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A Lower Bound for Odd Triperfects

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Abstract. If **n** is odd and $\sigma(n) = 3n$, then **n** exceeds 10^{50}.

A natural number **n** is called triperfect if $\sigma(n) = 3n$. Six triperfects are known, all even. The existence of an odd triperfect is an open question. Kanold [1] showed that an odd triperfect: (a) has at least nine distinct prime factors; (b) is a square; and, (c) exceeds 10^{20}. The purpose of this note is to improve the last result. We prove

THEOREM. *If* **n** *is odd triperfect, then* **n** *exceeds* 10^{50}.

The inequality $\sigma(p^a)/p^a < p/(p-1)$, where $x/(x-1)$ is monotone decreasing in **x**, yields the least number of primes so that $\sigma(n)/n \geqslant 3$. Thus,

LEMMA. *If n is triperfect and $(n, 6) = 1$, then* **n** *exceeds* 10^{108} *and has at least* 32 *distinct prime factors.*

Assume **n** an odd triperfect and $3 \mid n$. Since $\sigma(n) = 3n$, even powers of prime factors of $\sigma(n)$ divide **n**, up to, possibly, powers of 3. This caution is necessary since if $3^a \parallel n$, then $3^{a+1} \parallel \sigma(n)$. If $p^a \parallel n$, then $p^a\sigma(p^a) \mid n$, again, up to power of 3. As $3^{54} > 5 \cdot 10^{25}$, it suffices to examine **n** assuming $3^t \parallel n$, **t** even, $2 \leqslant t < 54$.

We construct all sequences of primes $P = \{3 = p_0, p_1, p_2, \ldots\}$ and of even naturals $N = \{t = n_0, n_1, n_2, \ldots\}$ so that (a) p_{i+1} is the largest prime divisor of $\sigma(p_i^{n_i})$, and (b) for each i, $n_i = 2, 4, 6, \ldots$. The sequences are extended until either: (a) the product of known factors of **n** exceeds 10^{50}, (b) powers of 3 among the σ's exceed $t + 1$, or (c) **n** is shown to be triperfect. For the most part, complete prime factorizations of the σ's were calculated, although this was not always necessary. Of the 221 cases that occurred, four were of type (b) and the remainder of type (a). No triperfects were found. This proves the theorem.

From known results on perfect and multiperfect numbers, we have

COROLLARY. *If* **n** *is odd multiperfect, the* **n** *exceeds* 10^{50}.

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A copy of the computational data will be deposited with UMT. The Appendix below contains the computations for the cases $3^{16} \parallel n$, $3^{24} \parallel n$, and $3^{44} \parallel n$.

Appendix.

Case 16–1

$$3^{16} = 43\,046\,721; \quad \sigma(3^{16}) = 64\,570\,081 = 1871 \cdot 34511.$$

$$34511^2 = 1\,191\,009\,121; \quad \sigma(34511^2) = 1\,191\,043\,033 = 13 \cdot 19 \cdot 4822\,039.$$

$$4822\,039^2 = 23\,252\,060\,117\,521;$$

$$\sigma(4822\,039^2) = 23\,252\,064\,939\,561 = 3 \cdot 331 \cdot 23\,415\,976\,777.$$

$$\text{Known factors include } \left[\sigma(3^{16})\right]^2 > 3 \cdot 10^{15}; \quad \left[\sigma(34511^2)\right]^2 > 10^{18};$$

$$\left[\sigma(4822\,039^2)\right]^2 > 4 \cdot 10^{26}. \quad \text{Exceed } 10^{59}.$$

Case 16–2

$$4822\,039^4 > 10^{26}. \quad \sigma(4822\,039^4) > 10^{26}. \quad \text{Exceed } 10^{52}.$$

Case 16–3

$$34511^4 = 1\,418\,502\,726\,305\,192\,641;$$

$$\sigma(34511^4) = 1\,418\,543\,830\,412\,011\,105 = 5 \cdot 11 \cdot 31 \cdot 71 \cdot 131 \cdot 89\,451\,727\,381.$$

$$\text{Known factors include } 34511^4 > 10^{18}; \quad \left[\sigma(34511^4)\right]^2 > 10^{36}.$$

$$\text{Exceed } 10^{54}.$$

Case 16–4

$$34511^6 > 10^{27}; \quad \sigma(34511^6) > 10^{27}. \quad \text{Exceed } 10^{54}.$$

End 3^{16}.

Case 24–1

$$3^{24} = 282\,429\,536\,481; \quad \sigma(3^{24}) = 423\,644\,304\,721 = 11^2 \cdot 8951 \cdot 391\,151.$$

$$391\,151^2 = 152\,999\,104\,801; \quad \sigma(391\,151^2) = 152\,999\,495\,953 = 313 \cdot 488\,816\,281.$$

$$\text{Known factors include } 3^{24} > 10^{11}; \quad \left[\frac{1}{11}\sigma(3^{24})\right]^2 > 10^{20};$$

$$\left[\sigma(391\,151^2)\right]^2 > 10^{22}. \quad \text{Exceed } 10^{53}.$$

Case 24–2

$$391\,151^4 = 23\,408\,726\,069\,907\,381\,249\,601;$$

$$\sigma(391\,151^4) = 23\,408\,785\,915\,813\,222\,761\,505 = 5 \cdot \underline{4\,681\,757\,183\,162\,644\,552\,301}.$$

The primality of the underscored integer is unknown. $3 \nmid \sigma(391\,151^4)$.

$$\text{Known factors include } 3^{24} > 10^{11}; \quad 391\,151^4 > 10^{22};$$

$$\sigma(391\,151^4) > 10^{22}. \quad \text{Exceed } 10^{55}.$$

Case 24–3

$$391\,151^6 > 10^{30}; \quad \sigma(391\,151^6) > 10^{30}. \quad \text{Exceed } 10^{60}.$$

End 3^{24}.

Case 44–1

$$3^{44} = 984\,770\,902\,183\,611\,232\,881;$$

$$\sigma(3^{44}) = 1\,477\,156\,353\,275\,416\,849\,321$$

$$= 11^2 \cdot 13 \cdot 181 \cdot 757 \cdot 1621 \cdot 4561 \cdot 927\,001.$$

Known factors include $3^{44} > 10^{20}$; $\left[\frac{1}{11}(3^{44})\right]^2 > 10^{40}$. Exceed 10^{60}.

End 3^{44}.

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1. H.-J. Kanold, "Über mehrfach volkommene Zahlen. II," *J. Reine Angew. Math.*, v. 197, 1957, pp. 82–96. MR **18**, 873.