# A General Method of Approximation. Part I

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**Abstract.** In this paper we study two families of functions, viz. $F$ and $H$, and show how to approximate the functions considered in the interval $[0, 1]$. The functions are assumed to be real when the argument is real.

We define

$F = \{ f;$ (i) $f\left(\frac{1}{2} + x\right) = f\left(\frac{1}{2} - x\right)$, (ii) $f(0) = f(1) = 0$,

(iii) $f(x)$ is analytic in a sufficiently large neighborhood of $x = 0 \}$,

$H = \{ h;$ (j) $h\left(\frac{1}{2} + x\right) = -h\left(\frac{1}{2} - x\right)$, (jj) $h(0) = h(1) = 0$,

(jjj) $h(x)$ is analytic in a sufficiently large neighborhood of $x = 0 \}$.

The approximations are defined in the interval $[0, 1]$ by

$$\min \int_0^1 \left( f(x) - \sum_{n=1}^{k} c_{n,k} [x(1 - x)]^n \right)^2 x^q (1 - x)^q \, dx$$

and

$$\min \int_0^1 \left( h(x) - (1 - 2x) \sum_{n=1}^{k} C_{n,k} [x(1 - x)]^n \right)^2 x^q (1 - x)^q \, dx,$$

where $q \in \{0, 1, 2, \dots\}$.

The associated matrices are analyzed using the theory of orthogonal polynomials, especially the Jacobi polynomials $G_n(p, q, x)$. We apply the general theory to the basic trigonometric functions $\sin(x)$ and $\cos(x)$.

**Introduction.** This paper traces its origin from a wish to determine simple, accurate and rapid approximations of the basic trigonometric functions $\sin(x)$ and $\cos(x)$. We encountered this problem when repeatedly applying the Box-Müller transformation for generating bivariate normally distributed pseudo-random variates. But as is often the case when starting with an analysis of a special example one discovers an underlying more general pattern. The method found by us in [6], when approximating $\sin(x)$ and $\cos(x)$, could thus be applied to a much wider class of functions; see [7].

When measuring the "distance" between the functions and their approximations we use the $L_2$-norm. The required coefficients can then be determined from the resulting linear equation system.

The calculation of accurate values of the required coefficients is difficult. The associated Hankel matrices are as usual almost singular. We solve this problem by explicit calculation of the inverse matrices. The necessary numerical values of the associated integrals are determined using high-precision techniques.

Once recognized as "moment-matrices" the resulting Hankel matrices can be analyzed using the theory of orthogonal polynomials (especially the Jacobi-polynomials $G_n(p, q, x)$ with associated weight function $w(x) = (1 - x)^{p-q}x^{q-1}$, $x \in [0, 1]$). Empirically observed peculiarities can then be proved, and results can be generalized.

An even more thorough analysis of the methods used in [7] revealed that Bernstein polynomials were involved. That made it possible to further generalize the methods in [7]. The results thus found (see [8]) will be presented in part II.

The approximation method presented here consists of two parts. One can be solved once and for all (i.e., inversion of the associated matrices), the other part involves calculation of some integrals associated with the approximated functions.

**1. Certain General Expansions.**

a. *The Symmetric Case.* We state the main result of this section in the following

THEOREM 1. *Let $f(x)$ be a function with the following characteristics*:
(i) $f(\frac{1}{2} + x) = f(\frac{1}{2} - x)$, $f(x)$ *is real when $x$ is real*,
(ii) $f(0) = f(1) = 0$,
(iii) $f(x)$ *may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than* 1.
*Then $f(x)$ has an expansion of the form $f(x) = \sum_{m=1}^{\infty} a_m[x(1 - x)]^m$ valid at least in the interval $[0, 1]$. Expressions for the coefficient $a_m$ are given by Eqs.* (1.3), (1.6), *and* (1.9).

To prove Theorem 1 we expand $f(x)$ in a Taylor series around $x = \frac{1}{2}$, i.e.,

$$(1.1) \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(2n)}(\frac{1}{2})}{(2n)!} (x - \tfrac{1}{2})^{2n},$$

where we have

$$(x - \tfrac{1}{2})^{2n} = \frac{1}{4^n}[1 - 4x(1 - x)]^n = \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k}(-1)^k 4^k [x(1 - x)]^k.$$

Putting

$$(1.2) \qquad f(x) = \sum_{m=1}^{\infty} a_m[x(1 - x)]^m,$$

we get

$$(1.3) \qquad a_m = (-1)^m \sum_{v=m}^{\infty} \frac{f^{(2v)}(\frac{1}{2})}{(2v)!} \binom{v}{m} 4^{m-v}.$$

To proceed we also need the identity

$$(1.4) \qquad (1 - x)^n + x^n = \sum_{k=0}^{[n/2]} \frac{n}{n - k} \binom{n - k}{k}(-1)^k [x(1 - x)]^k.$$

(1.4) is easily proved using Lagrange's inversion formula; see [6, p. 13].
From (i) we conclude that
$$(1.5) \qquad f(x) + f(1 - x) = 2f(x).$$

Using (1.5) and the expansion in (1.4) we get

$$2f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (-1)^k [x(1-x)]^k.$$

An alternative expression of the coefficient $a_m$ is then

(1.6) $$a_m = \frac{1}{2} \sum_{n=2m}^{\infty} \frac{n}{n-m} \binom{n-m}{m} (-1)^m \frac{f^{(n)}(0)}{n!}.$$

Writing

$$f(x) = \sum_{v=1}^{\infty} \frac{f^{(v)}(0)}{v!} x^v = \sum_{m=1}^{\infty} a_m [x(1-x)]^m$$

and identifying corresponding powers of $x$, we get the equation system

(1.7) $$\frac{f^{(n)}(0)}{n!} = \sum_{v=0}^{[n/2]} \binom{n-v}{n-2v} (-1)^v a_{n-v}, \qquad n = 1, 2, 3, \ldots .$$

The solution of this equation system may be obtained using Lagrange's inversion formula. Thus

(1.8) $$a_m = \frac{1}{m!} \left\{ \frac{d^{m-1}}{dx^{m-1}} \left[ \frac{\dot{f}(x)}{(1-x)^m} \right] \right\}_{x=0},$$

which may be written

(1.9) $$a_m = \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{(m+v-1)!}{m!(m-1)!} f^{(m-v)}(0).$$

Now suppose that $g(x)$ is any analytic function satisfying only the condition (iii). Then we may put $f(x) = g(x - \frac{1}{2}) + g(\frac{1}{2} - x) - g(-\frac{1}{2}) - g(\frac{1}{2})$. We notice that $f(x)$ is symmetric around $x = \frac{1}{2}$, and $f(0) = f(1) = 0$.

*Example*. We may expand

(1.10) $$e^{t(x-1/2)} + e^{-t(x-1/2)} = e^{t/2} + e^{-t/2} + \sum_{m=1}^{\infty} a_m [x(1-x)]^m,$$

where the coefficients $a_m$ satisfy the recurrence relation

(1.11) $$(n+2)(n+1)a_{n+2} = (2n+2)(2n+1)a_{n+1} + t^2 a_n,$$

with starting values $a_0 = 2\cosh(\frac{1}{2}t)$, $a_1 = -2t\sinh(\frac{1}{2}t)$. $\quad\square$

Another more general possibility is to consider

$$f(x) = G(g(x), g(1-x)) - G(g(0), g(1)),$$

where $G(x, y)$ is any regular and symmetric function (i.e. $G(x, y) = G(y, x)$). Evidently we have $f(\frac{1}{2} + x) = f(\frac{1}{2} - x)$ and an expansion of the form

(1.12) $$f(x) = \sum_{m=1}^{\infty} a_m [x(1-x)]^m.$$

Suppose that $g(x)$ satisfies a functional relation of the form $g(1-x) = H(g(x), F(x))$ where $H(x, y)$ and $F(x)$ are "simple" functions. Approximating in

the $L_2$-norm, as indicated in Section 2a, we get a relation of the form

$$G(g(x), H(g(x), F(x))) - G(g(0), g(1)) \approx \sum_{n=1}^{k} c_{n,k}[x(1-x)]^n.$$

If $G(x, y)$ and $H(x, y)$ are simple functions, we may sometimes solve the "identity" with respect to $g(x)$, which enables us to calculate $g(x)$ even if the function does not satisfy the conditions (i) and (ii).

*Example.* We put $G(x, y) = x + y$ and $g(x) = 1/\Gamma(1 + x)$. Then $g(x)$ satisfies the relation $g(x)g(1 - x) = \sin(\pi x)/\pi x(1 - x)$.

