MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 157
JANUARY 1982

Evaluation of Integrals of Howland Type Involving a Bessel Function

By Chih-Bing Ling and Ming-Jing Wu

Abstract. This paper presents a method of evaluation of four integrals of Howland type, which involve a Bessel function in the integrands. With the aid of tabulated values, they are evaluated to 10D. Two of the four Howland integrals needed in the evaluation are evaluated anew to 20D in order to provide adequate accuracy.

In a recent investigation of certain problems in elasticity concerning elliptic boundaries, four integrals of Howland type involving an additional Bessel function in the integrands were encountered. We believe that they deserve special consideration. The integrals are as follows:

$$\begin{aligned} &\begin{matrix} F_{n,k} \\ F^*_{n,k} \end{matrix}(a) = \frac{2^k}{k!}\int_0^\infty \frac{m^k J_n(ma)}{\sinh 2m \pm 2m}\,dm \qquad \begin{matrix} (n+k \geqslant 1), \\ (n+k \geqslant 3), \end{matrix} \\ &\begin{matrix} E_{n,k} \\ E^*_{n,k} \end{matrix}(a) = \frac{2^k}{k!}\int_0^\infty \frac{m^k J_n(ma)\coth m}{\sinh 2m \pm 2m}\,dm \qquad \begin{matrix} (n+k \geqslant 2), \\ (n+k \geqslant 4), \end{matrix} \end{aligned} \tag{1}$$

where J_n is a Bessel function of the first kind of integral order n. n and k are nonnegative integers restricted as indicated above in order to render each integral convergent at the lower limit. The constant a may be real or complex.

By using the usual series expression for J_n and integrating, the first integral becomes

$$F_{n,k}(a) = \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p}\binom{n+k+2p}{k}\left(\frac{a}{4}\right)^{n+2p} I_{n+k+2p}, \tag{2}$$

where

$$I_k = \frac{2^k}{k!}\int_0^\infty \frac{m^k\,dm}{\sinh 2m + 2m} \qquad (k \geqslant 1). \tag{3}$$

Since I_k tends asymptotically to unity as k tends to infinity, we write, with a view of improving convergence of the series,

$$\begin{aligned} F_{n,k}(a) = {} & K_{n,k}(a) \\ & - \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p}\binom{n+k+2p}{k}\left(1 - I_{n+k+2p}\right)\left(\frac{a}{4}\right)^{n+2p}, \end{aligned} \tag{4}$$

Received February 10, 1981; revised May 29, 1981.
1980 *Mathematics Subject Classification*. Primary 65A05, 65D20, 65D30.
Key words and phrases. Howland integrals, integrals involving Bessel function.

© 1982 American Mathematical Society
0025-5718/82/0000-0482/$03.00

where [7]

$$K_{n,k}(a) = \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} \left(\frac{a}{4}\right)^{n+2p} \tag{5}$$
$$= \frac{2^{k+1}}{k!} \int_0^\infty m^k e^{-2m} J_n(ma)\, dm = \frac{(n+k)!}{k!} P_k^{-n}(t) t^{k+1},$$

in which P_k^{-n} is an associated Legendre function of the first kind [1] and

$$t = 2/(a^2+4)^{1/2}. \tag{6}$$

Or, in terms of Gauss' hypergeometric function,

$$K_{n,k}(a) = \binom{n+k}{k} \left(\frac{1-t}{1+t}\right)^{n/2} t^{k+1} {}_2F_1\left(-k, k+1; n+1; \frac{1}{2} - \frac{t}{2}\right), \tag{7}$$

if t is real and $-1 < t < 1$. On the other hand, if t is complex, the factor $(1-t)^{n/2}$ is replaced by $(t-1)^{n/2}$.

