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Mesh Modification for Evolution Equations

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Abstract. Finite element methods for which the underlying function spaces change with time are studied. The error estimates produced are all in norms that are very naturally associated with the problems. In some cases the Galerkin solution error can be seen to be quasi-optimal. K. Miller's moving finite element method is studied in one space dimension; convergence is proved for the case of smooth solutions of parabolic problems. Most, but not all, of the analysis is done on linear problems. Although second order parabolic equations are emphasized, there is also some work on first order hyperbolic and Sobolev equations.

1. Introduction. Finite element methods usually fail to perform well on problems whose solutions are too rough to be approximated well in the space of trial functions. Typically, computed solutions will oscillate unacceptably near regions of rapid change when too coarse a mesh is used, or if sufficient dissipation is added to control the oscillations, then the front is smeared.

The most straightforward solution to this difficulty is to include sufficient flexibility to match the solution to a reasonable level of accuracy. This approach works well for problems whose roughness is concentrated in a fixed small part of the region being studied. For many important problems the solutions are rough in a very small fraction of the underlying domain, but the area of roughness sweeps out a substantial part of the total region over the life of the problem. For fixed-mesh finite element methods these problems would require the use of great flexibility over essentially the entire domain, and this is frequently too expensive to be a useful approach.

Consider the following problems as possible examples in which some form of time-dependent mesh might be useful.

The displacement in a porous medium of one fluid by another that is miscible with it is frequently simulated using an equation of the form

$$\varphi u_t + v \cdot \nabla u - \nabla \cdot D \nabla u = 0, \tag{1.1}$$

where $u(x, t)$ is the concentration of the displacing fluid, $\varphi = \varphi(x)$ is the porosity, ∇ is the gradient with respect to the spatial variables, $v = v(x, t)$ is an underlying flow field, and D is a diffusivity matrix. This equation is the simplest example of the many types of models that are used in petroleum engineering. For some realistic situations the solution u is very nearly piecewise constant; u is approximately one or zero over most of the region with a rather narrow transition region. The transition region is an area of roughness of u that sweeps out a large portion of the reservoir during the course of the problem.

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Conservation laws of the form

$$(1.2) \qquad u_t + (f(u))_x - \varepsilon u_{xx} = 0$$

are used to model a wide variety of phenomena. For such equations one or more "near shocks" can develop. It is necessary when $\varepsilon > 0$ is quite small to use a very fine mesh close to these near shocks or to add dissipation in some form in these regions. Direct application of the most elementary finite element methods without these precautions can give solutions that have properties that are qualitatively in error.

Two phase flow in pipes is a problem that has attracted much interest lately. There is no consensus as yet as to the proper equations to use in describing such flows, but there are many sets that have been proposed. (See [23] for a recent survey.) Since the physical systems involved have sharp fronts that sweep out long lengths of pipe, a good mathematical model should have that property too. Most of the models proposed consist of functional relations together with first order systems with small dissipation terms. Thus they have some of the properties of the conservation law (1.2).

Another motivation for considering changing meshes for evolution equations is that optimal or near optimal meshes for steady state problems can be computed if the rules by which the mesh evolves are properly chosen. This seems particularly interesting for singular perturbation problems.

In this paper I consider a combination of two fundamental techniques for mesh modification. The two methods can be characterized as *continuous* and *discontinuous* changes in the underlying space. The prototype of a continuous change is a finite element space in which the elements are being smoothly deformed with time. While in the case of purely discontinuous changes the underlying function space is held fixed for a period of time and then abruptly changed to another.

There has been a considerable amount of work, both theoretical and experimental, on changing meshes for time-dependent problems, An early practical demonstration of the utility of changing the function space was given by H. S. Price and R. S. Varga [20]. Shortly thereafter J. Douglas and I proved in [8] that a finite number of mesh changes could be tolerated without loss in the rate of convergence.

P. Jamet in [12] introduced a general class of Galerkin-like methods for parabolic problems. He proved optimal order convergence results under a mild constraint on the number of discontinuous changes in the function spaces. Jamet's work with R. Bonnerot on the Stefan problem [4], [5], [6] presents a chain of ideas that has culminated in a method that has a continuously moving mesh that tracks the fronts in a multiphase Stefan problem; their method also admits discontinuous mesh changes. They used a related approach in [7] for compressible flow calculations, and Jamet has analyzed a one-dimensional parabolic analog in [13].

K. Miller and R. Miller introduced the moving finite element method in [17] and K. Miller provided interesting examples of the application of this method in [18]. This technique gives a general principle by which nodes are to be moved and seems to be applicable to a very wide range of problems [2], [11].