Approximating in the $L_2$-norm we get a relation of the form

(1.13) $\quad \dfrac{1}{\Gamma(1 + x)} + \dfrac{\Gamma(1 + x)\sin(\pi x)}{\pi x(1 - x)} \approx 2 + \sum_{n=1}^{k} c_{n,k}[x(1 - x)]^n.$ $\quad \square$

b. *The Antisymmetric Case.* The theorem corresponding to Theorem 1 is

THEOREM 2. *Let $h(x)$ be a function satisfying the conditions*
(j) $h(\frac{1}{2} + x) = -h(\frac{1}{2} - x)$, $h(x)$ *is real when $x$ is real,*
(jj) $h(0) = h(1) = 0$,
(jjj) $h(x)$ *may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than 1.*
*Then $h(x)$ has an expansion of the form $h(x) = (1 - 2x)\sum_{m=1}^{\infty} b_m[x(1 - x)]^m$ valid at least in the interval $[0, 1]$. Different expressions for the coefficients $b_m$ are given by Eqs. (1.16) and (1.19).*

To prove Theorem 2 we expand $h(x)$ in a Taylor series around $x = \frac{1}{2}$, i.e.,

(1.14) $\qquad\qquad h(x) = \sum_{n=0}^{\infty} \dfrac{h^{(2n+1)}(\frac{1}{2})}{(2n+1)!}(x - \frac{1}{2})^{2n+1},$

where we have

$$(x - \tfrac{1}{2})^{2n+1} = -\tfrac{1}{2}(1 - 2x)\dfrac{1}{4^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k 4^k[x(1 - x)]^k.$$

Putting

(1.15) $\qquad\qquad h(x) = (1 - 2x)\sum_{m=1}^{\infty} b_m[x(1 - x)]^m,$

we get

(1.16) $\qquad\qquad b_m = -\tfrac{1}{2}(-1)^m \sum_{v=m}^{\infty} \dfrac{h^{(2v+1)}(\frac{1}{2})}{(2v+1)!}\binom{v}{m}4^{m-v}.$

To proceed we will need an identity similar to (1.4), viz.

(1.17) $\quad (1 - x)^n - x^n = (1 - 2x)\sum_{k=0}^{[(n-1)/2]}\binom{n - k - 1}{k}(-1)^k[x(1 - x)]^k.$

(1.17) follows easily from (1.4) when differentiating with respect to $x$.

From (j) we conclude that

(1.18) $\qquad\qquad h(x) - h(1 - x) = 2h(x).$

Using (jjj) and (1.18) we get

$$2h(x) = -\sum_{n=1}^{\infty} \frac{h^{(n)}(0)}{n!}(1-2x)^{[(n-1)/2]} \sum_{k=0}^{\infty} \binom{n-k-1}{k}(-1)^k[x(1-x)]^k.$$

The coefficient $b_m$ in the expansion (1.15) is then given by

$$(1.19) \qquad b_m = (-1)^{m+1}\frac{1}{2}\sum_{n=2m+1}^{\infty}\frac{h^{(n)}(0)}{n!}\binom{n-m-1}{m}.$$

Multiplying (1.15) with $(1-2x)$ we get, using Lagrange's inversion formula,

$$(1.20) \qquad b_m - 4b_{m-1} = \frac{1}{m!}\left\{\frac{d^{m-1}}{dx^{m-1}}\left[\frac{(1-2x)\dot{h}(x) - 2h(x)}{(1-x)^m}\right]\right\}_{x=0}.$$

When the function $g(x)$ does not satisfy the conditions (j) and (jj) we may put $h(x) = g(x - \frac{1}{2}) - g(-x + \frac{1}{2}) + (1-2x)[g(\frac{1}{2}) - g(-\frac{1}{2})]$. Then $h(x)$ is antisymmetric around $x = \frac{1}{2}$ and $h(0) = h(\frac{1}{2}) = h(1) = 0$.

If (jj) is not satisfied, then we have $h(0) = -h(1) \neq 0$ (which is the case for $h(x) = \cos(\pi x)$). We may avoid this difficulty by substituting $h_1(x) = h(x) - (1-2x)h(0)$.

Finally let

$$h(x) = G(g(x), g(1-x)) - (1-2x)G(g(0), g(1)),$$

where $G(x, y)$ is a regular, antisymmetric function of $x$ and $y$. Assuming that $G(x, y)$ is "regular enough", then $h(x)$ satisfies the conditions (j), (jj), and (jjj). All the remarks occurring in the end of Section 1a are relevant also in this case.

c. *The General Case.* From a study of the proofs of Theorems 1 and 2 it is obvious that we may formulate

THEOREM 3. *Let $g(x)$ be a function satisfying the conditions*
(k) *$g(x)$ is neither symmetric nor antisymmetric but real when $x$ is real,*
(kk) *$g(0) = g(1) = 0$,*
(kkk) *$g(x)$ may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than 1.*
*Then $g(x)$ has an expansion of the form*

$$g(x) = \sum_{m=1}^{\infty} a_m[x(1-x)]^m + (1-2x)\sum_{m=1}^{\infty} b_m[x(1-x)]^m$$

*valid at least in the interval $[0, 1]$. Expressions for the coefficients $a_m$ and $b_m$ are given by obvious generalizations of Eqs. (1.3) and (1.16).*

When $g(x)$ does not satisfy the condition (kk) we simply substitute

$$g_1(x) = g(x) - \left[\frac{1}{2}(g(0) + g(1)) + \frac{1}{2}(g(0) - g(1))(1-2x)\right].$$

## 2. Approximation in the $L_2$-norm.

a. *The Symmetric Case.* Let $f(x)$ be a function which satisfies the conditions (i), (ii), and (iii) in Section 1a. In many cases already the truncated series in Eq. (1.2) may be used for computational purposes. We may however improve this result by

approximating in the $L_2$-norm, i.e., we consider min $D_1^2(w)$, where

$$(2.1) \qquad D_1^2(w) = \int_0^1 \left\{ f(x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right\}^2 x^q(1-x)^q \, dx$$

and $q \in \{0, 1, 2, \ldots\}$.

The resulting linear equation system yielding the coefficients $c_{n,k}$ will be (with $r = 1, 2, \ldots, k$)

$$(2.2) \qquad \sum_{n=1}^k c_{n,k} u_{n,r+q} = \int_0^1 f(x) [x(1-x)]^{r+q} \, dx,$$

where

$$(2.3) \qquad u_{i,j} = \int_0^1 [x(1-x)]^{i+j} \, dx = \frac{1}{(2i+2j+1)\binom{2i+2j}{i+j}}.$$

The inversion of the matrices $\mathbf{D}_{n,s} = \{u_{i,j}\}$ (with $i = 0, 1, 2, \ldots, n$; $j = s, s+1, \ldots, s+n$; $s \in \{0, 1, 2, \ldots\}$) will be treated in Sections 4 and 5.

A remaining problem, with many solutions, is how to determine numerical high precision values of the integrals

$$(2.4) \qquad I_j = \int_0^1 f(x)[x(1-x)]^j \, dx; \qquad j = 1, 2, 3, \ldots.$$

That problem will be treated in Section 3.

b. *The Antisymmetric Case.* Let $h(x)$ be a function satisfying the conditions (j), (jj), and (jjj) in Section 1b. We consider min $D_2^2(w)$, where

$$(2.5) \quad D_2^2(w) = \int_0^1 \left\{ h(x) - (1-2x) \sum_{n=1}^k C_{n,k} [x(1-x)]^n \right\}^2 x^q(1-x)^q \, dx$$

and $q \in \{0, 1, 2, \ldots\}$.

The resulting linear equation system yielding the coefficients $C_{n,k}$ will be (with $r = 1, 2, \ldots, k$)

$$(2.6) \qquad \sum_{n=1}^k C_{n,k} U_{n,r+q} = \int_0^1 h(x)(1-2x)[x(1-x)]^{r+q} \, dx,$$

where

$$U_{i,j} = \int_0^1 (1-2x)^2 [x(1-x)]^{i+j} \, dx$$

$$(2.7) \qquad = \frac{1}{(2i+2j+3)(2i+2j+1)\binom{2i+2j}{i+j}}.$$

The inversion of the matrices $\mathbf{E}_{n,t} = \{U_{i,j}\}$ (with $i = 0, 1, 2, \ldots, n$; $j = t, t+1, \ldots, t+n$; $t \in \{0, 1, 2, \ldots\}$) will be treated in Sections 4 and 5.