Similarly, the other three integrals are

$$F_{n,k}^*(a) = K_{n,k}(a) + \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (I_{n+k+2p}^* - 1) \left(\frac{a}{4}\right)^{n+2p},$$

$$E_{n,k}(a) = K_{n,k}(a) - \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (1 - IV_{n+k+2p}) \left(\frac{a}{4}\right)^{n+2p}, \tag{8}$$

$$E_{n,k}^*(a) = K_{n,k}(a) + \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (IV_{n+k+2p}^* - 1) \left(\frac{a}{4}\right)^{n+2p},$$

where

$$I_k^* = \frac{2^k}{k!} \int_0^\infty \frac{m^k\, dm}{\sinh 2m - 2m} \qquad (k \geqslant 3), \tag{9}$$
$$\begin{matrix} IV_k \\ IV_k^* \end{matrix} = \frac{2^k}{k!} \int_0^\infty \frac{m^k \coth m\, dm}{\sinh 2m \pm 2m} \qquad \begin{matrix} (k \geqslant 2), \\ (k \geqslant 4). \end{matrix}$$

The evaluation of the last two integrals in (9) has been considered before by the first author [4]. It was found that

$$IV_k = I_k - (I_{k-1} + II_{k-1} - S_k)/k, \tag{10}$$
$$IV_k^* = I_k^* + (I_{k-1}^* + II_{k-1}^* - S_k)/k,$$

where

$$\begin{matrix} II_k \\ II_k^* \end{matrix} = \frac{2^k}{k!} \int_0^\infty \frac{m^k e^{-2m}\, dm}{\sinh 2m \pm 2m} \qquad \begin{matrix} (k \geqslant 1), \\ (k \geqslant 3), \end{matrix} \tag{11}$$
$$S_k = \sum_{n=1}^{\infty} \frac{1}{n^k} \qquad (k \geqslant 2).$$

The integrals I_k, I_k^*, II_k and II_k^* are the four ordinary Howland integrals. The first two were tabulated by the first author and Lin to 25D [6], [5] and the remaining two by Nelson to 9D [8]. The Riemann zeta function S_k was tabulated by Glaisher for integral k to 32D [2]. Among these tabulated values, it appears that Nelson's 9D values of II_k and II_k^* are inadequate for the present purpose. Consequently, the values are computed anew. The evaluation of these two integrals is described in the Appendix. Table 1 shows the values to 20D.

The relations in (4) and (8) are suitable for numercial computation. With the tabulated values, the four integrals can therefore be evaluated when a is given. In case of slow convergence of the series, the Euler transformation for alternating series [3] may be applied. The following recurrence relation for the first integral is mentioned:

$$F_{n-1,k}(a) + F_{n+1,k}(a) = \frac{4n}{ka} F_{n,k-1}(a), \tag{12}$$

which is derived by virtue of the relation connecting three Bessel functions of consecutive integral orders. Similar relations can be found for the other three integrals as well as for $K_{n,k}(a)$. These recurrence relations may be used for checking purposes. They can also be used for computing an unknown integral from two known integrals.

Values of the four integrals for the particular value $a = 1$ are computed. The expression in (7) is used for computing $K_{n,k}$. The results for $n = 0(1)3$ and $k = 0(1)10$ are shown in Table 2 to 10D.

Appendix. *Evaluation of II_k and II_k^*.* The two Howland integrals in (11) may be written as

$$\begin{matrix} II_k \\ II_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k e^{-w}\, dw}{\sinh w \pm w} \qquad \begin{matrix} (k \geqslant 1), \\ (k \geqslant 3). \end{matrix} \tag{13}$$

Expansion of the integrand yields

$$\frac{w^k e^{-w}}{\sinh w \pm w} = \frac{2w^k e^{-2w}}{1 \pm 2we^{-w} - e^{-2w}} = 2w^k e^{-2w} \sum_{n=0}^{\infty} (\mp 1)^n p_n(w) e^{-nw}, \tag{14}$$

where $p_n(w)$ is a polynomial in w of degree n related to the Gegenbauer polynomial of order unity. Its form is different according as n is even or odd. For $n \geqslant 0$,

$$p_{2n}(w) = \sum_{t=0}^{n} \binom{n+t}{2t} (2w)^{2t}, \qquad p_{2n+1}(w) = \sum_{t=0}^{n} \binom{n+t+1}{2t+1} (2w)^{2t+1}. \tag{15}$$