D. R. Lynch and W. G. Gray [16] derive finite element methods for deforming meshes and apply these to shallow water equations. They use the flow velocity of the fluid to move the knots as though they were neutral-density chips floating in the fluid. Their paper contains several pages of discussion of the history of moving meshes in the context of finite difference and finite element methods.

In [19] K. O'Neill and D. R. Lynch use a moving mesh procedure for a convection-diffusion equation. They used the given flow to move the mesh points near the local roughness in their one-dimensional example.

In [15] O. K. Jensen and B. A. Finlayson indicate the economies that can be had by moving the mesh in a chemical flooding problem. They actually choose to move the domain of the problem across a fixed mesh that covers a larger domain. In [14] they apply a moving coordinate system approach (translation of the domain) to solve transport equations. The rate at which the coordinates change can either be a fixed constant or adaptively defined.

Throughout the first six sections of this paper I use rather standard notation for the Sobolev spaces and their norms. For $1 \leqslant p \leqslant \infty$ and m a nonnegative integer, $W^{m,p}(\Omega)$ will be used to denote the usual Sobolev spaces [1]. Also, $H^m(\Omega)$ is the same as $W^{m,2}(\Omega)$. The norm on $H^0(\Omega) = L^2(\Omega)$ will be denoted by $\|\cdot\|$ or $\|\cdot\|_{L^2(\Omega)}$. The space $H^{-m}(\Omega)$ is defined to be the dual to $H^m(\Omega)$; this is not exactly the universal choice. No fractional order spaces are used in this paper.

For functions ψ from an interval J into a norm space X, with norm $\|\cdot\|_X$, we use the notation

$$\|\psi\|_{L^p(J;\,X)} = \left(\int_J \|\psi\|_X^p \, ds\right)^{1/p}$$

with the usual $p = \infty$ modification.

Section 2 treats parabolic Galerkin methods both in the case of continuous-time and discrete-time approximations. A theorem is proved that, for a particular norm, reduces the estimation of the error in the Galerkin solution to a question in approximation theory. The theorems in this section were constructed specifically for the case when the finite-dimensional function spaces are changing, but these particular theorems are new in the case of a fixed function space.

Section 3 contains asymptotic error bounds that are obtained from the results of Section 2.2. In Section 3.2 the results of Section 2.2 are generalized to include certain nonlinear parabolic problems with nonlinear Neumann boundary conditions.

Section 4 presents an example to show that mesh changes, when completely uncontrolled, can cause convergence to the wrong function. In Section 5 a Galerkin method for first order equations is examined.

Section 6 is devoted to an analysis of K. Miller's moving finite element method [17]. Only the continuous-time case is treated. An existence and stability result is given and then asymptotic error estimates are proved for smooth solutions. The order of convergence for this method on smooth solutions is optimal.

Section 7 looks briefly at Sobolev equations.

2. Basic Results for Parabolic Galerkin Methods. Let Ω be a bounded domain in R^d with piecewise smooth boundary. For $T > 0$ set $Q = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times (0, T)$. This section deals primarily with the approximate solution of the following parabolic problem:

$$(2.1)\qquad \begin{aligned} &u_t + bu + v \cdot \nabla u - \nabla \cdot (a \nabla u) = f \quad \text{on } Q, \\ &a\frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma, \qquad u(x,0) = u_0(x) \quad \text{on } \Omega, \end{aligned}$$

where ∇ is the spatial gradient operator and b, v, a, f, g, and u_0 are given smooth functions. The outward normal to $\partial\Omega$ is ν. Assume that the function a is uniformly positive on Q. Adopt the notation

$$(2.2)\qquad (\varphi, \psi) = \int_\Omega \varphi(x)\psi(x)\,dx, \qquad \langle \varphi, \psi \rangle = \int_{\partial\Omega} \varphi(x)\psi(x)\,d\sigma(x),$$

and let

$$(2.3)\qquad B(\varphi, \psi) = B(t; \varphi, \psi) = \int_\Omega [b\varphi\psi + (v \cdot \nabla\varphi)\psi + a\nabla\varphi \cdot \nabla\psi]\,dx.$$

The problem (2.1) can then be posed as

$$(2.4)\qquad \begin{aligned} &(u_t, \psi) + B(u, \psi) = (f, \psi) + \langle g, \psi \rangle, \qquad \psi \in H^1(\Omega), 0 < t \leqslant T, \\ &u(\cdot, 0) = u_0. \end{aligned}$$