In Section 3 we will consider different techniques to determine numerical values of the integrals

$$(2.8) \qquad I_j' = \int_0^1 h(x)(1-2x)[x(1-x)]^j \, dx, \qquad j = 1, 2, \ldots.$$

c. *The General Case.* Let the function $g(x)$ satisfy the conditions (k), (kk), and (kkk) in Section 1c and consider min $D_3^2(w)$, where $q \in \{0, 1, 2, \ldots\}$ and

$$
\begin{aligned}
&D_3^2(w) \\
(2.9) \quad &= \int_0^1 \left\{ g(x) - \sum_{n=1}^{k} (c'_{n,k} + (1 - 2x)C'_{n,k})[x(1 - x)]^n \right\}^2 x^q(1 - x)^q \, dx.
\end{aligned}
$$

The resulting equation systems yielding the coefficients $c'_{n,k}$ and $C'_{n,k}$ will be (with $r = 1, 2, \ldots, k$)

$$
(2.10) \qquad \sum_{n=1}^{k} c'_{n,k} u_{n,r+q} = \int_0^1 g(x)[x(1 - x)]^{r+q} \, dx
$$

and

$$
(2.11) \qquad \sum_{n=1}^{k} C'_{n,k} U_{n,r+q} = \int_0^1 g(x)(1 - 2x)[x(1 - x)]^{r+q} \, dx.
$$

The equations (2.10) and (2.11) will be considered in detail in part II in connection with Bernstein polynomials.

d. *Approximations in $L_2$ With Certain Restrictions.* In many cases it is natural to consider min $D_1^2(w)$, min $D_2^2(w)$, and min $D_3^2(w)$ given certain restrictions. We may, e.g., consider

$$
(2.12) \quad
\begin{cases}
\min \displaystyle\int_0^1 \left\{ f(x) - \sum_{n=1}^{k} c_{n,k}[x(1 - x)]^n \right\}^2 x^q(1 - x)^q \, dx \\[2ex]
\displaystyle\sum_{n=1}^{k} c_{n,k} \frac{1}{4^n} = f\left(\tfrac{1}{2}\right).
\end{cases}
$$

When $f(x)$ is a probability density function it is natural to consider

$$
(2.13) \quad
\begin{cases}
\min \displaystyle\int_0^1 \left\{ f(x) - \sum_{n=1}^{k} c_{n,k}[x(1 - x)]^n \right\}^2 x^q(1 - x)^q \, dx \\[2ex]
\displaystyle\sum_{n=1}^{k} c_{n,k} \frac{1}{(2n + 1)\binom{2n}{n}} = 1.
\end{cases}
$$

We will consider (2.12) and (2.13) in part II, when approximating $f(x) = \sin(\pi x)$ with restrictions.

Another natural approximation technique would be to equalize all the moments. We then get the equation system

$$
(2.14) \quad \int_0^1 x^j f(x) \, dx = \sum_{n=1}^{k} c_{n,k} \frac{1}{(2n + j + 1)\binom{2n + j}{n}}; \qquad j = 0, 1, 2, \ldots, k - 1.
$$

**3. Calculation of the Associated Integrals.** We will consider different techniques to determine numerical values of the associated integrals occurring in (2.4) and (2.8). We start with

a. *The Symmetric Case.* In some cases we may use a recursion formula for $I_j$ or a closed expression (See Application in Section 6, (6.22).) When it is easy to determine numerically the coefficients $a_m$ defined by (1.2), (e.g., by using a recursion formula) then

$$(3.1) \qquad I_j = \sum_{m=1}^{\infty} a_m \frac{1}{(2m + 2j + 1)\begin{pmatrix} 2m + 2j \\ m + j \end{pmatrix}}.$$

Expanding $f(x)$ in a Taylor series around $x = 0$, we get

$$(3.2) \qquad I_j = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{(n + 2j + 1)\begin{pmatrix} n + 2j \\ j \end{pmatrix}}.$$

Putting $u = x(1 - x)$ and making use of the symmetry, we get

$$(3.3) \qquad I_j = \frac{1}{2^{2j+1}} \int_0^1 f\left(\tfrac{1}{2} - \tfrac{1}{2}(1 - u)^{1/2}\right) \frac{u^j}{(1 - u)^{1/2}} \, du.$$

Making an obvious expansion of $f(\tfrac{1}{2} - \tfrac{1}{2}(1 - u)^{1/2})$ in (3.3), we obtain

$$(3.4) \qquad I_j = \sum_{n=0}^{\infty} \frac{f^{(2n)}\left(\tfrac{1}{2}\right)}{(2n)!} \cdot \frac{\begin{pmatrix} 2n \\ n \end{pmatrix}}{2^{2n-1}\begin{pmatrix} 2n + 2j + 2 \\ n + j + 1 \end{pmatrix}\begin{pmatrix} n + j + 1 \\ n + 1 \end{pmatrix}(n + 1)}.$$

In some cases we may also use a Fourier expansion. For reasons of symmetry we must have

$$(3.5) \qquad f(x) = \beta_0 + \sum_{n=1}^{\infty} \left[ \alpha_n \sin((2n - 1)\pi x) + \beta_n \cos(2n\pi x) \right],$$

where $\beta_0 + \sum_{n=1}^{\infty} \beta_n = 0$ and $x \in [0, 1]$.

The expansion in (3.5) is of special importance when $\alpha_n$ and $\beta_n$ converge fast towards zero. That is, e.g., the case when $f(x) = \mathrm{sn}(2K(k)x, k)$; see [7, p. 25].

To calculate $I_j$, making use of the expansion (3.5), we must know how to calculate the integrals

$$(3.6) \qquad S_{2n,j} = \int_0^1 \cos(2n\pi x)[x(1 - x)]^j \, dx$$

and

$$(3.7) \qquad T_{2n-1,j} = \int_0^1 \sin((2n - 1)\pi x)[x(1 - x)]^j \, dx.$$

However $S_{2n,j}$ and $T_{2n-1,j}$ satisfy the recurrence relations

$$(3.8) \qquad S_{2n,j} = \frac{2j(2j - 1)}{(2n\pi)^2} S_{2n,j-1} - \frac{j(j - 1)}{(2n\pi)^2} S_{2n,j-2}$$

and

(3.9)        $$T_{2n-1,j} = \frac{2\,j(2\,j-1)}{(2n-1)^2\pi^2}\,T_{2n-1,j-1} - \frac{j(j-1)}{(2n-1)^2\pi^2}\,T_{2n-1,j-2}.$$

Starting values are given by

$$S_{2n,0} = 0, \qquad S_{2n,1} = -\frac{2}{(2n\pi)^2},$$

$$T_{2n-1,0} = \frac{2}{(2n-1)\pi}, \qquad T_{2n-1,1} = \frac{4}{(2n-1)^3\pi^3}.$$

c. *The Antisymmetric Case.* We consider next the integrals $I_j'$ defined by (2.8). When it is easy to calculate the coefficients $b_m$ in the expansion (1.15) we easily get

(3.10)        $$I_j' = \sum_{m=1}^{\infty} b_m \frac{1}{(2m+2j+3)(2m+2j+1)\binom{2m+2j}{m+j}}.$$

But we may also use the expansion in (1.14). After a routine calculation we get

(3.11)    $$I_j' = -\sum_{n=0}^{\infty} \frac{h^{(2n+1)}\left(\frac{1}{2}\right)}{(2n+1)!} \cdot \frac{\binom{2n+2}{n+1}}{2^{2n}(n+2)\binom{2n+2j+4}{n+j+2}\binom{n+j+2}{j}}.$$

Expanding $h(x)$ in a Taylor series around $x = 0$, we obtain

(3.12)    $$I_j' = \sum_{n=1}^{\infty} \frac{h^{(n)}(0)}{n!\,(n+2j+2)} \left[ \frac{1}{\binom{n+2j+1}{j+1}} - \frac{1}{\binom{n+2j+1}{j}} \right].$$

Even in this case we may sometimes use a Fourier series. For reasons of asymmetry we must have

(3.13)        $$h(x) = \sum_{n=1}^{\infty} \left[ \gamma_n \sin(2n\pi x) + \delta_n \cos((2n-1)\pi x) \right],$$

where $\sum_{n=1}^{\infty} \delta_n = 0$ and $x \in [0, 1]$.

To calculate $I_j'$, using the expansion (3.13), we must be able to calculate

(3.14)        $$Q_{2n-1,j} = \int_0^1 (1-2x)\cos((2n-1)\pi x)[x(1-x)]^j\,dx$$

and

(3.15)        $$R_{2n,j} = \int_0^1 (1-2x)\sin(2n\pi x)[x(1-x)]^j\,dx.$$

However, $Q_{2n-1,j}$ and $R_{2n,j}$ satisfy the recurrence relations

(3.16)        $$Q_{2n-1,j} = \frac{2\,j(2\,j+1)}{(2n-1)^2\pi^2}\,Q_{2n-1,j-1} - \frac{j(j-1)}{(2n-1)^2\pi^2}\,Q_{2n-1,j-2}$$

and

$$(3.17) \qquad R_{2n,j} = \frac{2\,j(2\,j+1)}{(2n\pi)^2} R_{2n,j-1} - \frac{j(j-1)}{(2n\pi)^2} R_{2n,j-2}.$$

Starting values are given by

$$Q_{2n-1,0} = \frac{4}{(2n-1)^2\pi^2}, \qquad Q_{2n-1,1} = \frac{24}{(2n-1)^4\pi^4} - \frac{2}{(2n-1)^2\pi^2},$$

$$R_{2n,0} = \frac{1}{n\pi}, \qquad R_{2n,1} = \frac{3}{2n^3\pi^3}.$$

**4. Analysis of the Associated Matrices and Determinants.** We now turn to the most difficult section of this paper, namely the analysis of the associated Hankel matrices occurring in (2.2) and (2.6). A well-known and well-analyzed example of a Hankel matrix is the Hilbert matrix $\mathbf{H}_n$ with general element $h_{jk} = (j + k - 1)^{-1}$ ($j, k = 1, 2, \ldots, n$). Both the determinant $\det \mathbf{H}_n$ and the inverse matrix $\mathbf{H}_n^{-1}$ are exactly known. (See Savage and Lukacs [5].) The Hilbert matrix has a settled bad reputation in regard to numerical difficulties. The matrices occurring in (2.2) and (2.6) are no better.

