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Polynomial Type Padé Approximants

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Abstract. Some results are established giving conditions on $f(x)$ so that its main diagonal Padé approximation $R_n(x)$ is of the form $P_n(x)/P_n(-x)$, where $P_n(x)$ is a polynomial in x of degree n. A number of applications to special functions are presented. Numerical computations are given for the gamma function using the "bignum" arithmetical facilities of formula manipulation languages REDUCE2, FORMAC.

1. Introduction. Suppose $f(x)$ is represented by the at least formal series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{1}$$

then the main diagonal Padé approximation to $f(x)$ is of the form

$$R_n(x) = P_n(x)/Q_n(x), \tag{2}$$

where $P_n(x)$ and $Q_n(x)$ are polynomials in x of degree n. That is,

$$f(x) - R_n(x) = O(x^{2n+1}). \tag{3}$$

If

$$Q_n(x) = P_n(-x), \qquad R_n(x) = P_n(x)/P_n(-x), \tag{4}$$

then evaluation of $R_n(x)$ can be considerably simplified in the sense that the number of operations required is as though only a single polynomial needs to be evaluated instead of two polynomials. For a case in point, it is known (e.g., [7]) that if $f(x) = e^{-x}$, then (4) is true. The simplification results by separately evaluating the odd and even parts of $P_n(x)$. In this connection, see the discussion given by Luke [4, p. 192] and [5, p. 48].

In this paper, we present some more cases where the Padé approximant is of the form (4). We call such rational approximations polynomial type Padé approximants.

2. Functions with Polynomial Type Padé Approximants. We prove

THEOREM 1. *Let $f(x)$ be given by* (1), *and suppose that* (2) *and* (3) *hold. Further, suppose that*

$$f(x) \cdot f(-x) = 1. \tag{5}$$

Then (4) *is true.*

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Note 1. It is not difficult to see that (5) is equivalent to the condition

$$\ln f(x) = \sum_{k=1}^{\infty} g_k x^{2k-1}. \tag{6}$$

That is, the formal expansion of the logarithm of the function contains only odd powers of x.

Note 2. This result first appeared in [6].

Proof. It is known that if $R_n(x)$ as given by (2) is the Padé approximation of $f(x)$, then the reciprocal of (2) is the Padé approximation of $1/f(x)$; see [2]. So

$$\frac{Q_n(x)}{P_n(x)} = \frac{P_n(-x)}{Q_n(-x)}. \tag{7}$$

That is $Q_n(x) = \rho P_n(-x)$ and $P_n(x) = \rho Q_n(-x)$, where ρ is a constant. Since $Q_n(0) = P_n(0) = c_0$, $\rho = 1$ and (4) holds.

An interesting application of our result is the Stirling series. It is a well-known fact that

$$\Gamma(x) = x^x e^{-x}\sqrt{\frac{2\pi}{x}} \cdot F(x), \tag{8}$$

where

$$F(x) \sim 1 + \frac{1}{12x} - \frac{1}{288x^2} - \frac{1}{51840x^3} + \cdots. \tag{9}$$

But we also know that

$$\ln F(x) \sim \sum_{k=1}^{\infty} f_k x^{-2k-1}, \tag{10}$$

where $f_k = B_{2k}/2k(2k-1)$ and B_k are the Bernoulli numbers. Thus by Theorem 1 we get that the Padé approximants of $F(x)$ are of type (4). Thus

$$F_1(x) = \frac{24x+1}{24x-1}, \qquad F_2(x) = \frac{8640x^2+360x+293}{8640x^2-360x+293}. \tag{11}$$

The next two numerator polynomials of $F_3(x)$ and $F_4(x)$ in integer coefficients are

$$\begin{aligned} P_3(x) &= 425295360x^3 + 17720640x^2 + 120170160x + 4406147, \\ P_4(x) &= 153494651842560x^4 + 6395610493440x^3 + 124448535691200x^2 \\ &\quad + 4968467473872x + 2749505046083. \end{aligned} \tag{12}$$

In Section 3 we shall present the integer coefficients of the polynomials $P_n(x)$ for $n = 1, \ldots, 5$, and in the tables presented at the end of this paper for $n = 1, \ldots, 10$.

