

Simplified Forecasting by Polynomial Regression With Equally Spaced Values of the Independent Variable

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Abstract. A method is developed and tables are presented which yield polynomial least squares forecasts by computing the inner product of two n -component vectors, where n is the number of observed points.

1. Introduction. In this paper, a method is developed which makes it easy to forecast by polynomial regression when the values of the independent variable are equally spaced. More specifically, one can obtain the forecast yielded by a least squares polynomial by simply computing an inner product. This method of forecasting by least squares is easier than by any of the other simplified methods which have appeared in the literature. Many methods [1] avoid solving the normal equations. Instead the least squares polynomial is obtained by first computing moments from the data and then obtaining coefficients of the least squares polynomial from linear functions of these moments. Tables which give the coefficients of the linear functions are provided in those papers. Then a forecast is obtained by evaluating the polynomial at the desired value of the independent variable. The method presented in this paper avoids the computation of moments, the solving of the normal equations, and the evaluation of the polynomial. The tables presented are applicable for polynomials up to the seventh degree for 2 to 20 points with equally spaced values of the independent variable.

2. Forecasting Technique. Without loss of generality, the values of the independent variable, x , are to be taken $1, 2, 3, \dots, n$ and the corresponding values of the dependent variable are denoted by $y_1, y_2, y_3, \dots, y_n$. The vector formed by these y values will be denoted by Y . The coefficients of the k th degree least squares polynomial,

$$(1) \quad \hat{y} = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k,$$

are usually obtained by solving the normal equations

$$(2) \quad (X'X)A = b,$$

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where

$$X = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^k \\ 1 & 3 & 3^2 & \cdots & 3^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^k \end{pmatrix}, \quad A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}, \quad b = X'Y = \begin{pmatrix} \Sigma y \\ \Sigma xy \\ \Sigma x^2 y \\ \vdots \\ \Sigma x^k y \end{pmatrix}$$

and X' is the transpose of X . Therefore

$$(3) \quad A = (X'X)^{-1}b.$$

The solution for the a_j given by (3) yields linear forms in the y_i , say $L_j(y)$. By replacing the a_j in (1) by these linear forms, the k th degree least squares polynomial, \hat{y} , can be expressed as

$$(4) \quad \hat{y} = L_0(y) + L_1(y)x + L_2(y)x^2 + \cdots + L_k(y)x^k.$$

Then by letting $x = n + 1$, and combining the y_i terms in (4), the least squares forecast, \hat{y} , for $x = n + 1$, becomes a linear form in y_i , say

$$(5) \quad \hat{y} = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n.$$

Tables are presented which contain the values of c_1 to c_n for (n, k) pairs, where $n = 2, 3, 4, \dots, 20$ and $k = 1, 2, 3, \dots, 7$ with $k \leq n - 1$. The forecast for $x = n + 1$ is simply the inner product $Y \cdot C$, where C is the vector of the appropriate column of c_i values listed in the table.

These tables can also be used to forecast y_{n+2}, y_{n+3} , etc. This can be done because the forecast for y_{n+1} by a k th degree least squares polynomial yields a value which lies on that k th degree equation. Therefore, if a k th degree polynomial forecast is desired for y_{n+2} the forecast for y_{n+1} lies on the least squares polynomial implicitly employed, and so this new point can be included with the others to forecast y_{n+2} . Thus, the values $y_1, y_2, y_3, \dots, y_n, \hat{y}_{n+1}$ can be used with the $n + 1$ values of c_i taken from the $n + 1$ table to forecast y_{n+2} . Continuing in this manner, least squares polynomial forecasts can be obtained for other y_i . Repeated use of this procedure for forecasting at an x value far above n will usually cause a build up of rounding errors. This is because the forecast at an x value j intervals above n is obtained from the n observed values of y and from $j - 1$ forecasted values.

3. Tables of c_i . To use the tables of c_i values, first go to the table headed with the appropriate value of n . The c_i values for forecasting with a k th degree polynomial are in the column headed by that k value. Then $\sum_{i=1}^n c_i y_i$ gives the forecast at $x = n + 1$.

Example. Forecast the y values by a second degree least squares polynomial at $x = 8.5$ and at $x = 9.0$ given the following 10 points of data:

x	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
y	14.1	14.3	14.4	14.8	15.5	17.0	18.1	19.4	20.2	21.1

The forecast for $x = 8.5$ is obtained by using the c_i values from Table 9 in the column headed by $k = 2$. Then $\sum_{i=1}^{10} c_i y_i = 23.1$ is the forecast.

