MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 158
APRIL 1982, PAGES 589-591

An Upper Bound for the First Zero of Bessel Functions

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Abstract. It is shown, using the Rayleigh-Ritz method of the calculus of variations, that an upper bound for the first zero j_ν, of $z^{-\nu}J_\nu(z)$, $\nu > -1$, is given by

$$(\nu+1)^{1/2}\{(\nu+2)^{1/2}+1\},$$

and that for large ν, $j_\nu = \nu + O(\nu^{1/2})$.

1. The following upper bound is given by Watson [4] for the first zero j_ν of $J_\nu(x)$ $(\nu > 0)$

$$j_\nu < \left\{\frac{4}{3}(\nu+1)(\nu+5)\right\}^{1/2}. \tag{1}$$

It may be shown that a better bound may be obtained, valid for $\nu > -1$, namely

$$(\nu+1)^{1/2}\big((\nu+2)^{1/2}+1\big). \tag{2}$$

2. Consider the function

$$u(z) = \Gamma(\nu+1)(2/(\gamma z))^\nu J_\nu(\gamma z). \tag{3}$$

The differential equation satisfied by $u(z)$ is given by Watson [3] to be

$$z^2u'' + (2\nu+1)zu' + \gamma^2z^2u = 0 \tag{4}$$

with the boundary condition $u(0) = 1$, and if γ is a zero of J_ν, $u(1) = 0$.

Equation (4) can be written in Sturm-Liouville form

$$\frac{d}{dz}\left(z^{2\nu+1}\frac{du}{dz}\right) + \gamma^2z^{2\nu+1}u = 0. \tag{5}$$

Multiplying Eq. (5) by u and integrating over $0 \leqslant z \leqslant 1$, it follows that

$$\gamma^2 = \frac{\int_0^1 z^{2\nu+1}u'^2\,dz}{\int_0^1 z^{2\nu+1}u^2\,dx}. \tag{6}$$

On integration by parts, $uu'z^{2\nu+1}$ will vanish at $z = 0$, if $\nu > -\frac{1}{2}$, and $u(1)$ vanishes. Thus the relation (6) provides a variational formulation, as indicated by Irving and Mullineux [1], for γ^2 which is an eigenvalue for the differential equation (5). The first eigenvalue will be j_ν^2. The functional

$$\Lambda(\omega) = \frac{\int_0^1 z^{2\nu+1}\omega'^2\,dz}{\int_0^1 z^{2\nu+1}\omega^2\,dz}, \tag{7}$$

Received February 19, 1981; revised July 13, 1981.
1980 *Mathematics Subject Classification.* Primary 33A40, 49G10.

as indicated by Irving and Mullineux [2], obeys the following relations

(8a) $$\Lambda(u) = j_\nu^2,$$

(8b) $$\Lambda(\omega) > \Lambda(u), \qquad u \neq \omega.$$

Thus $\Lambda(\omega)$ provides an upper bound to j_ν^2 when $\omega(1) = 0$, by the Rayleigh-Ritz procedure.

3. Consider the approximating function

(9) $$\omega = 1 - z^p,$$

where p is as yet unspecified.

(10) $$\Lambda(\omega) = \frac{\int_0^1 z^{2\nu+1} p^2 z^{2p-2}\, dz}{\int_0^1 (1 - 2z^p + z^{2p}) z^{2\nu+1}\, dz}$$

(11) $$= p^2 \left\{ \frac{1/(2\nu + 2p)}{1/(2\nu+2) - 2/(2\nu+p+2) + 1/(2\nu+2p+2)} \right\}$$

(12) $$= \frac{(\nu+1)(2\nu+p+2)(\nu+p+1)}{\nu+p}$$

on simplification.

Up till now p has not been specified. $\Lambda(\omega)$ may be regarded as a function of p, and the best upper bound will follow by minimizing $\Lambda(\omega)$ with respect to p. It can be verified by differentiation that this happens when

(13) $$p + \nu = (\nu + 2)^{1/2}.$$

Although p is negative outside of $-1 < \nu < 2$, the process is still valid because all of the denominators in the expression (11) remain positive, and it can easily be seen that

$$\operatorname*{lt}_{z \to 0} z^{2\nu+1} \omega \omega' = 0,$$

so that the endpoint condition at $z = 0$ remains satisfied. It follows that

$$j_\nu^2 < \frac{(\nu+1)\left[(\nu+2) + (\nu+2)^{1/2}\right]\left[(\nu+2)^{1/2} + 1\right]}{(\nu+2)^{1/2}},$$

which gives

(14) $$j_\nu < (\nu+1)^{1/2}\left((\nu+2)^{1/2} + 1\right).$$

A straightforward reduction shows that, if $\nu + 1 > 0$,

$$(\nu+1)^{1/2}\left((\nu+2)^{1/2} + 1\right) < \left\{\frac{4}{3}(\nu+1)(\nu+5)\right\}^{1/2}.$$

(For $\nu = 7$ there is in fact equality.)

It thus follows that [4]

(15) $$\{\nu(\nu+2)\}^{1/2} < j_\nu < (\nu+1)^{1/2}\left((\nu+2)^{1/2} + 1\right), \qquad \nu > 1.$$

It follows from (15) that

(16) $$j_\nu = \nu + O(\nu^{1/2}) \quad \text{for large } \nu.$$

As an example, the bound for $\nu = 0$ is given by $\sqrt{2} + 1 = 2.4142$ in comparison with the true value 2.4048.

Acknowledgement. I am indebted to a referee for some improvements in the presentation of this paper.

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2. J. IRVING & N. MULLINEUX, Loc. cit., p. 39.

3. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, The University Press, Cambridge, 1944, p. 98.

4. G. N. WATSON, Loc. cit., p. 486.