2.1. *Continuous-Time Galerkin Approximations.* Partition $[0, T]$ using $T_0 = 0 < T_1 < \cdots < T_M = T$, and let $J_j = [T_{j-1}, T_j)$. Suppose that for each $t \in [0, T]$ $\mathfrak{M}(t)$ is a finite-dimensional subspace of $H^1(\Omega)$. Suppose further that $\mathfrak{M}(t)$ varies smoothly on each J_j in the following sense: for $j = 1, \ldots, M$, there exists $\{\psi_{k,j}(\cdot, t)$: $k = 1, \ldots, N_j\} \subset \mathfrak{M}(t)$ for $t \in J_j$ such that $\{\psi_{k,j}(\cdot, t)$: $k = 1, \ldots, N_j\}$ is a basis for $\mathfrak{M}(t)$, $t \to \psi_{k,j}(\cdot, t)$ is continuously differentiable as a map of J_j into $L^2(\Omega)$, and the derivative is bounded. Further suppose that there exists a constant $\tilde{C}$ such that for $0 \leqslant t \leqslant T$

$$(2.5\text{i})\qquad \|\varphi\|_{H^1(\Omega)} \leqslant \tilde{C}\|\varphi\|, \qquad \varphi \in \mathfrak{M}(t),$$

$$(2.5\text{ii})\qquad \sum_{k=1}^{N_j} \alpha_{k,j}^2 \leqslant \tilde{C}\left\|\sum_{k=1}^{N_j} \alpha_{k,j}\psi_{k,j}\right\|^2 \quad \text{on } J_j, j = 1, \ldots, M.$$

The constant $\tilde{C}$ in (2.5) will not enter into the estimates below except through its existence. I.e., the size of $\tilde{C}$ is not important, but if it were unbounded there would be technical complications. (When we look at the time-discrete versions of this process the existence of $\tilde{C}$ will play no role at all.)

Next we define a function space $\mathfrak{M}$ that will contain the approximate solution. $\mathfrak{M}$ consists of certain functions V defined on $[0, T)$ such that for each $t \in [0, T)$ $V(t) \in \mathfrak{M}(t)$. We suppose that $V|\,J_j$ for $V \in \mathfrak{M}$ is uniformly Lipschitz as a map into $L^2(\Omega)$. Further we suppose that each $V \in \mathfrak{M}$ is such that the jump in V at T_j, $V|_{T_j}$, is orthogonal to $\mathfrak{M}(T_j), j = 1, \ldots, M - 1$.

The continuous-time Galerkin approximation U of u is defined to be an element of $\mathfrak{M}$ such that

$$(2.6)\qquad \begin{aligned} &(U(0) - u(0), \chi) = 0, \qquad \chi \in \mathfrak{M}(0), \\ &(U_t, \chi) + B(U, \chi) = (f, \chi) + \langle g, \chi \rangle, \qquad \chi \in \mathfrak{M}(t), 0 \leqslant t \leqslant T. \end{aligned}$$

Under the above assumptions it is easy to see that U exists and that U is C^1 on each J_j.

2.2. *A Symmetric Error Estimate.* For functions ψ: $[0, T] \to H^1(\Omega)$ that are piecewise smooth let

$$\text{(2.7)} \quad |||\psi|||^2 = \|\psi\|^2_{L^\infty(0,T;\, L^2(\Omega))} + \|\psi\|^2_{L^2(0,T;\, H^1(\Omega))} + \sum_{j=1}^{M} \int_{J_j} \|\psi_t(t)\|^2_{H^{-1}(\Omega,\mathfrak{M}(t))}\, dt,$$

where

$$\text{(2.8)} \quad \|\psi\|_{H^{-1}(\Omega,\mathfrak{M}(t))} = \sup_{\substack{\|\chi\|_{H^1}=1 \\ \chi\in\mathfrak{M}(t)}} (\psi, \chi).$$

The seminorm in the sum in (2.7) makes the norm $|||\cdot|||$ depend on the space $\mathfrak{M}$. It would be preferable if we could use a norm that was independent of $\mathfrak{M}$. At present, doing so seems to require an "inverse assumption" on the spaces $\mathfrak{M}(t)$ and that is something I want to avoid here. The $H^{-1}(\Omega, \mathfrak{M}(t))$ seminorm is clearly no bigger than the $H^{-1}(\Omega)$ norm, where the $H^{-1}(\Omega)$ norm is defined by duality to all of $H^1(\Omega)$ instead of just the subspace $\mathfrak{M}(t)$.

The norm $|||\cdot|||$ is naturally associated with the error involved in (2.6). In this norm the Galerkin process does as well as it is possible to do (up to a constant factor) given that the approximate solution must be in $\mathfrak{M}$. This fact is expressed in the theorem below.