The forecast for $x = 9.0$ is obtained by using the c_i values from Table 10 ($k = 2$ column) including the 11th y value of 23.1 at $x = 8.5$. Then $\sum_{i=1}^{11} c_i y_i = 24.7$ is the forecast.

To check the accuracy of the method, the second-degree least squares polynomial was obtained by using the Statistical Package for the Social Sciences (SPSS). The polynomial is

$$\hat{y} = 15.39727 - 1.378633x + .2681814x^2.$$

The values obtained from this polynomial at $x = 8.5$ and $x = 9.0$ to three significant figures are 23.1 and 24.7, respectively.

4. Remarks About the Tables. If all y_i are set equal to a constant, w , the forecast for y_{n+1} must be w . It then follows that the sum of the c_i values for each (n, k) pair is 1.

When $k = 0$, the least squares polynomial for any n is the arithmetic mean of the y_i . For this value of k , all the c_i are equal to $1/n$, so these values are not listed in the tables.

For any (n, k) pair in the tables, it is seen that the values of the c_i lie on a k th degree polynomial in i . It is conjectured that this property holds also for all other (n, k) pairs for $k \leq n - 1$. Under this assumption the values of c_i for $k = n - 1$ are signed binomial coefficients. This follows by first expressing the $(k + 1)$ st differences [3] as

$$(6) \quad \Delta^{k+1}y_1 = \sum_{i=0}^{k+1} (-1)^{k-i+1} \binom{k+1}{i} y_{i+1}.$$

Since the $(k + 1)$ st differences of a k th degree polynomial are zero, it follows that

$$(7) \quad \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} y_{i+1} = 0.$$

Therefore

$$(8) \quad y_{n+1} = \sum_{i=0}^{n-1} (-1)^{n-i-1} \binom{n}{i} y_{i+1}.$$

5. Details of the Computation. The tables were computed at Herbert H. Lehman College using an Amdahl 470/V6 computer located at the Computer Center of the City University of New York. The programming language chosen was APL because this language is convenient for operations on matrices. The entries were rounded off to four decimal places. The program incorporated a check that the sum of the c_i for each (n, k) pair is 1 except for rounding.

The least accurate step in the computation of the table entries occurred in the process of obtaining the inverse of $X'X$ when $k = 7$. For this reason the inverse of $X'X$, which is a symmetric matrix, was also obtained for these cases by programming appropriate steps called for in the square root method [2]. In that method, a triangular matrix, S , is obtained such that $X'X = S'S$. Since $(X'X)^{-1} = S^{-1}(S')^{-1}$, this last expression was used to obtain $(X'X)^{-1}$. The c_i values resulting from this revised program agreed with those obtained from the original program when rounded off to four decimal places.

Since the c_i values for any (n, k) pair in the tables satisfy a k th degree polynomial, the entries were checked by differencing to make sure the $(k + 1)$ st differences are zero. This was done for all (n, k) pairs for $n \geq k + 2$.

TABLE 1
Values of c_i for $n = 2$

i	k=1
1	-1.0000
2	2.0000

TABLE 2
Values of c_i for $n = 3$

i	k=1	k=2
1	-.6667	1.0000
2	.3333	-3.0000
3	1.3333	3.0000

TABLE 3
Values of c_i for $n = 4$

i	k=1	k=2	k=3
1	-.5000	.7500	-1.0000
2	.0000	-1.2500	4.0000
3	.5000	-.7500	-6.0000
4	1.0000	2.2500	4.0000

TABLE 4
Values of c_i for $n = 5$

i	k=1	k=2	k=3	k=4
1	-.4000	.6000	-.8000	1.0000
2	-.1000	-.6000	2.2000	-5.0000
3	.2000	-.8000	-.8000	10.0000
4	.5000	.0000	-2.8000	-10.0000
5	.8000	1.8000	3.2000	5.0000

TABLE 5
Values of c_i for $n = 6$

i	k=1	k=2	k=3	k=4	k=5
1	-.3333	.5000	-.6667	.8333	-1.0000
2	-.1333	-.3000	1.3333	-3.1667	6.0000
3	.0667	-.6000	.3333	3.3333	-15.0000
4	.2667	-.4000	-1.3333	1.6667	20.0000
5	.4667	.3000	-1.3333	-5.8333	-15.0000
6	.6667	1.5000	2.6667	4.1667	6.0000