For another application, consider

$$G(x) = \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}. \tag{13}$$

In this instance

$$G(x) \sim x^{-1/2}\left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} + \cdots\right). \tag{14}$$

Put

$$z = \frac{1}{x}, \qquad xG(x) = H(z). \tag{15}$$

Then $H(z)$ is of type (6). So

$$H(z) \sim x^{1/2}\frac{S_n(z)}{S_n(-z)},$$

$$\begin{aligned} S_1(z) &= 16 - z, \\ S_2(z) &= 256 - 16z + 11z^2, \\ S_3(z) &= 45056 - 2816z + 13280z^2 - 709z^3, \\ S_4(z) &= 46465024 - 2904064z + 38648064z^2 - 2290720z^3 + 1072189z^4. \end{aligned} \tag{16}$$

If we take $z = k$, k an integer, we get a remarkable approximation to the asymptotic series in the Wallis formula for large k:

$$\frac{1 \cdot 3 \ldots (2k-1)}{1 \cdot 2 \ldots 2k} \sim \frac{1}{\sqrt{\pi k}}\frac{S_n(k)}{S_n(-k)}. \tag{17}$$

The third example is the binomial function $((1-x)/(1+x))^\sigma$. This function satisfies condition (6). So Padé approximants exist of the form

$$\left(\frac{1-x}{1+x}\right)^\sigma \sim \frac{D_n(-x)}{D_n(x)}, \qquad n = 0, 1, 2, \ldots, \tag{18}$$

where

$$\begin{aligned} D_0(x) &= 1, \\ D_1(x) &= 1 + \sigma x, \\ D_2(x) &= 1 + \sigma x - \frac{1-\sigma^2}{3}x^2, \\ D_3(x) &= 1 + \sigma x - \frac{3-2\sigma^2}{5}x^2 - \frac{\sigma}{15}(4-\sigma^2)x^3, \\ D_4(x) &= 1 + \sigma x - \frac{6-3\sigma^2}{7}x^2 - \frac{(11-2\sigma^2)}{21}x^3 + \frac{(1-\sigma^2)(9-\sigma^2)}{105}x^4. \end{aligned} \tag{19}$$

The polynomials $D_n(x)$ satisfy the recurrence relation

$$D_{n+1}(x) = D_n(x) - \frac{n^2-\sigma^2}{4n^2-1}x^2 D_{n-1}(x), \qquad n = 1, 2, 3, \ldots. \tag{20}$$

The function $((1-ix)/(1+ix))^{i\nu} = \exp\{-2\nu \arctan x\}$ can be approximated in a similar manner. The Padé approximants also exist and are of the form

$$\left(\frac{1-ix}{1+ix}\right)^{i\nu} \sim \frac{C_n(-x)}{C_n(x)}, \tag{21}$$

$$
\begin{aligned}
&C_0(x) = 1,\\
&C_1(x) = 1 - \nu x,\\
&C_2(x) = 1 - \nu x + \frac{1+\nu^2}{2}x^2,\\
&C_3(x) = 1 - \nu x + \frac{3+2\nu^2}{5}x^2 - \frac{\nu}{15}(4+\nu^2)x^3,\\
&C_4(x) = 1 - \nu x + \frac{6+3\nu^2}{7}x^2 - \frac{\nu}{21}(11+2\nu^2)x^3 + \frac{(1+\nu^2)(9+\nu^2)}{105}x^4.
\end{aligned} \tag{22}
$$

The polynomials $C_n(x)$ satisfy the recurrence relation

$$
C_{n+1}(x) = C_n(x) + \frac{n^2+\nu^2}{4n^2-1}x^2 C_{n-1}(x), \qquad n = 1,2,3,\ldots. \tag{23}
$$

Another result is

THEOREM 2. *Let $f(x, a)$ be given by* (1) *and suppose that* (2) *and* (3) *hold where $P_n(x)$ is replaced by $P_n(x, a)$, etc. Further, suppose that*

$$
f(x, a)f(x, -a) = 1. \tag{24}
$$

Then

$$
R_n(x, a) = \frac{P_n(x, a)}{Q_n(x, a)}, \qquad Q_n(x, a) = P_n(x, -a). \tag{25}
$$

The proof is much like that for Theorem 1, and we omit details.

