MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 158
APRIL 1982, PAGES 539-545

An Error Estimate for Stenger's Quadrature Formula

By S. Beighton and B. Noble

Abstract. The basis of this paper is the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \log q \sum_{m=-M}^{M} \frac{2q^m}{(1+q^m)^2} f\left(\frac{q^m-1}{q^m+1}\right),$$

where $q = \exp(2h)$, h being a chosen step length. This formula has been derived from the Trapezoidal Rule formula by F. Stenger.

An explicit form of the error is given for the case where the integrand has a factor of the form $(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$. Application is made to the evaluation of Cauchy principal value integrals with endpoint singularities and an appropriate error form is derived.

An alternative derivation is given for Stenger's quadrature formula for the finite interval $[-1, 1]$, with a more explicit form of the error in the case where the integrand has a factor of the form

$$(1-x)^\alpha(1+x)^\beta, \qquad \alpha, \beta > -1.$$

Application is made to the evaluation of Cauchy principal value integrals with endpoint singularities and an appropriate error form is derived.

An Alternative Derivation of Stenger's Formula With an Error Estimate. Stenger [2] derives the formula

$$(1) \qquad \int_{-1}^{+1} f(x)\,dx \approx \log q \sum_{m=-\infty}^{m=+\infty} \frac{2q^m}{(1+q^m)^2} f\left(\frac{q^m-1}{q^m+1}\right),$$

where $q = \exp(2h)$, h being a chosen step length.

Taking a finite sum gives the modified form

$$(2) \qquad \int_{-1}^{+1} f(x)\,dx \approx \log q \sum_{m=-M}^{m=+M} \frac{2q^m}{(1+q^m)^2} f\left(\frac{q^m-1}{q^m+1}\right),$$

and, throughout this paper, we refer to (2) as Stenger's quadrature formula. Stenger [2] states that (1) is accurate even if f has singularities at the endpoints. For the form

$$f(x) \equiv (1-x)^\alpha(1+x)^\beta g(x),$$

where $\alpha, \beta > -1$, and $g(x)$ possesses differential coefficients of all orders for $-1 \leqslant x \leqslant 1$, Stenger later gave the error in (2) as

$$O\left[\exp\left(-\pi(1+\gamma)^{1/2}M^{1/2}/2^{1/2}\right)\right],$$

where $\gamma = \min(\alpha, \beta)$.

Received April 30, 1981.
1980 *Mathematics Subject Classification.* Primary 65D30.

We first give an alternative derivation of (2) which will lead to a more explicit form of the error.

The substitution $x = \tanh u$ gives

$$\int_{-1}^{+1} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{f(\tanh u)}{\cosh^2 u}\,du \tag{3}$$

$$= \int_{-Mh}^{+Mh} \frac{f(\tanh u)}{\cosh^2 u}\,du + R_1 + R_2, \tag{4}$$

where

$$R_1 = \int_{Mh}^{\infty} \frac{f(\tanh u)}{\cosh^2 u}\,du, \qquad R_2 = \int_{-\infty}^{-Mh} \frac{f(\tanh u)}{\cosh^2 u}\,du. \tag{5}$$

Reverting to $x = \tanh u$, we find

$$R_1 = \int_{1-\varepsilon}^{1} f(x)\,dx, \tag{6}$$

where $\varepsilon = 1 - \tanh Mh$. If we assume that $f(x)$ behaves like $(1 - x)^\alpha g_1(x)$ near $x = 1$, where $g_1(x)$ can be expanded in a Taylor series about $x = 1$, then

$$R_1 = \frac{\varepsilon^{\alpha+1}}{\alpha + 1} g_1(1) + O(\varepsilon^{\alpha+2}). \tag{7}$$

Further,

$$\varepsilon = 1 - \tanh Mh \approx 2\exp(-2Mh), \tag{8}$$

for Mh large enough, giving

$$R_1 = \frac{2^{\alpha+1}}{\alpha + 1} g_1(1) e^{-2(\alpha+1)Mh} + O(e^{-2(\alpha+2)Mh}). \tag{9}$$

Similarly, if we assume that $f(x)$ behaves like $(1 + x)^\beta g_{-1}(x)$ near $x = -1$, where $g_{-1}(x)$ can be expanded in a Taylor series about $x = -1$, then

$$R_2 = \frac{2^{\beta+1}}{\beta + 1} g_{-1}(-1) e^{-2(\beta+1)Mh} + O(e^{-2(\beta+2)Mh}). \tag{10}$$