With the aid of the integral

$$\int_0^\infty w^t e^{-cw}\, dw = \frac{t!}{c^{t+1}} \qquad (c > 0) \tag{16}$$

we get

$$\begin{matrix} II_k \\ II_k^* \end{matrix} = \sum_{n=1}^{\infty} (\mp 1)^{n+1} \frac{q_n(k)}{(n+1)^{k+1}}, \tag{17}$$

where, for $n \geqslant 0$,

$$
(18)\qquad
\begin{aligned}
q_{2n+1}(k) &= \sum_{t=0}^{n} \binom{k+2t}{k} \frac{(n+t)!}{(n-t)!} \frac{1}{(n+1)^{2t}},\\
q_{2n+2}(k) &= \sum_{t=0}^{n} \binom{k+2t+1}{k} \frac{(n+t+1)!}{(n-t)!} \left(\frac{2}{2n+3}\right)^{2t+1}.
\end{aligned}
$$

The series in (17) is rapidly convergent when k is large but slowly convergent when k is small. For instance, an accuracy of 20D can be reached with the first six terms when $k \geqslant 30$, with the first twenty terms when $k \geqslant 22$, and with the first eighty terms when $k \geqslant 18$. To reach an accuracy of 25D, the corresponding values of k are not less than 35, 26, and 21, respectively.

It is noted that these two integrals, unlike I_k and I_k^*, cannot be evaluated by Plana's method. In terms of I_k and I_k^*, the following relations are obtained by expanding e^{-w} into series of w and then applying the Kummer transformation [3]:

$$
(19)\qquad
\begin{aligned}
\frac{1}{2^{k+1}} - II_k &= \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{n} (1 - I_{n+k}),\\
II_k^* - \frac{1}{2^{k+1}} &= \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{n} (I_{n+k}^* - 1).
\end{aligned}
$$

They are suitable for computation only for the first few values of k. When k increases, accuracy is rapidly lost due to the binomial coefficient involved in the series. However, it appears that the loss can be reduced if the computation is carried out in several steps through some intermediate integrals. Suppose that s steps are taken and in each step a factor $e^{-w/s}$ is expanded into series instead of e^{-w}. Denote the intermediate integrals in the rth step by

$$
(20)\qquad
\left.\begin{matrix} I_k^{(r)} \\ I_k^{*(r)} \end{matrix}\right\} = \frac{1}{2(k!)} \int_0^{\infty} \frac{w^k e^{-rw/s}\, dw}{\sinh w \pm w} \qquad \begin{matrix} (k \geqslant 1), \\ (k \geqslant 3), \end{matrix}
$$

where $r = 0, 1, 2, \ldots, s$. These expressions give I_k and I_k^* when $r = 0$, and II_k and II_k^* when $r = s$. In a similar manner, it is found that

$$
(21)\qquad
\begin{aligned}
\left(\frac{s}{s+r}\right)^{k+1} - I_k^{(r)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{s^n} \binom{n+k}{n} \left\{ \left(\frac{s}{s+r-1}\right)^{n+k+1} - I_{n+k}^{(r-1)} \right\},\\
I_k^{*(r)} - \left(\frac{s}{s+r}\right)^{k+1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{s^n} \binom{n+k}{n} \left\{ I_{n+k}^{*(r-1)} - \left(\frac{s}{s+r-1}\right)^{n+k+1} \right\}.
\end{aligned}
$$

It is seen that the binomial coefficient involved in the series is now divided by a factor s^n. Hence, the loss of accuracy is considerably reduced, especially when s is large. If the computation is carried out recurrently s times, the integrals II_k and II_k^* are obtained. Also, it turns out that the total loss of accuracy in s steps together is smaller than that by (19) in a single step. For instance, when $k = 20$, the total loss is 10S if $s = 2$, 7S if $s = 4$, and 5S if $s = 7$. The loss is less when k is smaller. On the other hand, the computing time required for s steps is increased s times. Furthermore, analogous to (17), the expansions of the intermediate integrals in the rth step are

