as q tends to zero.

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