Next, consider the integral in expression (4): using the Euler-Maclaurin summation formula,

$$\begin{aligned}\int_{-Mh}^{Mh} \frac{f(\tanh u)}{\cosh^2 u}\,du = h \sum_{r=-M}^{M} \frac{f(\tanh rh)}{\cosh^2 rh} - \frac{h}{2}\{F(-Mh) + F(Mh)\} \\ - \frac{h^2}{12}\{F'(-Mh) - F'(Mh)\} + O(h^4),\end{aligned} \tag{11}$$

where $F(u) \equiv f(\tanh u)/\cosh^2 u$, so that

$$F'(u) \equiv \frac{f'(\tanh u)}{\cosh^2 u} - 2\frac{f(\tanh u)}{\cosh^2 u}\tanh u. \tag{12}$$

With the assumption above concerning the behavior of $f(x)$ near $x = 1$,

$$f(\tanh Mh) = \varepsilon^\alpha g_1(1) + O(\varepsilon^{\alpha+1}), \tag{13}$$

and

$$f'(\tanh Mh) = -\varepsilon^{\alpha-1}\alpha g_1(1) + O(\varepsilon^\alpha),$$

so that

$$F'(Mh) = -4\varepsilon^{\alpha+1}g_1(1)(1+\alpha) + O(\varepsilon^{\alpha+2}). \tag{14}$$

Similarly, we obtain

$$F'(-Mh) = 4\varepsilon^{\beta+1}g_{-1}(-1)(1+\beta) + O(\varepsilon^{\beta+2}). \tag{15}$$

Finally, the sum of (11) may be transformed into Stenger's formula by taking $q = \exp(2h)$, so that

$$\tanh jh = \frac{q^j - 1}{q^j + 1}, \qquad \cosh^2 jh = \frac{(q^j+1)^2}{4q^j},$$

giving

$$h \sum_{r=-M}^{M} \frac{f(\tanh rh)}{\cosh^2 rh} = \log q \sum_{r=-M}^{M} \frac{2q^r}{(1+q^r)^2} f\left(\frac{q^r-1}{q^r+1}\right). \tag{16}$$

To simplify the error form, we will modify Stenger's formula by combining the term $-h/2\{F(-Mh) + F(Mh)\}$ with the sum in (11). Thus, combining (9), (10), (14), (15) and (16) in (4), we obtain

$$\int_{-1}^{1} f(x)\,dx = \log q \sum_{r=-M}^{M}{}'' \frac{2q^r}{(1+q^r)^2} f\left(\frac{q^r-1}{q^r+1}\right) + E, \tag{17}$$

where Σ'' has the usual meaning, namely that the first and last terms are halved, and

$$\begin{aligned} E = {} & \frac{2^{\alpha+1}}{\alpha+1} g_1(1) e^{-2(\alpha+1)Mh} + \frac{2^{\beta+1}}{\beta+1} g_{-1}(-1) e^{-2(\beta+1)Mh} \\ & + O(e^{-2(\gamma+2)Mh}) - h^2\{O(e^{-2(\gamma+1)Mh})\} + O(h^4), \end{aligned} \tag{18}$$

with $\gamma = \min(\alpha, \beta)$. Thus, for h small and Mh large enough to satisfy the approximation (8), we have the dominant part of the error term E for the modified form of Stenger's formula (17):

$$\frac{2^{\alpha+1}}{\alpha+1} g_1(1) e^{-2(\alpha+1)Mh} + \frac{2^{\beta+1}}{\beta+1} g_{-1}(-1) e^{-2(\beta+1)Mh}. \tag{19}$$

Numerical Example. The errors incurred in using formula (17) when $f(x) \equiv (1-x)^{3/4}$ are given in Table 1 for a variety of values of M and h ($h < 1$). The values of the error expression (19) for the same values of M and h are given in Table 2. Below the dotted lines ($Mh \geqslant 1$) the two tables are very similar, whilst below the continuous line ($Mh \geqslant 8$) the error is less than $\frac{1}{2} \times 10^{-6}$.

To clarify the relationship between h and M (to give minimum error) we give in Table 3 values of the actual errors for the arguments h and Mh. These results clearly indicate that the error depends on the product Mh rather than M or h separately.

The pattern of these results was found to be substantially the same for the integral

$$\int_{-1}^{+1} (1-x)^{\alpha}\,dx,$$

with $\alpha = \pm\frac{1}{2}, \pm\frac{1}{4}$ and $-\frac{3}{4}$. Naturally, the different values of α resulted in changes in the critical value of the product Mh.

