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# A Note on the Stable Decomposition of Skew-Symmetric Matrices*

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**Abstract.** Computationally stable decompositions for skew-symmetric matrices, which take advantage of the skew-symmetry in order to halve the work and storage, are presented for solving linear systems of equations.

**1. Introduction.** We shall consider here the problem of solving $Ax = b$ on the computer, where $A$ is either skew-symmetric ($A^T = -A$) or skew-Hermitian ($\bar{A}^T = -A$). We seek a generalization of the $LU$ decomposition in order to obtain a stable decomposition which takes advantage of $A^T = -A$ (or $\bar{A}^T = -A$) so that the work and storage are halved. Although skew matrices do not occur as frequently as symmetric matrices, they are occasionally of interest [7], [9], [10], [12].

If $A$ is $n \times n$ (real or complex) skew-symmetric, then the diagonal of $A$ is null. Since $\det A = \det A^T = \det(-A) = (-1)^n \det A$, we have $\det A = 0$ if $n$ is odd. If $A^{-1}$ exists, then $A^{-1}$ is also skew-symmetric.

If $A$ is $n \times n$ skew-Hermitian, then the diagonal of $A$ is purely imaginary but need not be null, e.g.,

$$A = \begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix},$$

where $i = \sqrt{-1}$. Since $\overline{\det A} = \det(\bar{A}^T) = \det(-A) = (-1)^n \det A$, we have $\mathrm{Re}(\det A) = 0$ if $n$ is odd and $\mathrm{Im}(\det A) = 0$ if $n$ is even. If $A^{-1}$ exists, then $A^{-1}$ is skew-Hermitian.

**2. Decomposition of Skew-Symmetric Matrices.** Let $A$ be a real or complex skew-symmetric matrix. We may generalize the diagonal pivoting method for symmetric matrices [2], [4], [5], [6], [8] as follows. First, partition $A$ as

$$\left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right],$$

where $S$ is $k \times k$, $C$ is $(n-k) \times k$, and $B$ is $(n-k) \times (n-k)$; clearly, $S$ and $B$ are skew-symmetric. If $S$ and $C$ are null, then we go on to $B$. If $S$ is nonsingular, then

$$A = \left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline CS^{-1} & I \end{array}\right]\left[\begin{array}{c|c} S & 0 \\ \hline 0 & B + CS^{-1}C^T \end{array}\right]\left[\begin{array}{c|c} I & -S^{-1}C^T \\ \hline 0 & I \end{array}\right].$$

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But $B + CS^{-1}C^T$ is once again skew-symmetric. Hence, we need store only the strictly lower (or upper) triangular part of $A$ and can overwrite those elements with the multipliers in $CS^{-1}$ (or $-S^{-1}C^T$) and the strictly lower (or upper) triangular part of $B + CS^{-1}C^T$. Note that $(CS^{-1})^T = -S^{-1}C^T$ since $S^{-T} = -S^{-1}$.

Since $\operatorname{diag}(A) = 0$, we cannot take $k = 1$ unless the first column of $A$ (and hence the first row) is null. Otherwise, we have $k = 2$ and

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix};$$

if $a_{21} \neq 0$, then $S$ is nonsingular. If $a_{21} = 0$ but $a_{i1} \neq 0$ for some $i$, $2 \leqslant i \leqslant n$, then we can interchange the $i$th and second row and column of $A$, so that

$$A = P_1 \left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right] P_1$$

and $S$ is nonsingular; $P_1 = P_1^T$ is obtained by interchanging the $i$th and first column of the identity matrix.

Thus, if the first column of $A$ is null, take $P_1 = I$, $S = 0$ is $1 \times 1$, $C = 0$ is an $(n - 1)$-vector, and we go directly to $B$. If the first column of $A$ is not null, then $A$ is $2 \times 2$ and nonsingular, $C$ is $(n - 2) \times 2$, and the reduced matrix is $B + CS^{-1}C^T$. Then we repeat this procedure for $B = -B^T$ of order $n - 1$ in the former case and for $B + CS^{-1}C^T = -(B + CS^{-1}C^T)^T$ of order $n - 2$ in the latter case.