TABLE 6
Values of c_i for $n = 7$

i	k=1	k=2	k=3	k=4	k=5	k=6
1	-.2857	.4286	-.5714	.7143	-.8571	1.0000
2	-.1429	-.1429	.8571	-2.1429	4.1429	-7.0000
3	.0000	-.4286	.5714	1.0000	-6.8571	21.0000
4	.1429	-.4286	-.4286	2.1429	2.1429	-35.0000
5	.2857	-.1429	-1.1429	-.7143	7.1429	35.0000
6	.4286	.4286	-.5714	-3.5714	-9.8571	-21.0000
7	.5714	1.2857	2.2857	3.5714	5.1429	7.0000

TABLE 7
Values of c_i for $n = 8$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.2500	.3750	-.5000	.6250	-.7500	.8750	-1.0000
2	-.1429	-.0536	.5714	-1.5179	3.0000	-5.1250	8.0000
3	-.0357	-.3036	.5714	.0893	-3.2500	11.3750	-28.0000
4	.0714	-.3750	.0000	1.4464	-1.5000	-9.6250	56.0000
5	.1786	-.2679	-.6429	.8036	3.7500	-4.3750	-70.0000
6	.2857	.0179	-.8571	-1.3393	2.0000	16.6250	56.0000
7	.3929	.4821	-.1429	-2.2321	-6.7500	-14.8750	-28.0000
8	.5000	1.1250	2.0000	3.1250	4.5000	6.1250	8.0000

TABLE 8
Values of c_i for $n = 9$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.2222	.3333	-.4444	.5556	-.6667	.7778	-.8889
2	-.1389	.0000	.3889	-1.1111	2.2500	-3.8889	6.1111
3	-.0556	-.2143	.5079	-.2778	-1.5000	6.4444	-16.8889
4	.0278	-.3095	.1905	.8333	-1.9167	-1.5556	21.7778
5	.1111	-.2857	-.2857	1.0000	1.0000	-6.2222	-6.2222
6	.1944	-.1429	-.6429	.0000	2.7500	3.1111	-20.2222
7	.2778	.1190	-.6032	-1.3889	-.1667	7.7778	31.1111
8	.3611	.5000	.1111	-1.3889	-4.7500	-10.8889	-20.8889
9	.4444	1.0000	1.7778	2.7778	4.0000	5.4444	7.1111

TABLE 9
Values of c_i for $n = 10$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.2000	.3000	-.4000	.5000	-.6000	.7000	-.8000
2	-.1333	.0333	.2667	-.8333	1.7333	-3.0333	4.8000
3	-.0667	-.1500	.4333	-.4167	-.6000	3.7333	-10.6000
4	.0000	-.2500	.2667	.4167	-1.6000	1.0000	8.0000
5	.0667	-.2667	-.0667	.8333	-.2667	-3.7333	5.6000
6	.1333	-.2000	-.4000	.5000	1.6000	-1.8667	-11.2000
7	.2000	-.0500	-.5667	-.4167	1.6000	4.2000	-2.8000
8	.2667	.1833	-.4000	-1.2500	-1.0667	3.2667	17.6000
9	.3333	.5000	.2667	-.8333	-3.4000	-8.1667	-16.0000
10	.4000	.9000	1.6000	2.5000	3.6000	4.9000	6.4000

TABLE 10
Values of c_i for $n = 11$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1818	.2727	-.3636	.4545	-.5455	.6364	-.7273
2	-.1273	.0545	.1818	-.6364	1.3636	-2.4182	3.8545
3	-.0727	-.1030	.3636	-.4545	-.1212	2.1636	-6.8364
4	-.0182	-.2000	.2879	.1515	-1.1818	1.6545	2.2000
5	.0364	-.2364	.0606	.6061	-.7273	-1.6727	5.9636
6	.0909	-.2121	-.2121	.6061	.6061	-2.5455	-2.5455
7	.1455	-.1273	-.4242	.1212	1.4545	.5091	-7.1273
8	.2000	.0182	-.4697	-.6061	.7273	3.5636	3.0182
9	.2545	.2242	-.2424	-1.0606	-1.3939	.8909	9.8909
10	.3091	.4909	.3636	-.4545	-2.4545	-6.2364	-12.5091
11	.3636	.8182	1.4545	2.2727	3.2727	4.4545	5.8182