For an illustration of this theorem, we have

$$
(1+x)^\alpha = \frac{P_n(x, \alpha)}{P_n(x, -\alpha)} - \frac{\alpha \dfrac{(1-\alpha)_n(1+\alpha)_n}{2n!}\displaystyle\int_0^x \frac{(x-t)^n t^n}{(1+t)^{n+1-\alpha}}dt}{P_n(x, -\alpha)}, \tag{26}
$$

where

$$
P_n(x, \alpha) = {}_2F_1(-n, -n-\alpha; -2n; -x), \tag{27}
$$

a result given by Luke in [4, p. 170].

3. Numerical Results. We have computed the coefficients of the polynomial type Padé approximants for $F(x)$ in Stirling's formula for $\Gamma(z)$ in (8).

If the asymptotic series of $F(x)$ is

$$
F(x) \sim \sum_{i=0}^{\infty} c_i x^{-i}, \tag{28}
$$

the $p_i^{(n)}$ and $q_i^{(n)}$ coefficients of the Padé approximation of $F(x)$,

$$
F(x) \sim \frac{P_n(x)}{Q_n(x)} = \frac{P_n(x)}{P_n(-x)}, \tag{29}
$$

can be computed from the equations

$$(30) \qquad \sum_{i=1}^{n} c_{n+k-i} q_i^{(n)} = -c_{n+k}, \qquad k = 1,2,\ldots,n,$$

and

$$(31) \qquad p_i^{(n)} = \sum_{k=0}^{i} q_k^{(n)} c_{i-k}, \qquad q_0^{(n)} = 1, i = 0,1,\ldots,n.$$

Because in our case $p_i^{(n)} = (-1)^i q_i^{(n)}$, Eqs. (31) are used only for checking the correctness of the computation of $q_i^{(n)}$ by (30).

Equation (30) can be solved for $q_i^{(n)}$ by a simplified version of Trench's method [9].

For $n = 0$

$$(32) \qquad q_0^{(0)} = 1, \qquad \lambda_0 = 1.$$

For $n > 0$

$$(33) \qquad \begin{aligned} \lambda_n &= \sum_{i=0}^{n-1} c_{n+i} q_{n-1-i}^{(n-1)}, \qquad n = 1,2,\ldots, \\ q_0^{(n)} &= 1, \qquad q_1^{(n)} = -\frac{1}{24}, \\ q_i^{(n)} &= q_i^{(n-1)} - \frac{\lambda_n}{\lambda_{n-1}} q_{i-2}^{(n-2)}, \quad \text{for } i = 2,3,\ldots,n, n = 2,3,\ldots, \end{aligned}$$

where

$$(34) \qquad \begin{aligned} q_{n+1}^{(n)} &= 0, \\ q_j^{(k)} &= 0, \quad \text{if } j < 0 \text{ or } k < 0. \end{aligned}$$

In the numerical computation of the Padé coefficients we have made use of the fact that algebraic manipulation languages offer a facility for calculation with large integer and rational numbers ("bignum" arithmetics).

The $p_i^{(n)}$ rational coefficients were computed from Eqs. (32) and (33). The Padé approximant was then taken in the form

$$(35) \qquad F(x) \sim \frac{P_n(x)}{P_n(-x)} = \frac{P_n^*(x)}{P_n^*(-x)},$$

where $P_n^*(x)$ has *integer* coefficients, and we computed the $p_i^{*(n)}$'s by multiplying the $p_i^{(n)}$'s by their common denominators.

The numerical values of the c_i coefficients of the asymptotic series of $F(z)$, for $n = 1,\ldots,20$, were taken from the paper by J. W. Wrench [10].

The $p_i^{*(n)}$ integer coefficients are listed in Table 1 for $n = 1,\ldots,5$, and in the tables presented at the end of this paper for $n = 1,\ldots,10$.