In conclusion, we have

$$A = P_1M_1P_2M_2 \cdots P_{n-1}M_{n-1}D\tilde{M}_{n-1}P_{n-1} \cdots \tilde{M}_2P_2\tilde{M}_1P_1,$$

where $P_j$ is the identity matrix or a permutation matrix, $M_j$ is the identity matrix or a block unit lower triangular matrix containing two columns of multipliers in its $j$th and $(j + 1)$st columns and $(j + 2)$nd through $n$th rows, $\tilde{M}_j = M_j^T$, and $D$ is skew-symmetric block diagonal with $1 \times 1$ and $2 \times 2$ diagonal blocks—all $1 \times 1$ blocks are zero and all $2 \times 2$ blocks are nonsingular. (If $n$ is odd, then there is at least one $1 \times 1$ block.) Thus, we have reduced the skew matrix $A$ to a block diagonal skew matrix $D$ by a sequence of permutations and congruence transformations. Of course, all relevant elements of the $M_j$ (or $\tilde{M}_j$) and $D$ could be stored in the corresponding strictly lower (or upper) triangular part of $A$. One $n$-vector could store the relevant information in the permutations $P_j$.

Counting divisions as multiplications, the decomposition requires $\frac{1}{6}n^3 - \frac{1}{4}n^2 - \frac{1}{6}n$ multiplications and $\frac{1}{6}n^3 - \frac{3}{4}n^2 + \frac{5}{6}n$ additions if $n$ is even, and $\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{5}{6}n - \frac{1}{2}$ multiplications and $\frac{1}{6}n^3 - n^2 + \frac{11}{6}n - 1$ if $n$ is odd. The number of comparisons is at most $\frac{1}{2}n^2 - \frac{1}{2}n$. Given the decomposition of $A$, we can now solve $Ax = b$ with $n^2 + \mathcal{O}(n)$ multiplications and additions.

**3. Stability of the Decomposition.** In order to have a stable decomposition, we need to ensure that catastrophic element growth in the reduced matrices does not occur from one step to the next [2], [13], [14]. No element growth occurred whenever $S$ was $1 \times 1$. Let us now consider the case when $S$ is $2 \times 2$.

Let

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix}.$$

Then a row of $CS^{-1}$ is

$$[a_{i1}, a_{i2}]\begin{bmatrix} 0 & \dfrac{1}{a_{21}} \\ -\dfrac{1}{a_{21}} & 0 \end{bmatrix} = [-a_{i2}/a_{21}, a_{i1}/a_{21}],$$

and an element of $A^{(3)} \equiv B + CS^{-1}C^T$ is of the form

$$a_{ij}^{(3)} = a_{ij} - \left(\frac{a_{i2}}{a_{21}}\right)a_{j1} + \left(\frac{a_{i1}}{a_{21}}\right)a_{j2}.$$

Thus, if $|a_{21}| = \max_{2 \leqslant i \leqslant n}\{|a_{i1}|, |a_{i2}|\}$, then $|a_{ij}^{(3)}| \leqslant 3\max_{r,s}|a_{rs}|$ and $\max_{i,j}|(B + CS^{-1}C^T)_{ij}| \leqslant 3\max_{r,s}|a_{rs}|$. We can ensure this by interchanging the $k$th and second row and column if $|a_{k1}| = \max_{2 \leqslant i \leqslant n}\{|a_{i1}|, |a_{i2}|\}$ or by interchanging the second and first row and column and then the $k$th and second row and column if $|a_{k2}| = \max_{2 \leqslant i \leqslant n}\{|a_{i1}|, |a_{i2}|\}$.