TABLE 11
Values of c_i for $n = 12$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1667	.2500	-.3333	.4167	-.5000	.5833	-.6667
2	-.1212	.0682	.1212	-.4924	1.0909	-1.9621	3.1515
3	-.0758	-.0682	.3030	-.4470	.1364	1.2197	-4.4848
4	-.0303	-.1591	.2828	-.0126	-.8182	1.6439	-.2424
5	.0152	-.2045	.1313	.4040	-.8182	-.4242	4.2121
6	.0606	-.2045	-.0808	.5556	.0000	-1.9697	1.2121
7	.1061	-.1591	-.2828	.3535	.9091	-1.0606	-4.2424
8	.1515	-.0682	-.4040	-.1313	1.0909	1.4848	-3.1515
9	.1970	.0682	-.3737	-.6692	.1364	2.5985	4.4848
10	.2424	.2500	-.1212	-.8712	-1.4545	-.3712	5.3333
11	.2879	.4773	.4242	-.1894	-1.7727	-4.8258	-9.9394
12	.3333	.7500	1.3333	2.0833	3.0000	4.0833	5.3333

TABLE 12
Values of c_i for $n = 13$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1538	.2308	-.3077	.3846	-.4615	.5385	-.6154
2	-.1154	.0769	.0769	-.3846	.8846	-1.6154	2.6154
3	-.0769	-.0420	.2517	-.4196	.2727	.6364	-2.9650
4	-.0385	-.1259	.2657	-.1119	-.5350	1.4196	-1.2028
5	.0000	-.1748	.1678	.2448	-.7552	.2448	2.5175
6	.0385	-.1888	.0070	.4545	-.3147	-1.2238	2.2727
7	.0769	-.1678	-.1678	.4196	.4196	-1.3986	-1.3986
8	.1154	-.1119	-.3077	.1399	.9091	.0000	-3.4965
9	.1538	-.0210	-.3636	-.2867	.7133	1.7133	-.5594
10	.1923	.1049	-.2867	-.6643	-.2413	1.7133	4.3357
11	.2308	.2657	-.0280	-.6993	-1.3916	-1.0280	2.5734
12	.2692	.4615	.4615	.0000	-1.2692	-3.7692	-8.0000
13	.3077	.6923	1.2308	1.9231	2.7692	3.7692	4.9231

TABLE 13
Values of c_i for $n = 14$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1429	.2143	-.2857	.3571	-.4286	.5000	-.5714
2	-.1099	.0824	.0440	-.3022	.7253	-1.3462	2.1978
3	-.0769	-.0220	.2088	-.3846	.3407	.2692	-1.9560
4	-.0440	-.0989	.2438	-.1698	-.3237	1.1503	-1.4945
5	-.0110	-.1484	.1838	.1254	-.6384	.5629	1.2747
6	.0220	-.1703	.0639	.3472	-.4496	-.6119	2.1978
7	.0549	-.1648	-.0809	.4046	.0749	-1.2238	.2747
8	.0879	-.1319	-.2158	.2697	.5994	-.6993	-2.1978
9	.1209	-.0714	-.3057	-.0225	.7742	.6119	-2.1978
10	.1538	.0165	-.3157	-.3741	.3896	1.5909	.8791
11	.1868	.1319	-.2108	-.6244	-.4705	1.0035	3.6484
12	.2198	.2747	.0440	-.5495	-1.2747	-1.3462	.8791
13	.2527	.4451	.4835	.1374	-.8901	-2.9615	-6.5055
14	.2857	.6429	1.1429	1.7857	2.5714	3.5000	4.5714