$$
(22)\qquad
\left.\begin{matrix} I_k^{(r)} \\ I_k^{*(r)} \end{matrix}\right\} = \sum_{n=1}^{\infty} (\mp 1)^{n+1} q_n^{(r)}(k) \left(\frac{s}{ns+r}\right)^{k+1},
$$

where, for $n \geqslant 0$,

$$(23)\qquad \begin{aligned} q^{(r)}_{2n+1}(k) &= \sum_{t=0}^{n} \binom{k+2t}{k} \frac{(n+t)!}{(n-t)!} \left\{ \frac{2s}{(2n+1)s+r} \right\}^{2t}, \\ q^{(r)}_{2n+2}(k) &= \sum_{t=0}^{n} \binom{k+2t+1}{k} \frac{(n+t+1)!}{(n-t)} \left\{ \frac{2s}{(2n+2)s+r} \right\}^{2t+1}. \end{aligned}$$

Note that when $r = s$, these expressions become those for II_k and II_k^* as given in (17) and (18). Suppose that in the present computation, an accuracy not less than 20D is prescribed to the values of II_k and II_k^*. With the 25D values of I_k and I_k^*, the loss of accuracy must therefore be kept within 5S. In each step, the relations in (21) with $s = 7$ are used for k up to 20. For larger k, the relations in (22) are used instead. The values are also computed by both methods for some overlapping values of k to serve as a check. Further checks on the final results are provided by the following relations:

$$(24)\qquad \begin{aligned} &\sum_{k=0}^{\infty} II_{2k+1} = \frac{1}{2} - II_1, \qquad &&\sum_{k=1}^{\infty} kII_{2k} = \frac{1}{4} - II_2, \\ &\sum_{k=1}^{\infty} II^*_{2k+1} = \frac{1}{2}, &&\sum_{k=2}^{\infty} kII^*_{2k} = \frac{1}{4}. \end{aligned}$$

The values rounded to 20D are shown in Table 1. No discrepancy is noticed in Nelson's 9D values when they are compared with the present values.

Corrigenda. The factor $(-1)^{m+n}$ in (3) and (7) of the paper [6] and also in (3) of the paper [5] should be deleted. The expressions (6) in the paper [6] should be revised accordingly. These corrections, however, do not effect the numerical results of the Howland integrals I_k and I_k^* in these two papers.

TABLE I
Values of II_k and II_k^*

k	II_k	II_k^*
1	0.22011 95814 42489 13267	-
2	0.08792 72351 54623 64461	-
3	0.04334 78620 32253 03164	0.46071 37190 35659 66356
4	0.02258 30042 91563 34696	0.09931 55321 31928 69508
5	0.01192 34729 94286 16469	0.03241 26902 77966 60244
6	0.00628 79721 18696 33642	0.01261 69084 78437 55392
7	0.00329 50133 54822 42863	0.00539 11167 37422 13901
8	0.00171 32982 31483 05028	0.00243 29999 83832 20264
9	0.00088 41479 97290 09130	0.00113 59968 30348 45397
10	0.00045 32145 74349 76403	0.00054 22017 96217 86263
11	0.00023 10097 74078 29735	0.00026 25896 85716 89211
12	0.00011 72094 40904 63806	0.00012 84309 57923 71136
13	0.00005 92535 02802 31223	0.00006 32389 09903 71072
14	0.00002 98701 48236 51008	0.00003 12834 43520 29328
15	0.00001 50252 72536 49084	0.00001 55254 16440 55194
16	0.00000 75457 13757 46890	0.00000 77222 95797 22959
17	0.00000 37848 78742 55501	0.00000 38470 73132 06282
18	0.00000 18967 75182 64532	0.00000 19186 28537 52976
19	0.00000 09499 41039 13811	0.00000 09576 01987 42877
20	0.00000 04755 23929 87590	0.00000 04782 03676 61263