Consider the finite square Hankel matrix $\mathbf{D}_{n,s}$ of order $(n + 1)$ defined by

$$(4.1) \qquad \mathbf{D}_{n,s} = \begin{vmatrix} u_{0,s} & u_{0,s+1} & \cdots & u_{0,s+n} \\ u_{1,s} & u_{1,s+1} & \cdots & u_{1,s+n} \\ u_{2,s} & u_{2,s+1} & \cdots & u_{2,s+n} \\ \vdots & \vdots & & \vdots \\ u_{n,s} & u_{n,s+1} & \cdots & u_{n,s+n} \end{vmatrix} ; \qquad s \in \{0, 1, 2, \ldots\},$$

where

$$u_{i,j} = \int_0^1 [x(1-x)]^{i+j}\,dx = \frac{1}{(2i+2j+1)\dbinom{2i+2j}{i+j}}.$$

We wish to calculate explicitly $\mathbf{D}_{n,s}^{-1} = \{d_{j,k}\}$ ($j, k = 0, 1, \ldots, n$) and $D_{n,s} = \mathrm{Det}(\mathbf{D}_{n,s})$. This may be done noting that the Hankel matrix $\mathbf{D}_{n,s}$ is also a moment matrix. To see this we rewrite the coefficients $u_{i,j}$ in the following way, putting $x(1-x) = z$,

$$
\begin{aligned}
u_{i,j} &= \int_0^1 [x(1-x)]^{i+j}\,dx = 2\int_0^{1/2} [x(1-x)]^{i+j}\,dx \\
&= \int_0^{1/4} \frac{z^{i+j-s}z^s}{\left(\frac{1}{4}-z\right)^{1/2}}\,dz.
\end{aligned}
$$
(4.2)

The coefficients $u_{i,j}$ may thus be interpreted as moments belonging to the weight function $w(z) = z^s(\frac{1}{4} - z)^{-1/2}$; $z \in [0, \frac{1}{4}]$, $i + j \geqslant s$.

Let $Q_{n,s}(x)$ and $q_{n,s}(x)$ be the associated orthogonal and orthonormal polynomials defined by the determinants (see, e.g., Cramér [3, p. 132])

$$(4.3) \qquad Q_{n,s}(x) = \begin{vmatrix} u_{0,s} & u_{0,s+1} & \cdots & u_{0,s+n} \\ u_{1,s} & u_{1,s+1} & \cdots & u_{1,s+n} \\ \vdots & & & \vdots \\ u_{n-1,s} & & \cdots & u_{n-1,s+n} \\ 1 & x & & x^n \end{vmatrix},$$

$$(4.4) \qquad q_{n,s}(x) = \frac{Q_{n,s}(x)}{\left(D_{n,s} D_{n-1,s}\right)^{1/2}}; \qquad D_{n,s} = \mathrm{Det}(\mathbf{D}_{n,s}).$$

The polynomials $Q_{n,s}(x)$ and $q_{n,s}(x)$ satisfy the following equations, where $i \neq j$,

$$(4.5) \quad 0 = \int_0^{1/4} Q_{i,s}(x) Q_{j,s}(x) \frac{x^s}{\left(\frac{1}{4} - x\right)^{1/2}} dx = \int_0^1 Q_{i,s}\left(\frac{x}{4}\right) Q_{j,s}\left(\frac{x}{4}\right) \frac{x^s}{(1-x)^{1/2}} dx.$$

$$(4.6) \quad 1 = \int_0^{1/4} q_{i,s}^2(x) \frac{x^s}{\left(\frac{1}{4} - x\right)^{1/2}} dx = \frac{1}{2^{2s+1}} \int_0^1 q_{i,s}^2\left(\frac{x}{4}\right) \frac{x^s}{(1-x)^{1/2}} dx.$$

The orthogonal polynomials associated with the weight function $w(x) = (1-x)^{p-q} x^{q-1}$ are known as Jacobi polynomials and are denoted by $G_n(p, q, x)$. They may be standardized so that the coefficients of $x^n$ equal 1 [1, p. 774]. The explicit expression is then

$$(4.7) \quad G_n(p, q, x) = \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)} x^{n-m}.$$

The normalizing constant $h_n$ is in this case determined by

$$(4.8) \quad h_n = \int_0^1 G_n^2(p, q, x) w(x)\, dx = \frac{n! \Gamma(n+q) \Gamma(n+p) \Gamma(n+p-q+1)}{(2n+p) \Gamma^2(2n+p)}.$$

Using (4.5) and (4.6), we may identify the polynomials and get

$$(4.9) \qquad G_n\left(s + \tfrac{1}{2}, s+1, x\right) = \frac{4^n}{D_{n-1,s}} Q_{n,s}\left(\frac{x}{4}\right),$$

$$(4.10) \quad \frac{1}{2^{s+1/2}} q_{n,s}\left(\frac{x}{4}\right) = \frac{Q_{n,s}(x/4)}{2^{s+1/2}\left(D_{n,s} D_{n-1,s}\right)^{1/2}} = \frac{1}{h_n^{1/2}} G_n\left(s + \tfrac{1}{2}, s+1, x\right).$$

An identification of the coefficients of $x^n$ in (4.10) yields

$$(4.11) \qquad \frac{1}{2^{2s+1} 4^{2n}} \frac{D_{n-1,s}}{D_{n,s}} = \frac{\left(2n + s + \tfrac{1}{2}\right) \Gamma^2\left(2n + s + \tfrac{1}{2}\right)}{n! \Gamma(n+s+1) \Gamma\left(n+s+\tfrac{1}{2}\right) \Gamma\left(n+\tfrac{1}{2}\right)}.$$

We note the special case $s = 2$, which corresponds to $q = 0$ in (2.1) ($w(x) = x^q(1-x)^q = 1$). We get, after a little algebra,

$$(4.12) \qquad D_{n,2} = D_{n-1,2} \frac{1}{(2n+3)\binom{4n+5}{2n+2}\binom{4n+4}{2n}},$$

i.e.,

$$(4.13) \qquad D_{n,2}^{-1} = \prod_{r=0}^{n} (2r+3)\binom{4r+5}{2r+2}\binom{4r+4}{2r}.$$

We will now show how to find explicit expressions of the elements of $\mathbf{D}_{n,s}^{-1}$. Of special importance is the case $s = 2$, which we will use to find approximations of $\sin(\pi x)$ when $x \in [0, 1]$; see Section 6.

Some of the elements of $\mathbf{D}_{n,s}^{-1}$ may be obtained quite easily. Let $D_{n,s}^{i,j}$ be the first minor obtained by deleting the $i$th row and the $j$th column of the determinant $D_{n,s}$. We expand the polynomial $Q_{n,s}(x)$ (defined by the determinant in (4.3)) with respect to the last row, i.e., with respect to powers of $x$. Then

$$(4.14) \qquad Q_{n,s}(x) = \sum_{v=0}^{n} (-1)^{n+v} D_{n,s}^{n+1,v+1} x^v; \qquad D_{n,s}^{n+1,n+1} = D_{n-1,s}.$$

Using (4.9) and (4.11), we get, upon identifying corresponding powers of $x$,

$$(4.15) \quad \frac{D_{n,s}^{n+1,k+1}}{D_{n,s}} = \frac{(4n+2s+1)4^{s+k+n}\Gamma(2n+s+\frac{1}{2})\Gamma(s+\frac{1}{2}+n+k)}{n!\Gamma(n+s+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(s+1+k)}\binom{n}{k}.$$

Now define

$$(4.16) \qquad \begin{vmatrix} q_{0,s}(x) \\ q_{1,s}(x) \\ q_{2,s}(x) \\ \vdots \\ q_{n,s}(x) \end{vmatrix} = \begin{vmatrix} z_{00} & 0 & 0 & \cdots & 0 \\ z_{10} & z_{11} & 0 & \cdots & 0 \\ z_{20} & z_{21} & z_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ z_{n0} & z_{n1} & z_{n2} & \cdots & z_{nn} \end{vmatrix} \begin{vmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{vmatrix}.$$