TABLE 14
Values of c_i for $n = 15$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1333	.2000	-.2667	.3333	-.4000	.4667	-.5333
2	-.1048	.0857	.0190	-.2381	.6000	-1.1333	1.8667
3	-.0762	-.0066	.1729	-.3480	.3692	.0359	-1.2718
4	-.0476	-.0769	.2205	-.2015	-.1692	.8974	-1.4872
5	-.0190	-.1253	.1875	.0383	-.5119	.6821	.4513
6	.0095	-.1516	.0996	.2501	-.4825	-.1795	1.7436
7	.0381	-.1560	-.0176	.3546	-.1399	-.8974	1.0256
8	.0667	-.1385	-.1385	.3147	.3147	-.8974	-.8974
9	.0952	-.0989	-.2374	.1349	.6294	-.1282	-2.0513
10	.1238	-.0374	-.2886	-.1382	.5944	.8974	-1.0256
11	.1524	.0462	-.2667	-.4159	.1343	1.3282	1.5590
12	.1810	.1516	-.1458	-.5678	-.6000	.4667	2.8513
13	.2095	.2791	.0996	-.4212	-1.1385	-1.4718	-.1641
14	.2381	.4286	.4952	.2381	-.6000	-2.3333	-5.3333
15	.2667	.6000	1.0667	1.6667	2.4000	3.2667	4.2667

TABLE 15
Values of c_i for $n = 16$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1250	.1875	-.2500	.3125	-.3750	.4375	-.5000
2	-.1000	.0875	.0000	-.1875	.5000	-.9625	1.6000
3	-.0750	.0054	.1429	-.3125	.3750	-.1125	-.8000
4	-.0500	-.0589	.1978	-.2163	-.0577	.6798	-1.3538
5	-.0250	-.1054	.1841	-.0240	-.3942	.6933	-.0615
6	.0000	-.1339	.1209	.1683	-.4615	.1010	1.2308
7	.0250	-.1446	.0275	.2933	-.2596	-.5721	1.2308
8	.0500	-.1375	-.0769	.3125	.0962	-.8413	.0000
9	.0750	-.1125	-.1731	.2163	.4327	-.5048	-1.3462
10	.1000	-.0696	-.2418	.0240	.5769	.2644	-1.5385
11	.1250	-.0089	-.2637	-.2163	.4135	.9760	-.1538
12	.1500	.0696	-.2198	-.4279	-.0577	1.0298	1.7846
13	.1750	.1661	-.0907	-.5048	-.6635	.0740	2.1077
14	.2000	.2804	.1429	-.3125	-1.0000	-1.4875	-.8000
15	.2250	.4125	.5000	.3125	-.3750	-1.8375	-4.4000
16	.2500	.5625	1.0000	1.5625	2.2500	3.0625	4.0000

TABLE 16
Values of c_i for $n = 17$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1176	.1765	-.2353	.2941	-.3529	.4118	-.4706
2	-.0956	.0882	-.0147	-.1471	.4191	-.8235	1.3824
3	-.0735	.0147	.1176	-.2794	.3676	-.2059	-.4706
4	-.0515	-.0441	.1765	-.2206	.0221	.5000	-1.1765
5	-.0294	-.0882	.1765	-.0679	-.2919	.6493	-.3620
6	-.0074	-.1176	.1324	.1018	-.4146	.2692	.7783
7	.0147	-.1324	.0588	.2319	-.3156	-.3009	1.1584
8	.0368	-.1324	-.0294	.2862	-.0560	-.6810	.5068
9	.0588	-.1176	-.1176	.2489	.2489	-.6335	-.6335
10	.0809	-.0882	-.1912	.1244	.4666	-.1584	-1.3462
11	.1029	-.0441	-.2353	-.0622	.4853	.5000	-.9593
12	.1250	.0147	-.2353	-.2658	.2506	.9344	.4253
13	.1471	.0882	-.1765	-.4208	-.1968	.7443	1.7557
14	.1691	.1765	-.0441	-.4412	-.6838	-.2059	1.4706
15	.1912	.2794	.1765	-.2206	-.8676	-1.4412	-1.1765
16	.2132	.3971	.5000	.3676	-.1985	-1.4412	-3.6471
17	.2353	.5294	.9412	1.4706	2.1176	2.8824	3.7647