TABLE 1

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 5)$

i	$p_i^{*(n)}, n = 1$
0	24
1	1

i	$p_i^{*(n)}, n = 2$
0	8640
1	360
2	293

i	$p_i^{*(n)}, n = 3$
0	4252 95360
1	177 20640
2	1201 70160
3	44 06147

i	$p_i^{*(n)}, n = 4$
0	15349 46518 42560
1	639 56104 93440
2	12444 85356 91200
3	496 84674 73872
4	274 95050 46083

i	$p_i^{*(n)}, n = 5$
0	11379 02961 26750 17531 39200
1	474 12623 38614 59063 80800
2	20671 39141 92943 96799 69280
3	845 22938 94009 92516 45760
4	3437 87425 22832 65242 23240
5	118 57916 88790 37605 99781

The programming was done in the REDUCE2 language for symbolic computing [3]. We have reprogrammed the algorithm in PL/I-FORMAC [1], [8] too and obtained the same results. Both versions computed the $p_i^{*(n)}$ coefficients recursively from (32) and (33).

A possibility was provided by the REDUCE2 facility for matrix calculations to do a further check by solving the linear equations (30).

The computation was carried out on the ES-1040 computer of the Central Research Institute for Physics, Budapest. (The ES-1040 has a 1 MByte memory and a processing speed of $\sim$ 380 thousand instruction/sec.)

The computation of the $p_i^{*(n)}$ coefficients took about 10 minutes of central processing unit time.

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Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$

i	$p_i^{*(n)}$, n = 1
0	24
1	1

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (*continued*)

i	$p_i^{*(n)}, n = 2$
0	8640
1	360
2	293

i	$p_i^{*(n)}, n = 3$
0	4252 95360
1	177 20640
2	1201 70160
3	44 06147

i	$p_i^{*(n)}, n = 4$
0	15349 46518 42560
1	639 56104 93440
2	12444 85356 91200
3	496 84674 73872
4	274 95050 46083

i	$p_i^{*(n)}, n = 5$
0	11379 02961 26750 17531 39200
1	474 12623 38614 59063 80800
2	20671 39141 92943 96799 69280
3	845 22938 94009 92516 45760
4	3437 87425 22832 65242 23240
5	118 57916 88790 37605 99781

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (continued)

i	$p_i^{*(n)}, n = 6$
0	15 31135 77051 07166 30501 78502 60553 72800
1	63797 32377 12798 59604 24104 27523 07200
2	51 11765 09760 31526 73375 05987 54239 48800
3	2 10826 71519 35358 75410 99293 16696 98560
4	23 51901 05663 44753 80119 88210 62350 13760
5	91384 25656 15255 03190 11710 17580 23640
6	41741 46161 72700 37964 16734 51741 93029

i	$p_i^{*(n)}, n = 7$
0	1 29355 29017 40411 82710 94233 14504 83807 01883 34645 24800
1	5389 80375 72517 15946 28926 38104 36825 29245 13943 55200
2	7 24460 96683 90226 51100 77492 52958 58508 14379 66090 24000
3	30003 09439 36543 28891 66854 38437 55911 24086 55695 25760
4	7 30244 32820 07543 91373 30571 25229 19149 02667 37828 71040
5	29454 82808 45615 89732 41737 38826 08430 55595 45429 15200
6	91928 61370 85936 78528 29568 14982 88558 09030 35852 54304

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (*continued*)

7	3049 16757 63003 94450 76807 28320 78171 02832 23960 34201

i	$p_i^{*(n)}, n = 8$
0	331 80846 90893 60927 32223 81557 12270 03743 60423 23596 01539 34071 33286 40000
1	13 82535 28787 23371 97175 99231 54677 91822 65017 63483 16730 80586 30553 60000
2	2859 08633 35038 69498 18030 20093 45514 30141 91115 09530 59729 11709 83510 01600
3	118 65975 13504 54522 14782 21141 06488 94889 70503 61890 09803 28194 87883 26400
4	5214 28031 96695 34421 16426 03394 76215 39965 40048 97355 95883 02798 13395 45600
5	213 35426 22466 26618 55429 49596 15812 59065 50903 67664 19137 02733 73586 48320
6	1773 04733 36056 16427 45549 52112 10349 85712 69267 36413 63684 42350 59266 89920
7	67 55167 73415 93757 64844 03338 35478 60485 02937 00602 02234 57634 90347 66880
8	27 28291 63531 26616 35315 75522 48668 92179 70880 77256 97658 79941 98033 27763