Putting

$$(4.17) \qquad \mathbf{L}_{n,s}^{-1} = \{z_{ij}\},$$

we may use the Choleski factorization [6, p. 8] and get the important relation

$$(4.18) \qquad \mathbf{D}_{n,s}^{-1} = \mathbf{L}_{n,s}^{-T}\mathbf{L}_{n,s}^{-1}.$$

Let the general element of $\mathbf{D}_{n,s}^{-1}$ be $d_{j,k}$ ($j, k = 0, 1, 2, \ldots, n$). Then

$$(4.19) \qquad d_{j,k} = \sum_{r=\max(j,k)}^{n} z_{rj} z_{rk}.$$

We therefore have to determine the elements $z_{rk}$. This may be done using (4.4) and (4.15). After some calculation we get

$$(4.20) \quad z_{nk} = \left(\frac{n!\Gamma(n+s+1)(4n+2s+1)}{\Gamma(n+s+\frac{1}{2})\Gamma(n+\frac{1}{2})}\right)^{1/2} \frac{2^{2k+s}\Gamma(s+\frac{1}{2}+n+k)}{\Gamma(s+1+k)n!}\binom{n}{k}(-1)^{n+k}.$$

Inserted in (4.19), this yields

$$(4.21) \quad \begin{aligned} d_{j,k} = &\sum_{r=\max(j,k)}^{n} \frac{\Gamma(r+s+1)(4r+2s+1)4^{k+j+s}\Gamma(s+\frac{1}{2}+r+k)\Gamma(s+\frac{1}{2}+r+j)}{\Gamma(r+s+\frac{1}{2})\Gamma(r+\frac{1}{2})\Gamma(s+1+k)\Gamma(s+1+j)r!} \\ &\times \binom{r}{k}\binom{r}{j}(-1)^{j+k}. \end{aligned}$$
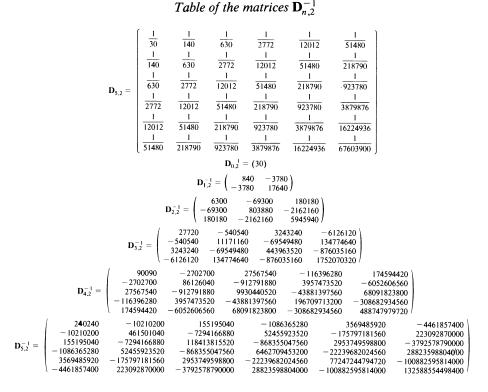
For $s = 2$ the elements $d_{j,k}$ (and $d_{k,j}$) are given by

$$d_{j,k} = (-1)^{j+k} \sum_{r=\max(j,k)}^{n} (4r + 5)$$

(4.22)
$$\times \frac{\binom{r+k+2}{k+2}\binom{r+j+2}{j+2}\binom{2r+2k+4}{r+k+2}\binom{2r+2j+4}{r+j+2}\binom{r}{k}\binom{r}{j}}{\binom{2r}{r}\binom{2r+4}{r+2}}.$$

(Accurate values of $D_{n,2}^{-1}$ ($n = 0, 1, \ldots, 6$) and $\mathbf{D}_{n,2}^{-1}$ ($n = 0, 1, \ldots, 9$) are given in [7, pp. 30–34].) In Table I we give the matrices $\mathbf{D}_{n,2}^{-1}$ for $n = 0, 1, \ldots, 5$.

<div align="center">

TABLE I

*Table of the matrices $\mathbf{D}_{n,2}^{-1}$*

</div>

$$\mathbf{D}_{5,2} = \begin{pmatrix} \frac{1}{30} & \frac{1}{140} & \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} \\ \frac{1}{140} & \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} \\ \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} \\ \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} \\ \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} & \frac{1}{16224936} \\ \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} & \frac{1}{16224936} & \frac{1}{67603900} \end{pmatrix}$$

$$\mathbf{D}_{0,2}^{-1} = (30)$$

$$\mathbf{D}_{1,2}^{-1} = \begin{pmatrix} 840 & -3780 \\ -3780 & 17640 \end{pmatrix}$$

$$\mathbf{D}_{2,2}^{-1} = \begin{pmatrix} 6300 & -69300 & 180180 \\ -69300 & 803880 & -2162160 \\ 180180 & -2162160 & 5945940 \end{pmatrix}$$

$$\mathbf{D}_{3,2}^{-1} = \begin{pmatrix} 27720 & -540540 & 3243240 & -6126120 \\ -540540 & 11171160 & -69549480 & 134774640 \\ 3243240 & -69549480 & 443963520 & -876035160 \\ -6126120 & 134774640 & -876035160 & 1752070320 \end{pmatrix}$$

$$\mathbf{D}_{4,2}^{-1} = \begin{pmatrix} 90090 & -2702700 & 27567540 & -116396280 & 174594420 \\ -2702700 & 86126040 & -912791880 & 3957473520 & -6052606560 \\ 27567540 & -912791880 & 9930440520 & -43881397560 & 68091823800 \\ -116396280 & 3957473520 & -43881397560 & 196709713200 & -308682934560 \\ 174594420 & -6052606560 & 68091823800 & -308682934560 & 488747979720 \end{pmatrix}$$

$$\mathbf{D}_{5,2}^{-1} = \begin{pmatrix} 240240 & -10210200 & 155195040 & -1086365280 & 3569485920 & -4461857400 \\ -10210200 & 461501040 & -7294166880 & 52455923520 & -175797181560 & 223092870000 \\ 155195040 & -7294166880 & 118413815520 & -868355047560 & 2953749598800 & -3792578790000 \\ -1086365280 & 52455923520 & -868355047560 & 6462709453200 & -22239682024560 & 28823598804000 \\ 3569485920 & -175797181560 & 2953749598800 & -22239682024560 & 77247244794720 & -100882595814000 \\ -4461857400 & 223092870000 & -3792578790000 & 28823598804000 & -100882595814000 & 132588554498400 \end{pmatrix}$$

In a similar way we consider the matrix $\mathbf{E}_{n,t}$ defined by

(4.23)
$$\mathbf{E}_{n,t} = \begin{vmatrix} U_{0,t} & U_{0,t+1} & \cdots & U_{0,t+n} \\ U_{1,t} & U_{1,t+1} & \cdots & U_{1,t+n} \\ U_{2,t} & U_{2,t+1} & \cdots & U_{2,t+n} \\ \vdots & & & \\ U_{n,t} & U_{n,t+1} & \cdots & U_{n,t+n} \end{vmatrix}, \qquad t \in \{0, 1, \ldots\},$$

where

$$U_{i,j} = \int_0^1 (1 - 2x)^2 [x(1-x)]^{i+j} \, dx = \frac{1}{(2i + 2j + 3)(2i + 2j + 1)\binom{2i + 2j}{i + j}}.$$

Let the general element of $\mathbf{E}_{n,2}^{-1}$ be $e_{j,k}$ ($j$, $k = 0, 1, 2, \ldots, n$). We may then prove, in a similar way as for $\mathbf{D}_{n,2}^{-1}$, that

$$
\begin{aligned}
(4.24) \quad e_{j,k} = (-1)^{j+k} \sum_{r=\max(j,k)}^{n} \\
\times \frac{(4r+7)(r+1)\binom{2r+2k+6}{r+k+3}\binom{2r+2j+6}{r+j+3}\binom{r+k+3}{k+2}\binom{r+j+3}{j+2}\binom{r}{k}\binom{r}{j}}{(r+3)\binom{2r+6}{r+3}\binom{2r+2}{r+1}}.
\end{aligned}
$$

(Accurate values of $E_{n,2}^{-1}$ ($n = 0, 1, \ldots, 4$) and $\mathbf{E}_{n,2}^{-1}$ ($n = 0, 1, \ldots, 9$) are given in [7, pp. 35–40].)

In Table II we give the matrices $\mathbf{E}_{n,2}^{-1}$ for $n = 0, 1, \ldots, 5$. Those interested in details may read Wrigge, Fransén and Borenius [6, pp. 20–22, 26–28].