TABLE 17
Values of c_i for $n = 18$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1111	.1667	-.2222	.2778	-.3333	.3889	-.4444
2	-.0915	.0882	-.0261	-.1144	.3529	-.7092	1.2026
3	-.0719	.0221	.0964	-.2492	.3529	-.2631	-.2386
4	-.0523	-.0319	.1569	-.2181	.0784	.3546	-.9935
5	-.0327	-.0735	.1667	-.0980	-.2059	.5801	-.5229
6	-.0131	-.1029	.1373	.0490	-.3575	.3599	.4183
7	.0065	-.1201	.0801	.1757	-.3311	-.0941	.9673
8	.0261	-.1250	.0065	.2492	-.1538	-.4953	.7320
9	.0458	-.1176	-.0719	.2516	.0995	-.6194	-.0915
10	.0654	-.0980	-.1438	.1797	.3318	-.3871	-.9150
11	.0850	-.0662	-.1977	.0449	.4480	.1065	-1.1209
12	.1046	-.0221	-.2222	-.1266	.3801	.6170	-.4444
13	.1242	.0343	-.2059	-.2941	.1124	.8297	.7712
14	.1438	.1029	-.1373	-.4020	-.2941	.4918	1.5948
15	.1634	.1838	-.0049	-.3799	-.6765	-.4003	.9477
16	.1830	.2770	.2026	-.1430	-.7451	-1.3611	-1.3856
17	.2026	.3824	.4967	.4085	-.0588	-1.1209	-3.0327
18	.2222	.5000	.8889	1.3889	2.0000	2.7222	3.5556

TABLE 18
Values of c_i for $n = 19$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1053	.1579	-.2105	.2632	-.3158	.3684	-.4211
2	-.0877	.0877	-.0351	-.0877	.2982	-.6140	1.0526
3	-.0702	.0279	.0784	-.2219	.3344	-.2962	-.0743
4	-.0526	-.0217	.1391	-.2116	.1176	.2384	-.8220
5	-.0351	-.0609	.1558	-.1182	-.1352	.5021	-.5944
6	-.0175	-.0898	.1378	.0077	-.2988	.3989	.1486
7	.0000	-.1084	.0939	.1264	-.3220	.0542	.7430
8	.0175	-.1166	.0333	.2090	-.2110	-.3215	.7802
9	.0351	-.1146	-.0351	.2374	-.0124	-.5346	.2601
10	.0526	-.1022	-.1022	.2043	.2043	-.4768	-.4768
11	.0702	-.0795	-.1589	.1135	.3633	-.1589	-.9536
12	.0877	-.0464	-.1963	-.0206	.3994	.2890	-.8127
13	.1053	-.0031	-.2054	-.1729	.2755	.6517	-.0372
14	.1228	.0506	-.1770	-.3070	-.0005	.6971	.9474
15	.1404	.1146	-.1022	-.3762	-.3591	.2781	1.3746
16	.1579	.1889	.0281	-.3225	-.6517	-.5310	.5294
17	.1754	.2735	.2229	-.0774	-.6336	-1.2642	-1.4861
18	.1930	.3684	.4912	.4386	.0526	-.8596	-2.5263
19	.2105	.4737	.8421	1.3158	1.8947	2.5789	3.3684

TABLE 19
Values of c_i for $n = 20$

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1	-.1000	.1500	-.2000	.2500	-.3000	.3500	-.4000
2	-.0842	.0868	-.0421	-.0658	.2526	-.5342	.9263
3	-.0684	.0325	.0632	-.1974	.3140	-.3132	.0421
4	-.0526	-.0132	.1230	-.2025	.1443	.1463	-.6687
5	-.0368	-.0500	.1447	-.1309	-.0779	.4242	-.6102
6	-.0211	-.0781	.1354	-.0241	-.2429	.4047	-.0446
7	-.0053	-.0974	.1024	.0845	-.2989	.1549	.5276
8	.0105	-.1079	.0528	.1696	-.2393	-.1739	.7282
9	.0263	-.1096	-.0060	.2141	-.0912	-.4226	.4551
10	.0421	-.1026	-.0669	.2090	.0966	-.4789	-.1214
11	.0579	-.0868	-.1226	.1533	.2656	-.3099	-.6675
12	.0737	-.0623	-.1659	.0542	.3595	.0282	-.8495
13	.0895	-.0289	-.1897	-.0729	.3361	.4015	-.5006
14	.1053	.0132	-.1866	-.2045	.1789	.6327	.2601
15	.1211	.0640	-.1494	-.3090	-.0901	.5575	1.0068
16	.1368	.1237	-.0710	-.3466	-.3997	.1024	1.1368
17	.1526	.1921	.0559	-.2696	-.6164	-.6144	.2006
18	.1684	.2693	.2386	-.0219	-.5333	-1.1605	-1.5158
19	.1842	.3553	.4842	.4605	.1421	-.6447	-2.1053
20	.2000	.4500	.8000	1.2500	1.8000	2.4500	3.2000

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