TABLE I (*continued*)

k	II_k	II^*_k
21	0.00000 02379 58146 32115	0.00000 02388 93568 18747
22	0.00000 01190 48321 96864	0.00000 01193 74219 70259
23	0.00000 00595 48513 89530	0.00000 00596 61850 89582
24	0.00000 00297 82793 38128	0.00000 00298 22142 27687
25	0.00000 00148 94380 42812	0.00000 00149 08020 42756
26	0.00000 00074 48230 48966	0.00000 00074 52951 81493
27	0.00000 00037 24477 12334	0.00000 00037 26109 13887
28	0.00000 00018 62364 19592	0.00000 00018 62927 61929
29	0.00000 00009 31225 63641	0.00000 00009 31419 91799
30	0.00000 00004 65627 88197	0.00000 00004 65694 80066
31	0.00000 00002 32819 14519	0.00000 00002 32842 17086
32	0.00000 00001 16411 36812	0.00000 00001 16419 28317
33	0.00000 00000 58206 30278	0.00000 00000 58209 02107
34	0.00000 00000 29103 36435	0.00000 00000 29104 29710
35	0.00000 00000 14551 75540	0.00000 00000 14552 07520
36	0.00000 00000 07275 90285	0.00000 00000 07276 01241
37	0.00000 00000 03637 96006	0.00000 00000 03637 99757
38	0.00000 00000 01818 98299	0.00000 00000 01818 99582
39	0.00000 00000 00909 49251	0.00000 00000 00909 49690
40	0.00000 00000 00454 74660	0.00000 00000 00454 74810
41	0.00000 00000 00227 37342	0.00000 00000 00227 37393
42	0.00000 00000 00113 68675	0.00000 00000 00113 68693
43	0.00000 00000 00056 84339	0.00000 00000 00056 84345
44	0.00000 00000 00028 42170	0.00000 00000 00028 42172
45	0.00000 00000 00014 21085	0.00000 00000 00014 21086
46	0.00000 00000 00007 10543	0.00000 00000 00007 10543
47	0.00000 00000 00003 55271	0.00000 00000 00003 55271
48	0.00000 00000 00001 77636	0.00000 00000 00001 77636
49	0.00000 00000 00000 88818	0.00000 00000 00000 88818
50	0.00000 00000 00000 44409	0.00000 00000 00000 44409
51	0.00000 00000 00000 22204	0.00000 00000 00000 22204
52	0.00000 00000 00000 11102	0.00000 00000 00000 11102
53	0.00000 00000 00000 05551	0.00000 00000 00000 05551
54	0.00000 00000 00000 02776	0.00000 00000 00000 02776
55	0.00000 00000 00000 01388	0.00000 00000 00000 01388
56	0.00000 00000 00000 00694	0.00000 00000 00000 00694
57	0.00000 00000 00000 00347	0.00000 00000 00000 00347
58	0.00000 00000 00000 00173	0.00000 00000 00000 00173
59	0.00000 00000 00000 00087	0.00000 00000 00000 00087
60	0.00000 00000 00000 00043	0.00000 00000 00000 00043
61	0.00000 00000 00000 00022	0.00000 00000 00000 00022
62	0.00000 00000 00000 00011	0.00000 00000 00000 00011
63	0.00000 00000 00000 00005	0.00000 00000 00000 00005
64	0.00000 00000 00000 00003	0.00000 00000 00000 00003
65	0.00000 00000 00000 00001	0.00000 00000 00000 00001
66	0.00000 00000 00000 00001	0.00000 00000 00000 00001