<div align="center">

**TABLE II**

*Table of the matrices $\mathbf{E}_{n,2}^{-1}$*

</div>

$$
\mathbf{E}_{5,2} = \begin{pmatrix}
\frac{1}{210} & \frac{1}{1260} & \frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} \\
\frac{1}{1260} & \frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} \\
\frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} \\
\frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} \\
\frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} & \frac{1}{405623400} \\
\frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} & \frac{1}{405623400} & \frac{1}{1825305300}
\end{pmatrix}
$$

$$
\mathbf{E}_{0,2}^{-1} = (210)
$$

$$
\mathbf{E}_{1,2}^{-1} = \begin{pmatrix} 2520 & -13860 \\ -13860 & 83160 \end{pmatrix}
$$

$$
\mathbf{E}_{2,2}^{-1} = \begin{pmatrix}
13860 & -180180 & 540540 \\
-180180 & 2522520 & -7927920 \\
540540 & -7927920 & 25765740
\end{pmatrix}
$$

$$
\mathbf{E}_{3,2}^{-1} = \begin{pmatrix}
51480 & -1158300 & 7876440 & -16628040 \\
-1158300 & 27953640 & -198661320 & 432329040 \\
7876440 & -198661320 & 1456266240 & -3242467800 \\
-16628040 & 432329040 & -3242467800 & 7349593680
\end{pmatrix}
$$

$$
\mathbf{E}_{4,2}^{-1} = \begin{pmatrix}
150150 & -5105100 & 58198140 & -271591320 & 446185740 \\
-5105100 & 185825640 & -2211529320 & 10630860240 & -17847429600 \\
58198140 & -2211529320 & 27120333240 & -133273740600 & 227554727400 \\
-271591320 & 10630860240 & -133273740600 & 666174709200 & -1152943952160 \\
446185740 & -17847429600 & 227554727400 & -1152943952160 & 2017651916280
\end{pmatrix}
$$

$$
\mathbf{E}_{5,2}^{-1} = \begin{pmatrix}
371280 & -17635800 & 296281440 & -2271491040 & 8112468000 & -10951831800 \\
-17635800 & 895898640 & -15702916320 & 123958511040 & -452270091000 & 620603802000 \\
296281440 & -15702916320 & 283456686240 & -2286499105800 & 8481585294000 & -11791472238000 \\
-2271491040 & 123958511040 & -2286499105800 & 18753267776880 & -70486800711600 & 99048366799200 \\
8112468000 & -452270091000 & 8481585294000 & -70486800711600 & 267797436160800 & -379685406063600 \\
-10951831800 & 620603802000 & -11791472238000 & 99048366799200 & -379685406063600 & 542407722948000
\end{pmatrix}
$$

The result of this section may be summed up in

**THEOREM 4.** *Let $u_{i,j} = \int_0^1 [x(1-x)]^{i+j} \, dx$ and $U_{i,j} = \int_0^1 (1-2x)^2 [x(1-x)]^{i+j} \, dx$. Define the matrices $\mathbf{D}_{n,s} = \{(u_{i,j})\}$, $i = 0, 1, 2, \ldots, n$; $j = s, s+1, \ldots, s+n$, and $\mathbf{E}_{n,t} = \{(U_{i,j})\}$, $i = 0, 1, \ldots, n$; $j = t, t+1, \ldots, t+n$; $s, t \in \{0, 1, 2, \ldots, n\}$. Let $d_{j,k}^{n,s}$ and*

$e_{j,k}^{n,t}$ be the general elements of $\mathbf{D}_{n,s}^{-1}$ and $\mathbf{E}_{n,t}^{-1}$, with $j, k = 0, 1, 2, \ldots, n$. Then

$$d_{j,k}^{n,s} = \sum_{r=\max(j,k)}^{n} \frac{\Gamma(r+s+1)(4r+2s+1)4^{k+j+s}\Gamma\left(s+\frac{1}{2}+r+k\right)\Gamma\left(s+\frac{1}{2}+r+j\right)}{\Gamma\left(r+s+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)\Gamma(s+1+k)\Gamma(s+1+j)r!}$$

$$\times \binom{r}{k}\binom{r}{j}(-1)^{k+j},$$

$$e_{j,k}^{n,t} = \sum_{r=\max(j,k)}^{n} \frac{\Gamma(r+t+1)(4r+2t+3)4^{k+j+t}\Gamma(t+3/2+r+k)\Gamma(t+3/2+r+j)}{\Gamma(r+t+3/2)\Gamma(r+3/2)\Gamma(t+1+k)\Gamma(t+1+j)r!}$$

$$\times \binom{r}{k}\binom{r}{j}(-1)^{k+j}.$$

Define the determinants $D_{n,s} = \text{Det}(\mathbf{D}_{n,s})$ and $E_{n,t} = \text{Det}(\mathbf{E}_{n,t})$. Then

$$D_{n,s} = \frac{n!\Gamma(n+s+1)\Gamma\left(n+s+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2^{2s+1}4^{2n}\left(2n+s+\frac{1}{2}\right)\Gamma^2\left(2n+s+\frac{1}{2}\right)}D_{n-1,s},$$

$$E_{n,t} = \frac{n!\Gamma(n+t+1)\Gamma(n+t+3/2)\Gamma(n+3/2)}{2^{2t+1}4^{2n}(2n+t+3/2)\Gamma^2(2n+t+3/2)}E_{n-1,t}.$$

## 5. Calculation of Determinants and Inverse Matrices.
The determinants $D_{n,2}$ and $E_{n,2}$ are calculated using the formulae

(5.1)
$$D_{n,2}^{-1} = \prod_{r=0}^{n}(2r+3)\binom{4r+5}{2r+2}\binom{4r+4}{2r},$$

(5.2)
$$E_{n,2}^{-1} = \prod_{r=0}^{n}(r+2)\binom{4r+7}{2r+2}\binom{4r+5}{2r+1}.$$

See [6, p. 22].

When calculating the inverse matrices we may choose one of several methods. We may, e.g., use a recursion technique (i.e., Householder's method).

Let $\mathbf{C}_{n,2}$ denote any one of the square matrices $\mathbf{D}_{n,2}$ and $\mathbf{E}_{n,2}$ of order $(n+1)$. We start making a partition of $\mathbf{C}_{n,2}$, i.e.,

(5.3)
$$\mathbf{C}_{n,2} = \begin{bmatrix} \mathbf{C}_{n-1,2} & \hat{c}_n \\ \hat{c}_n^T & \gamma_n \end{bmatrix}.$$

Here $\hat{c}_n$ is a column vector (of order $n$) and $\gamma_n$ a rational number. Denoting $C_{n,2} = \det \mathbf{C}_{n,2}$, we finally get the recursion formula

(5.4)
$$\mathbf{C}_{n,2}^{-1} = \begin{bmatrix} \mathbf{C}_{n-1,2}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{C_{n-1,2}}{C_{n,2}}\begin{bmatrix} -\mathbf{C}_{n-1,2}^{-1}\hat{c}_n \\ 1 \end{bmatrix}\begin{bmatrix} -\hat{c}_n^T\mathbf{C}_{n-1,2}^{-1} & 1 \end{bmatrix}.$$

(See e.g. F. Ayres [2, pp. 56–58].)

Thus we may start with $\mathbf{C}_{1,2}^{-1} = \mathbf{D}_{1,2}^{-1}$ or $\mathbf{E}_{1,2}^{-1}$. A repeated use of (5.4) then yields the matrices $\mathbf{D}_{k,2}^{-1}$ and $\mathbf{E}_{k,2}^{-1}$ ($k = 2, 3, \ldots, n$). (See [7, pp. 29–40], where $n = 9$.)

Alternatively, we may also use the formulae (4.22) and (4.24) which give explicit expressions of the general elements of $\mathbf{D}_{n,2}^{-1}$ and $\mathbf{E}_{n,2}^{-1}$.

But we can do better. The drawback of the recursion method given by (5.4) is that to calculate, e.g., $\mathbf{D}_{6,2}^{-1}$ we have to calculate all $\mathbf{D}_{k,2}^{-1}$ with $k \leqslant 5$. Therefore a direct method is to be preferred. Such a method may be obtained by a clever use of the Christoffel-Darboux formula [1, p. 785], which applied to the orthonormal functions

$q_{n,s}(x)$ (Eq. (4.4)) may be stated as

$$(5.5) \quad \sum_{m=0}^{n} q_{m,s}(x)q_{m,s}(y) = \frac{z_{n,n}}{z_{n+1,n+1}} \frac{\left[q_{n+1,s}(x)q_{n,s}(y) - q_{n,s}(x)q_{n+1,s}(y)\right]}{x - y}.$$

Multiplying both sides of (5.5) with $(x - y)$ and identifying the coefficients of $x^j y^k$, we get

$$(5.6) \quad d^{n,s}_{j-1,k} - d^{n,s}_{j,k-1} = \frac{z_{n,n}}{z_{n+1,n+1}}(z_{n+1,j}z_{n,k} - z_{n,j}z_{n+1,k}),$$

where we have written $d_{j,k} = d^{n,s}_{j,k}$ to point out the dependence of the parameters $n$ and $s$. Putting $s = 2$, we get

$$d_{j-1,k} - d_{j,k-1} = (-1)^{k+j}$$

$$(5.7) \quad \times \frac{(k-j)(n+1)(n+3)\binom{n}{j}\binom{n}{k}\binom{2n+2j+4}{n+j+2}\binom{2n+2k+4}{n+k+2}\binom{n+j+2}{n}\binom{n+k+2}{n}}{2(n+1-j)(n+1-k)\binom{2n+4}{n+2}\binom{2n}{n}}.$$