If we do this at each step, then the element growth factor, the largest element (in modulus) in all the reduced matrices divided by $\max_{r,s}|a_{rs}|$, is bounded by

$$\begin{Bmatrix} 3^{n/2-1} & \text{if } n \text{ is even} \\ 3^{(n-1)/2-1} & \text{if } n \text{ is odd} \end{Bmatrix} \leqslant \left(\sqrt{3}\right)^{n-2} < (1.7321)^{n-2}.$$

This requires $\frac{1}{2}n^2 - \frac{1}{2}n$ comparisons, and is a partial pivoting strategy; cf. [4], [5], [13], [14]. The partial pivoting strategy for the diagonal pivoting method in the symmetric case gives a bound of $(2.57)^{n-1}$ [4], [5].

We can obtain a smaller bound on the element growth factor by employing a complete pivoting strategy. If $|a_{pq}| = \max_{r>s}\{|a_{rs}|\}$, then we can move the $(p, q)$ element to the $(2, 1)$ position symmetrically by interchanging the $q$th and first row and column and then the $p$th and second row and column. This requires at most $\frac{1}{12}n^3 + \frac{1}{8}n^2 - \frac{1}{12}n$ comparisons. By an analysis identical to Wilkinson's for Gaussian elimination with complete pivoting [13], we obtain the same bound as his on the element growth factor: $< \sqrt{n}f(n)$, where

$$f(n) = \left(\prod_{k=2}^{n} k^{1/(k-1)}\right)^{1/2} < 1.8n^{(\ln n)/4},$$

$f(100) \approx 330$. This compares with the bound of $3nf(n)$ for the complete pivoting strategy for the diagonal pivoting method in the symmetric case [2], [6].

**4. Another Stable Decomposition for Skew-Symmetric Matrices.** The other well-known stable decomposition for symmetric matrices is the tridiagonal decomposition developed by Aasen [1] and Parlett and Reid [11]. It decomposes $A = A^T$ as

$$A = P_2L_2 \cdots P_nL_nTL_n^TP_n \cdots L_2^TP_2,$$

where the $P_j$ are permutation matrices, the $L_j$ are unit lower triangular, and $T$ is symmetric tridiagonal. It requires $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\frac{1}{2}n^2 + \mathcal{O}(n)$ comparisons; the bound on element growth is $4^{n-2}$ [3, p. 525].

If $A$ is skew-symmetric, then, by modifying Aasen's algorithm in a manner similar to Section 2, we obtain

$$A = P_2L_2 \cdots P_nL_nT\tilde{L}_nP_n \cdots \tilde{L}_2P_2,$$

where the $P_j$ and $L_j$ are as above, $\tilde{L}_j = L_j^T$, but $T$ is now skew-symmetric tridiagonal (with a null diagonal). It requires $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\frac{1}{2}n^2 + \mathcal{O}(n)$ comparisons; but now the bound on element growth is $3^{n-2}$ (this follows from [3, p. 525], since the diagonal of $A$ is null).

**5. Stable Decomposition of Skew-Hermitian Matrices.** If $A$ is skew-Hermitian $(\bar{A}^T = -A)$, Aasen's algorithm gives

$$A = P_2 L_2 \cdots P_n L_n T \tilde{L}_n P_n \cdots \tilde{L}_2 P_2,$$

where the $P_j$ and $L_j$ are as above, $\tilde{L}_j = \bar{L}_j^T$, but $T$ is now skew-Hermitian. Since the diagonal of $A$ is not necessarily null, element growth is bounded by $4^{n-2}$.