TABLE 2
Values of four integrals for a = 1

n	k	$F_{n,k}(1)$	$F^*_{n,k}(1)$	$E_{n,k}(1)$	$E^*_{n,k}(1)$
0	1	0.54084 46591	-	-	-
	2	0.34194 08513	-	0.63469 34541	-
	3	0.18795 53763	1.15707 58719	0.27562 06093	-
	4	0.04801 62464	0.33813 39129	0.08078 15488	0.64599 26848
	5	-0.06316 63029	0.04109 33656	-0.05054 94481	0.09969 22234
	6	-0.13339 10175	-0.09670 05792	-0.12874 07789	-0.08136 34422
	7	-0.16004 74673	-0.14907 54405	-0.15851 54846	-0.14476 57105
	8	-0.14933 52743	-0.14758 67013	-0.14894 97848	-0.14645 35865
	9	-0.11299 26003	-0.11398 91334	-0.11297 69522	-0.11377 37369
	10	-0.06461 18449	-0.06600 73063	-0.06468 26817	-0.06602 70468
1	0	0.16069 18196	-	-	-
	1	0.26110 17772	-	0.41598 35324	-
	2	0.33007 03722	1.14211 61369	0.40565 04816	-
	3	0.34825 03408	0.72059 05605	0.39028 30431	1.05051 69033
	4	0.31430 77838	0.51523 39857	0.33771 73885	0.60270 22003
	5	0.24089 51857	0.35123 15466	0.25357 33417	0.38323 09369
	6	0.14764 05342	0.20700 47013	0.15426 00266	0.22007 55146
	7	0.05455 61086	0.08540 39244	0.05787 80057	0.09096 34111
	8	-0.02254 70977	-0.00718 85853	-0.02094 79499	-0.00480 47337
	9	-0.07432 31433	-0.06705 20664	-0.07358 68206	-0.06604 11886
	10	-0.09816 79533	-0.09493 09795	-0.09784 54429	-0.09451 33296
2	0	0.03700 02514	-	0.05672 84626	-
	1	0.10192 26191	0.31247 28392	0.12168 33587	-
	2	0.18026 27032	0.33106 76512	0.19727 36106	0.45768 54746
	3	0.25213 84533	0.36574 56439	0.26524 66995	0.41202 63376
	4	0.30023 40944	0.38245 66476	0.30950 14944	0.40452 42185
	5	0.31461 25299	0.37109 38229	0.32072 33589	0.38246 95368
	6	0.29398 78080	0.33085 49436	0.29778 96734	0.33685 07335
	7	0.24441 34868	0.26736 38413	0.24666 40712	0.27052 31475
	8	0.17661 33286	0.19028 86636	0.17788 87876	0.19193 52921
	9	0.10297 16679	0.11079 42066	0.10366 67523	0.11163 82998
	10	0.03488 25876	0.03918 64797	0.03524 79535	0.03961 05714
3	0	0.00940 63033	0.02727 77106	0.01109 03091	-
	1	0.03490 02343	0.06101 33066	0.03784 41688	0.08236 46842
	2	0.07762 01043	0.10777 52201	0.08108 29533	0.11967 76422
	3	0.13245 02011	0.16225 65094	0.13577 99185	0.16997 76958
	4	0.18996 91228	0.21625 73021	0.19277 60106	0.22135 04748
	5	0.23947 93654	0.26069 90895	0.24162 90493	0.26400 78127
	6	0.27184 28389	0.28778 70626	0.27337 11186	0.28988 38678
	7	0.28142 99577	0.29271 60111	0.28245 30496	0.29400 88557
	8	0.26696 05846	0.27455 24265	0.26761 20212	0.27532 78811
	9	0.23131 27687	0.23619 75451	0.23171 01874	0.23665 03371
	10	0.18054 52877	0.18356 63448	0.18077 88447	0.18382 39694

Dpartment of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

Institute of Mathematics
Academia Sinica
Taipei, Taiwan, Republic of China

1. A. Erdélyi et al., *Higher Transcendental Functions*, vol. 1, Chapter 3, McGraw-Hill, New York, 1953.

2. J. W. L. Glaisher, "Tables of $1 \pm 2^{-n} + 3^{-n} \pm 4^{-n}$ + etc., etc. to 32 places of decimals," *Quart. J. Math.*, v. 45, 1914, pp. 141–158.

3. K. Knopp, *Infinite Series*, Hafner, New York, 1947.

4. C. B. Ling, "Tables of values of 16 integrals of algebraic-hyperbolic type," *MTAC.*, v. 11, 1957, pp. 160–166.

5. C. B. Ling, "Further evaluation of Howland integrals," *Math. Comp.*, v. 32, 1978, pp. 900–904.

6. C. B. Ling & J. Lin, "A new method of evaluation of Howland integrals," *Math. Comp.*, v. 25, 1971, pp. 331–337.

7. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962.

8. C. W. Nelson, "New tables of Howland's and related integrals," *Math. Comp.*, v. 15, 1961, pp. 12–18.