The corresponding formula for $e_{j,k}$ (with $t = 2$) is

$$e_{j-1,k} - e_{j,k-1} = (-1)^{k+j}$$

$$(5.8) \quad \times \frac{(k-j)(n+1)^2\binom{n}{j}\binom{n}{k}\binom{2n+2j+6}{n+j+3}\binom{2n+2k+6}{n+k+3}\binom{n+j+3}{j+2}\binom{n+k+3}{k+2}}{2(n+1-j)(n+1-k)\binom{2n+6}{n+3}\binom{2n+2}{n+1}}.$$

The calculation of $d_{j,k}$ may be given by

*Algorithm* D. Let the general matrix element of $\mathbf{D}_{n,2}^{-1}$ be $d_{j,k}$ ($j, k = 0, 1, 2, \ldots, n$). The steps in the calculation of $d_{j,k}$ are then as follows:

I.

$$d_{n,k+1} = -\frac{2(n-k)(2n+2k+5)}{(k+1)(k+3)}d_{n,k}; \qquad k = 0, 1, 2, \ldots, n-1.$$

Starting values are given by

$$d_{n,0} = (-1)^n \frac{(4n+5)\binom{4n+4}{2n+2}\binom{2n+2}{n+2}\binom{n+2}{2}}{\binom{2n}{n}}.$$

II. Calculate the elements of the lower right-half of the matrix using

$$d_{j-1,k} - d_{j,k-1} = (-1)^{k+j}$$

$$\times \frac{(k-j)(n+1)(n+3)\binom{n}{j}\binom{n}{k}\binom{2n+2j+4}{n+j+2}\binom{2n+2k+4}{n+k+2}\binom{n+j+2}{n}\binom{n+k+2}{n}}{2(n+1-j)(n+1-k)\binom{2n+4}{n+2}\binom{2n}{n}}.$$

III. Calculate $d_{j,0} = d_{0,j}$ ($j = 0, 1, 2, \ldots, n$) using

$$d_{j,0} = (-1)^j \frac{\Pi_{i=-1}^{j+1}(2n+3+2i)(2n+4-2i)}{4(j+3)!j!}.$$

IV. Use step II to calculate the elements of the upper left-half of the matrix.
V. In all steps we make use of the symmetry $d_{j,k} = d_{k,j}$.

A similar scheme of calculation exists for the elements $e_{j,k}$ of the matrix $\mathbf{E}_{n,2}^{-1}$, viz,

*Algorithm* E. Let the general matrix element of $\mathbf{E}_{n,2}^{-1}$ be $e_{j,k}$ ($j, k = 0, 1, 2, \ldots, n$). The steps in the calculation of $e_{j,k}$ are then as follows:

I.

$$e_{n,k+1} = -\frac{2(n-k)(2n+2k+7)}{(k+1)(k+3)} e_{n,k}, \qquad k = 0, 1, 2, \ldots, n-1.$$

Starting values are given by

$$e_{n,0} = (-1)^n \frac{(4n+7)\binom{4n+6}{2n+3}\binom{2n+3}{n+3}\binom{n+3}{2}}{\binom{2n+2}{n+1}}.$$

II. Calculate the elements of the lower right-half of the matrix using

$$e_{j-1,k} - e_{j,k-1} = (-1)^{k+j}$$

$$\times \frac{(k-j)(n+1)^2\binom{n}{j}\binom{n}{k}\binom{2n+2j+6}{n+j+3}\binom{2n+2k+6}{n+k+3}\binom{n+j+3}{j+2}\binom{n+k+3}{k+2}}{2(n+1-j)(n+1-k)\binom{2n+6}{n+3}\binom{2n+2}{n+1}}.$$

III. Calculate $e_{j,0} = e_{0,j}$ using

$$e_{j,0} = (-1)^j \frac{\prod_{i=-1}^{j+1}(2n+5+2i)(2n+4-2i)}{4j!(j+3)!}.$$

IV. Use step II to calculate the elements of the upper left-half of the matrix.

V. In all steps we make use of the symmetry $e_{j,k} = e_{k,j}$.

We managed to find still another method to calculate the elements of $\mathbf{D}_{n,2}^{-1}$ and $\mathbf{E}_{n,2}^{-1}$ making use of a special partition technique. The great advantage of this method is that we can describe both cases with one formula using an idempotent number $\beta$ (i.e. $\beta^2 = \beta$). This result may be summarized in

THEOREM 5. *Let* $\mathbf{C}_{n,2}^{-1}$ *denote any one of the matrices* $\mathbf{D}_{n,2}^{-1}$ *and* $\mathbf{E}_{n,2}^{-1}$, *and let the general element of* $\mathbf{C}_{n,2}^{-1}$ *be* $y_{j,k}$ ($j, k = 0, 1, 2, \ldots, n$). *Then*

$$y_{j,k} = (-1)^{j+k} \frac{\prod_{i=-1}^{j+1}(2n+3+2\beta+2i)(2n+4-2i)}{2(j+k+1)!(j+k+3)!k!(k+2)!} x_{j,k},$$

*where*

$$x_{j,k} = \sum_{v=0}^{k} R_v T_v (j-k+1+2v)$$

$$\times \prod_{i=v+1}^{k} (j+1+i)(j+3+i) \prod_{i=0}^{v-1} (k-i)(k+2-i)$$

*and*

$$R_m = \prod_{i=j+2}^{j+1+m} (2n+3+2\beta+2i)(2n+4-2i);$$

$$T_m = \prod_{i=2}^{k+1-m} (2n+1+2\beta+2i)(2n+6-2i).$$

*The case* $\beta = 0$ *corresponds to* $\mathbf{D}_{n,2}^{-1}$, *and the case* $\beta = 1$ *corresponds to* $\mathbf{E}_{n,2}^{-1}$.

(The products in Theorem 5 are interpreted as 1 if the upper index is less than the lower.)

**6. Application.** In this section we will apply our method to the case $f(x) = \sin(\pi x)$, $f(x) = \cos(2\pi x)$ and $h(x) = \sin(2\pi x)$, $h(x) = \cos(\pi x)$; $x \in [0, 1]$. (In [7, pp. 23–26] we successfully approximated the Jacobian elliptic function $sn(2K(k)x, k)$ for $0 \leqslant k \leqslant 0.25$.) Consider the expansions

$$(6.1) \qquad\qquad \sin(\pi x) = \sum_{n=1}^{\infty} a_n [x(1-x)]^n;$$

(see Lyusternik et al. [4, p. 82]) and

$$(6.2) \qquad\qquad \sin(2\pi x) = (1 - 2x) \sum_{n=1}^{\infty} b_n [x(1-x)]^n.$$

We will also need the coefficients $A_n$ defined by

$$(6.3) \qquad\qquad \sin^2(\pi x) = \sum_{n=2}^{\infty} A_n [x(1-x)]^n.$$

Since $d^2 \sin(\pi x)/dx^2 = -\pi^2 \sin(\pi x)$, we get, differentiating (6.1) twice and identifying the coefficients of $[x(1-x)]^n$, the recurrence relation

$$(6.4) \quad (n+2)(n+1)a_{n+2} = (2n+2)(2n+1)a_{n+1} - \pi^2 a_n; \qquad a_1 = a_2 = \pi.$$

Exact and approximate values of $a_i$ ($i = 1(1)10$ and $i = 1(1)20$) are given in [6, p. 35] and [7, p. 41]. Note that $a_n$ converges fast towards zero. We have, e.g., $a_{20} \approx 9.1 \times 10^{-29}$. We solve the recurrence relation (6.4) using (1.9). Thus

$$(6.5) \qquad a_{2n+1} = \frac{\pi}{2n+1} \sum_{v=0}^{n} \binom{2n+2v}{2v} \frac{(-1)^{n-v}\pi^{2n-2v}}{(2n-2v)!},$$

and

$$(6.6) \qquad a_{2n+2} = \frac{\pi}{2n+2} \sum_{v=0}^{n} \binom{2n+2v+2}{2v+1} \frac{(-1)^{n-v}\pi^{2n-2v}}{(2n-2v)!}.$$

Using (1.3) we obtain

$$(6.7) \qquad a_n = (-1)^n \sum_{k=n}^{\infty} \pi^{2k} \binom{k}{n} \frac{(-1)^k 4^{n-k}}{(2k)!}.$$

(1.6) yields the alternative form

$$(6.8) \quad a_n = \frac{(-1)^n}{2 \cdot n!} \sum_{k=n}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k)(2k-1)\cdots(2k-n+1)(2k-2n+1)!}.$$