However, when $A$ is skew-Hermitian, we cannot use the techniques of Sections 2 and 3 since the diagonal of $A$ is now not necessarily null. But, if $A$ is skew-Hermitian, then $B = iA$ is Hermitian since $\bar{B}^T = -i\bar{A}^T = -i(-A) = iA = B$. Since $B = iA$ is Hermitian, we can use the stable decomposition for Hermitian matrices [4], [5], [6] and the subroutines in LINPACK [8], obtaining a stable decomposition with $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\geqslant \frac{1}{2}n^2$ but $\leqslant n^2$ comparisons with a partial pivoting strategy as implemented in LINPACK [8], or $\geqslant \frac{1}{12}n^3$ but $\leqslant \frac{1}{6}n^3$ comparisons with a complete pivoting strategy [2], [6]. The element growth factor is bounded by $(2.57)^{n-1}$ for the partial pivoting strategy and $3nf(n)$ for the complete pivoting strategy. The decomposition can now be used to solve $Ax = b$ with $n^2 = \mathcal{O}(n)$ multiplications and additions (by solving $Bx = ib$).

**6. Remarks.** We could do the same thing when $A$ is real skew-symmetric, but $B = iA$ is then *complex* (Hermitian). The algorithms in Sections 2–4 show how we may stay in real arithmetic with stable decompositions based on congruence transformations. Since the nonzero eigenvalues of a real skew-symmetric matrix occur in purely imaginary complex conjugate pairs ($\pm i\mu_j$ where the $\mu_j$ are positive), the "inertia" $(\pi, \nu, \zeta)$ of $A$ (defined to be the number of positive, negative, and zero imaginary parts of the eigenvalues of $A$) is $((n - \zeta)/2, (n - \zeta)/2, \zeta)$. If $A$ is also nonsingular then its "inertia" is $(n/2, n/2, 0)$. This fixed inertia property is why skew-symmetric matrices are easier to decompose than symmetric indefinite matrices. We have an immediate modification of Sylvester's Inertia Theorem to skew-symmetric matrices: if $A$ is skew-symmetric, then $B = MAM^T$ is skew-symmetric and $B$ has the same "inertia" as $A$, where $M$ is nonsingular.

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J. O. Aasen, "On the reduction of a symmetric matrix to tridiagonal form," *BIT*, v. 11, 1971, pp. 233–242.

2. J. R. Bunch, "Analysis of the diagonal pivoting method," *SIAM J. Numer. Anal.*, v. 8, 1971, pp. 656–680.

3. J. R. Bunch, "Partial pivoting strategies for symmetric matrices," *SIAM J. Numer. Anal.*, v. 11, 1974, pp. 521–528.

4. J. R. Bunch & L. Kaufman, "Some stable methods for calculating inertia and solving symmetric linear systems," *Math. Comp.*, v. 31, 1977, pp. 163–179.

5. J. R. Bunch, L. Kaufman & B. N. Parlett, "Decomposition of a symmetric matrix," *Numer. Math.*, v. 27, 1976, pp. 95–109.

6. J. R. Bunch & B. N. Parlett, "Direct methods for solving symmetric indefinite systems of linear equations," *SIAM J. Numer. Anal.*, v. 8, 1971, pp. 639–655.

7. F. Delale & F. Erdogan, "The effect of transverse shear in a cracked plate under skew-symmetric loading," *Trans. ASME*, v. 46, 1979, pp. 618–624.

8. J. J. Dongarra, J. R. Bunch, C. B. Moler & G. W. Stewart, *LINPACK User's Guide*, SIAM, Philadelphia, Pa., 1979.

9. W. Graeff, W. Bauspiess, U. Bonse, M. Schlenker & H. Rauch, "Phase imaging with a skew symmetric LLL neutron interferometer," *Acta Cryst. Sect. A*, v. 34, 1978, p. 239.

10. L. Mirsky, *An Introduction to Linear Algebra*, Clarendon Press, Oxford, 1955.

11. B. N. Parlett & J. K. Reid, "On the solution of a system of linear equations whose matrix is symmetric but not definite," *BIT*, v. 10, 1970, pp. 386–397.

12. R. C. Thompson, "Principal minors of complex symmetric and skew-matrices," *Linear Algebra Appl.*, v. 28, 1979, pp. 249–255.

13. J. H. Wilkinson, "Error analysis of direct methods of matrix inversion," *J. ACM*, v. 8, 1961, pp. 281–330.

14. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.