Evaluation of the Integral $\int_{-1}^{+1}(1-x)^{3/4}\,dx$ Using (17).

TABLE 1

Actual errors incurred

M \ h	1/2	1/4	1/8	1/16	1/32
4	0.067	0.436	1.0006	1.433	1.674
8	0.0012	0.063	0.431	1.805	1.433
16	0	0.0012	0.062	0.429	1.004
32	0	0	0.0011	0.062	0.429
64	0	0	0	0.0011	0.062

TABLE 2

Values of the approximate error expression (19)

M \ h	1/2	1/4	1.8	1/16	1/32
4	0.063	0.513	1.571	2.841	3.860
8	0.0011	0.063	0.513	1.571	2.841
16	0	0.0011	0.063	0.513	1.571
32	0	0	0.0011	0.063	0.513
64	0	0	0	0.0011	0.063

TABLE 3

Actual errors for values of the product Mh

Mh \ h	1/2	1/4	1/8	1/16	1/32
1/2	1.032	1.016	1.006	1.005	1.004
1	0.440	0.436	0.431	0.430	0.429
2	0.067	0.063	0.062	0.062	0.062
4	0.0012	0.0012	0.0011	0.0011	0.0011
8	0	0	0	0	0

The zeros in the tables above represent numbers that are at most $\frac{1}{2}\times 10^{-6}$ in modulus.

An Application to Cauchy Principal Value Integrals With Endpoint Singularities. Consider

$$I(x) = P\int_{-1}^{+1}\frac{(1-y)^{\alpha}(1+y)^{\beta}g(y)}{y-x}\,dy, \tag{20}$$

where the integral is defined in the Cauchy principal value sense, α and β are greater than -1, $-1 < x < 1$, and $g(y)$ possesses differential coefficients of all orders for $-1 \leqslant y \leqslant 1$. 'Subtracting out the singularity' gives

$$I(x) = \int_{-1}^{+1} F(x, y)\, dy + g(x) P \int_{-1}^{+1} \frac{(1-y)^{\alpha}(1+y)^{\beta}}{y-x}\, dy, \tag{21}$$

where

$$F(x, y) = \frac{(1-y)^{\alpha}(1+y)^{\beta}}{y-x}[g(y) - g(x)]. \tag{22}$$

The first integral is evaluated by the modified Stenger formula (17), whilst the second may be directly evaluated using Erdélyi [1, p. 250].

To estimate the error we follow the procedure given earlier and observe that $F(x, y)$ behaves like $(1-y)^{\alpha} g_1(y, x)$ near $y = 1$ and like $(1+y)^{\beta} g_{-1}(y, x)$ near $y = -1$, where

$$g_1(y, x) = (1+y)^{\beta}[g(y) - g(x)]/(y-x)$$

and

$$g_{-1}(y, x) = (1-y)^{\alpha}[g(y) - g(x)]/(y-x).$$

We assume that $g_1(y, x)$ and $g_{-1}(y, x)$ may be expanded in Taylor series about $y = 1$ and $y = -1$, respectively, for given x. We may therefore develop an error term from (18) giving the dominant part as

$$\begin{aligned} E_s(x) &= \frac{2^{\alpha+\beta+1}}{\alpha+1} e^{-2(\alpha+1)Mh}\left[\frac{g(1) - g(x)}{1-x}\right] \\ &\quad + \frac{2^{\alpha+\beta+1}}{\beta+1} e^{-2(\beta+1)Mh}\left[\frac{g(-1) - g(x)}{-1-x}\right]. \end{aligned} \tag{23}$$

It will be observed that $E_s(x)$ remains finite as $x \to \pm 1$.

Numerical Example. We evaluate the special case $\alpha = \beta = \frac{1}{2}$ for two typical functions $g(y)$, namely $\cos y$ and $\exp(-y)$, taking $x = -0.4$. Thus

$$E_s(-0.4) = \frac{8}{3} e^{-3Mh}\left[\frac{5}{7}\{g(1) - g(x)\} - \frac{5}{3}\{g(-1) - g(x)\}\right], \tag{24}$$

so that

$$|E_s(-0.4)| \leqslant \frac{80}{9} e^{-3Mh} M(g), \quad \text{where } M(g) = \max_{-1 \leqslant y \leqslant 1} |g(y)|.$$

The values of e^{-3Mh}, for some typical values of the product Mh, are

Mh	1	2	4	8
e^{-3Mh}	0.049787	0.0024787	6.14×10^{-6}	3.77×10^{-11}

so that we can expect results correct to five decimal places when

$$Mh \geqslant 4.$$

Using the two functions $g(y)$, we give a pair of tables in each case, the first recording the errors incurred in using formula (21) for a variety of values of M and h, and the second giving the corresponding values of the error expression (24).