To obtain a recurrence relation for the coefficients $b_n$ we use the coefficients $A_n$ defined by (6.3). We note that

$$\frac{d^2}{dx^2} \sin^2(\pi x) = 2\pi^2 - 4\pi^2 \sin^2(\pi x).$$

Then

$$(6.9) \qquad \begin{aligned} (n+2)(n+1)A_{n+2} &= (2n+2)(2n+1)A_{n+1} - 4\pi^2 A_n; \\ A_2 &= \pi^2, \qquad A_3 = 2\pi^2. \end{aligned}$$

Differentiating (6.3) with respect to $x$ and identifying coefficients, we get

(6.10)
$$b_{n-1} = \frac{n}{\pi} A_n.$$

$b_n$ thus satisfies the recurrence relation

(6.11) $\quad n(n+1)b_{n+1} = 2n(2n+1)b_n - 4\pi^2 b_{n-1}; \quad b_1 = 2\pi, \quad b_2 = 6\pi.$

Using (1.8), we get

(6.12) $\quad A_{2n+1} = \frac{\pi}{2n+1} \sum_{v=1}^{n} \binom{2n+2v-1}{2v-1} \frac{(-1)^{n-v}(2\pi)^{2n-2v+1}}{(2n-2v+1)!}$

and

(6.13) $\quad A_{2n+2} = \frac{\pi}{2n+2} \sum_{v=0}^{n} \binom{2n+2v+1}{2v} \frac{(-1)^{n-v}(2\pi)^{2n-2v+1}}{(2n-2v+1)!}.$

From (6.10), (6.12), and (6.13) we get finite expressions for $b_{2n}$ and $b_{2n+1}$.

The expansions for $\cos(\pi x)$ and $\cos(2\pi x)$ are deduced in a similar way. We put

(6.14)
$$\cos(\pi x) = (1 - 2x) \sum_{n=0}^{\infty} \beta_n [x(1-x)]^n$$

and

(6.15)
$$\cos(2\pi x) = \sum_{n=0}^{\infty} \alpha_n [x(1-x)]^n.$$

Differentiating (6.1) with respect to $x$, we get

(6.16)
$$\beta_{n-1} = na_n/\pi.$$

Using (6.4), it is easy to prove that $\beta_n$ satisfies the recurrence relation

(6.17) $\quad n(n+1)\beta_{n+1} = 2n(2n+1)\beta_n - \pi^2\beta_{n-1}; \quad \beta_0 = 1, \quad \beta_1 = 2.$

A differentiation of (6.15) with respect to $x$ yields

(6.18)
$$b_{n-1} = -n\alpha_n/2\pi.$$

Using (6.11), we get the recurrence relation

(6.19) $\quad (n+2)(n+1)\alpha_{n+2} = (2n+2)(2n+1)\alpha_{n+1} - 4\pi^2\alpha_n; \quad \alpha_0 = 1, \quad \alpha_1 = 0.$

We next turn our interest to approximations in $L_2$, i.e., we consider min $D_1^2(w)$ and min $D_2^2(w)$, where

(6.20)
$$D_1^2(w) = \int_0^1 \left\{ \sin(\pi x) - \sum_{n=1}^{k} c_{n,k} [x(1-x)]^n \right\}^2 dx$$

and

(6.21)
$$D_2^2(w) = \int_0^1 \left\{ \sin(2\pi x) - (1-2x) \sum_{n=1}^{k} C_{n,k} [x(1-x)]^n \right\}^2 dx.$$

The associated integrals

$$I_j = \int_0^1 \sin(\pi x)[x(1-x)]^j \, dx$$

and

$$I_j' = \int_0^1 (1 - 2x)\sin(2\pi x)[x(1 - x)]^j\, dx$$

are calculated from the recurrence formulae

(6.22)    $$I_j = \frac{2j(2j - 1)}{\pi^2} I_{j-1} - \frac{j(j - 1)}{\pi^2} I_{j-2}; \quad I_0 = \frac{2}{\pi}, \quad I_1 = \frac{4}{\pi^3},$$

and

(6.23)    $$I_j' = \frac{2j(2j + 1)}{4\pi^2} I_{j-1}' - \frac{j(j - 1)}{4\pi^2} I_{j-2}'; \quad I_0' = \frac{1}{\pi}, \quad I_1' = \frac{3}{2\pi^3}.$$

Solving the equation systems (2.2) and (2.6) by inverting the matrices as described in Sections 4 and 5 we may formulate

THEOREM 6. *Let* $x \in [0, 1]$, *then we have the approximations*

$$\sin(\pi x) \approx \begin{cases} 3.141583993[x(1 - x)] + 3.141891945[x(1 - x)]^2 \\ +1.112123058[x(1 - x)]^3 + 0.219850867[x(1 - x)]^4 \end{cases}$$

*and*

$$\sin(\pi x) \approx \begin{cases} 3.141592715257[x(1 - x)] + 3.141589575603[x(1 - x)]^2 \\ +1.115524716287[x(1 - x)]^3 + 0.204430015076[x(1 - x)]^4 \\ +0.024416348195[x(1 - x)]^5. \end{cases}$$

*The absolute errors are less than* $8 \times 10^{-8}$, *respectively* $4 \times 10^{-10}$.

The corresponding relations for $\sin(2\pi x)$ are formulated in

THEOREM 7. *Let* $x \in [0, 1]$, *then we have the approximations*

$$\sin(2\pi x) \approx \begin{cases} (1 - 2x)\{6.281856[x(1 - x)] + 18.902201[x(1 - x)]^2\} \\ + (1 - 2x)\{20.829857[x(1 - x)]^3 + 16.439719[x(1 - x)]^4\} \end{cases}$$

*and*

$$\sin(2\pi x) \approx \begin{cases} (1 - 2x)\{6.283217166[x(1 - x)] + 18.847760765[x(1 - x)]^2\} \\ + (1 - 2x)21.523970874[x(1 - x)]^3 \\ + (1 - 2x)\{12.922874461[x(1 - x)]^4 + 6.154478369[x(1 - x)]^5\}. \end{cases}$$

*The absolute errors are less than* $10^{-5}$, *respectively* $2 \times 10^{-7}$

To carry through the corresponding approximations for $\cos(\pi x)$ and $\cos(2\pi x)$ we must calculate the integrals

(6.24)               $$I_j = \int_0^1 [\cos(2\pi x) - 1][x(1 - x)]^j\, dx$$

and

(6.25)    $$I_j' = \int_0^1 (1 - 2x)[\cos(\pi x) - (1 - 2x)][x(1 - x)]^j\, dx.$$

We introduce the complementary integrals $J_j$ and $J_j'$ defined by

(6.26) $$J_j = \int_0^1 \cos(2\pi x)[x(1-x)]^j\,dx$$

and

(6.27) $$J_j' = \int_0^1 (1-2x)\cos(\pi x)[x(1-x)]^j\,dx.$$

The integrals $J_j$ and $J_j'$ satisfy the recurrence relations

(6.28) $$J_j = \frac{2j(2j-1)}{4\pi^2}J_{j-1} - \frac{j(j-1)}{4\pi^2}J_{j-2}$$

and

(6.29) $$J_j' = \frac{2j(2j+1)}{\pi^2}J_{j-1}' - \frac{j(j-1)}{\pi^2}J_{j-2}'.$$

Starting values are given by

$$J_0 = 0, \quad J_1 = -\frac{1}{2\pi^2} \quad \text{and} \quad J_0' = \frac{4}{\pi^2}, \quad J_1' = \frac{24}{\pi^4} - \frac{2}{\pi^2}.$$

We finally get the following expressions for the integrals $I_j$ and $I_j'$

(6.30) $$I_j = J_j - \frac{1}{(2j+1)\binom{2j}{j}}; \quad I_j' = J_j' - \frac{1}{(2j+3)(2j+1)\binom{2j}{j}}.$$

The approximations for $\cos(\pi x)$ are given in

THEOREM 8. *Let* $x \in [0,1]$, *then*

$$\cos(\pi x) \approx \begin{cases} (1-2x)\{1 + 1.999999230[x(1-x)] + 1.065228532[x(1-x)]^2\} \\ + (1-2x)\{0.260400939[x(1-x)]^3 + 0.038640515[x(1-x)]^4\} \end{cases}$$

*and*

$$\cos(\pi x)$$

$$\approx \begin{cases} (1-2x)\{1 + 2.000000004489[x(1-x)] + 1.065197545425[x(1-x)]^2\} \\ + (1-2x)\{0.260796014285[x(1-x)]^3 + 0.036638801083[x(1-x)]^4\} \\ + (1-2x)0.003502999395[x(1-x)]^5. \end{cases}$$

*The absolute errors are less than* $0.6 \times 10^{-8}$, *respectively* $0.3 \times 10^{-10}$.

We have found the approximations of $\cos(2\pi x)$ of less value and prefer to compute $\cos(2\pi x)$ from the formula $\cos(2\pi x) = 2\cos^2(\pi x) - 1$, thereby using Theorem 8 to compute $\cos(\pi x)$.

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