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (*continued*)

i	$p_i^{*(n)}$, n = 9
0	50 37574 45424 79881 71741 21851 85422 15201 11741 27602 87909 52432 99201 98373 81403 70994 68535 36563 20000
1	2 09898 93559 36661 73822 55077 16059 25633 37989 21983 45329 56351 37466 74932 24225 15458 11188 97356 80000
2	636 46447 42396 50180 58316 96682 40268 35837 87044 23098 54229 13915 82960 98739 66133 04724 35289 42387 20000
3	26 44817 20885 39886 93525 93757 87695 74617 02002 90297 95409 82962 81781 04142 49202 53781 42827 06288 64000
4	1925 15342 69264 19187 25703 74449 85608 12690 86644 52661 28973 25490 05334 63609 70244 91908 30998 34687 48800
5	79 33551 35252 81866 35120 71010 68326 01384 57556 95258 41650 01068 09173 62733 17012 12848 65775 92742 91200
6	1411 74820 26924 14792 97827 95734 90901 12156 90240 24252 56845 92962 87243 46409 73238 06669 90369 32670 77120
7	56 34167 59487 77673 38915 22941 42987 78206 36961 64499 08037 88618 49877 89860 11536 00785 80562 44690 18240

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (*continued*)

8	147 97626 35348 29576 85183 24439 84400 74837
	12063 72278 04263 29836 70109 60677 58352 90346
	43993 79043 58760
9	4 77081 43015 06694 89354 13965 88195 46639
	37058 85971 61613 70041 68247 82383 56868 38860
	94299 25414 87979

i	$p_i^{*(n)}, n = 10$
0	3 46580 88441 81447 12259 39718 25990 79665
	71114 82803 78351 32553 90681 17724 88053 79079
	48341 17814 91801 76982 36030 22683 43910 40000
1	14440 87018 40893 63010 80821 59416 28319
	40463 11783 49097 97189 74611 71571 87002 24128
	31180 88242 28825 07374 26501 25945 14329 60000
2	61 15426 73242 41720 79858 98288 67714 89551
	00348 84434 19491 47687 85851 77285 31678 98926
	94975 00123 61083 42781 27246 82307 66592 00000
3	2 54319 72785 98106 20993 87349 90702 36011
	66385 40398 58186 48448 03627 94045 38844 07169
	45677 66225 63766 69993 12481 94297 31860 48000
4	282 08654 97668 29581 80029 76531 74422 28462
	16086 72782 32994 92673 98896 62346 72512 61758
	41374 11674 69118 39173 94233 77234 60485 12000
5	11 66857 89021 71135 01275 87552 55205 43833
	31853 07089 48833 94491 09329 38395 62853 84224
	41554 91812 36934 38429 75149 42057 94918 40000

Integer coefficients of the polynomial type Padé approximants for $\Gamma(z)$ $(n = 1, 2, \ldots, 10)$ (*continued*)

6	370 02981 79422 05250 74478 33817 86363 76195
	22687 93218 12855 27151 22833 76624 67249 38042
	98545 16299 50457 67982 02781 03622 31587 43040
7	15 04270 22336 60452 59140 77817 11854 77847
	79460 61832 49698 68427 75379 03742 59192 81242
	77722 40264 99963 00763 68884 69370 14612 88960
8	102 97762 05123 30727 50949 22467 11914 53883
	76168 50444 82140 75329 76395 05441 24423 77573
	99650 53249 40801 72286 00048 09858 52584 20800
9	3 86371 82378 81242 14570 23922 21472 84694
	16779 80308 43903 39655 64180 29399 73174 65049
	87523 75284 70310 68830 02552 00007 63491 24808
10	1 42792 13772 55921 17305 18405 85080 40124
	53660 94330 97888 76281 73196 04891 86283 41663
	65876 76865 00663 37014 12048 24529 57198 12037