Evaluation of the Integral $\int_{-1}^{+1}((1-y^2)^{1/2}\cos y/(y-x))\,dy$ **Using (21) With** $x = -0.4$.

Table 4

Actual errors incurred

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	0.00273	0.04045	0.12944	0.20815	0.25467	0.27897
8	0	0.00243	0.03951	0.12905	0.20808	0.25466
16	0	0	0.00235	0.03928	0.12896	0.20806
32	0	0	0	0.00233	0.03922	0.12893
64	0	0	0	0	0.00233	0.03921

Table 5

Values of the error expression (24)

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	0.00240	0.04814	0.21577	0.45678	0.66461	0.80168
8	0	0.00240	0.04814	0.21577	0.45678	0.66461
16	0	0	0.00240	0.04814	0.21577	0.45678
32	0	0	0	0.00240	0.04814	0.21577
64	0	0	0	0	0.00240	0.04814

Evaluation of the Integral $\int_{-1}^{+1}((1-y^2)^{1/2}e^{-y}/(y-x))\,dy$ **Using (21) With** $x = -0.4$.

Table 6

Actual errors incurred

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	-0.02118	-0.29833	-0.90271	-1.40823	-1.70055	-1.85231
8	-0.00006	-0.01889	-0.29209	-0.90049	-1.40786	-1.70049
16	0	-0.00005	-0.01830	-0.29053	-0.89994	-1.40777
32	0	0	-0.00005	-0.01815	-0.29014	-0.89981
64	0	0	0	-0.00005	-0.01812	-0.29004

Table 7

Values of the error expression (24)

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	-0.01882	-0.37797	-1.69395	-3.58610	-5.21774	-6.29380
8	-0.00005	-0.01882	-0.37797	-1.69395	-3.58610	-5.21774
16	0	-0.00005	-0.01882	-0.37797	-1.69395	-3.58610
32	0	0	-0.00005	-0.01882	-0.37797	-1.69395
64	0	0	0	-0.00005	-0.01882	-0.37797

We observe that, as with Tables 1 and 2, the tables are very similar below the dotted lines ($Mh \geqslant 1$) and below the continuous lines ($Mh \geqslant 4$); the error is confined to the 6th decimal place at most.

We give a further example illustrating the effectiveness of the method and the error expression in the potentially more difficult situation

$$\alpha = \beta = -\tfrac{1}{2}.$$

Again $x = -0.4$, and we take $g(y) = \cos y$.

In this instance, the crucial factor in $E_s(x)$ is

$$e^{-Mh},$$

so that we can expect results accurate to 5 decimal places when $Mh \geqslant 12$. This is confirmed by the numerical results.

Evaluation of the Integral $\int_{-1}^{+1}(\cos y/(1-y^2)^{1/2}(y-x))\,dy$ Using (21) With $x = -0.4$.

TABLE 8

Actual errors incurred

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	0.09979	0.26007	0.40645	0.49719	0.54547	0.57001
8	0.01363	0.09832	0.25930	0.40628	0.49717	0.54547
16	0.00032	0.01342	0.09795	0.25910	0.40623	0.49716
32	0.00000	0.00031	0.01337	0.09786	0.25905	0.40622
64	0.00000	0.00000	0.00031	0.01336	0.09783	0.25904

TABLE 9

Values of the error expression (23)

h / M	1/2	1/4	1/8	1/16	1/32	1/64
4	0.09815	0.26681	0.43989	0.56483	0.64003	0.68131
8	0.01328	0.09815	0.26681	0.43889	0.56483	0.64003
16	0.00024	0.01328	0.09815	0.26681	0.43989	0.56483
32	0.00000	0.00024	0.01328	0.09815	0.26681	0.43989
64	0.00000	0.00000	0.00024	0.01328	0.09815	0.26681

N. E. London Polytechnic
Longbridge Road
Barking, Essex, England

1. A. ERDÉLYI, *Tables of Integral Transforms*, Vol. II, McGraw-Hill, New York, 1954.

2. F. STENGER, "Integration formulae based on the trapezoidal formula," *J. Inst. Math. Appl.*, v. 12, 1973, pp. 103–114.