THEOREM 2.1. *There is a constant C, dependent on Q and the functions a, b, and v but independent of u and $\mathfrak{M}$, such that if u and U solve* (2.1) *and* (2.6), *respectively, then*

$$\text{(2.9)} \quad |||u - U||| \le C \inf\{|||u - V|||: V \in \mathfrak{M}\}.$$

Proof. Let V be in $\mathfrak{M}$, and define $\vartheta = U - V$ and $\eta = u - V$. Then for $t \in J_j$

$$\text{(2.10)} \quad (\vartheta_t, \chi) + B(\vartheta, \chi) = (\eta_t, \chi) + B(\eta, \chi), \qquad \chi \in \mathfrak{M}(t).$$

Use $\chi = \vartheta$ to see that for some positive constant $\underline{a}$

$$\text{(2.11)} \quad \frac{d}{dt}\|\vartheta\|^2 + \underline{a}\|\vartheta\|^2_{H^1(\Omega)} \le C\left[\|\eta_t\|^2_{H^{-1}(\Omega,\mathfrak{M}(t))} + \|\eta\|^2_{H^1(\Omega)} + \|\vartheta\|^2\right].$$

Next note that for $j > 1$

$$\text{(2.12)} \quad \|\vartheta(T_{j-1} - 0)\|^2 - \|\vartheta(T_{j-1})\|^2 \ge 0,$$

because $\vartheta(T_{j-1})$ is the $L^2(\Omega)$-projection into $\mathfrak{M}(T_{j-1})$ of $\vartheta(T_{j-1} - 0)$.

From (2.11) and (2.12) it follows easily that

$$\text{(2.13)} \quad \|\vartheta\|^2_{L^\infty(0,T;\, L^2(\Omega))} + \|\vartheta\|^2_{L^2(0,T;\, H^1(\Omega))} \le C\{|||\eta|||^2 + \|\vartheta(0)\|^2\}.$$

Next,the fact that

$$\vartheta = U - V = U - u + u - V = U - u + \eta$$

when combined with the choice of $U(0)$ gives

$$\text{(2.14)} \quad \|\vartheta(0)\| \le \|\eta(0)\|;$$

just inner product with $\vartheta(0)$ in the first part of (2.6) and apply the Cauchy inequality. Thus (2.13) becomes

$$\|\vartheta\|^2_{L^\infty(0,T;\,L^2(\Omega))} + \|\vartheta\|^2_{L^2(0,T;\,H^1(\Omega))} \leqslant C|||\eta|||^2. \tag{2.15}$$

The relation (2.10) implies that

$$\|\vartheta_t\|_{H^{-1}(\Omega,\mathfrak{M}(t))} \leqslant C\left[\|\vartheta\|_{H^1(\Omega)} + \|\eta_t\|_{H^{-1}(\Omega,\mathfrak{M}(t))} + \|\eta\|_{H^1(\Omega)}\right].$$

This together with (2.15) gives

$$|||\vartheta|||^2 \leqslant C|||\eta|||^2.$$

Now the triangle inequality and taking the infimum over V complete the proof. □

Most parabolic Galerkin error bounds give asymptotic rates of convergence. Such results almost always put a stronger norm on the solution u than on the error and express the difference in the strength of the two norms as h to some power, where h measures the size of the elements. Thus, while these asymptotic results may be properly balanced, they are almost never symmetric in the sense of Theorem 2.1.

One other symmetric error estimate can be found in my work with Jim Douglas, Jr., [8, Theorem 3.2]. There are also some results by A. Schatz, V. Thomée and L. Wahlbin that are asymptotic in nature but almost capture a symmetric result [21, relation 0.15]. Schatz, Thomée and Wahlbin show, under appropriate hypotheses, that the solution of a parabolic Galerkin process approximates the solution of the parabolic problem as well as possible up to a certain factor in the L^2 and L^∞ norms. If the factor were a constant this would be a symmetric or quasi-optimal result; however the constant involves a logarithm of the parameter h.

2.3. *Discrete-Time Error Estimates.* In some ways these discrete-time procedures are more elementary than the continuous-time process introduced before. There is however a time truncation term which makes the error estimates nonsymmetric.

Let $\tau = \{t_j\}_{j=0}^K$, with $0 = t_0 < t_1 < \cdots < t_K = T$, be a partition of $[0, T]$, and denote by Δt_j the difference $t_j - t_{j-1}$. Assume for each $j = 0, 1, \ldots, K$, that $\mathfrak{M}_j$ is a finite-dimensional subspace of $H^1(\Omega)$. The discrete-time solutions will be sequences $\{U_j\}_{j=0}^K$ where $U_j \in \mathfrak{M}_j$, $j = 0, \ldots, K$.

The two discrete-time procedures treated here are based on the first and second order correct backward difference formulas. For the first order correct backward difference the sequence $\{U_j\}$ will satisfy

$$\begin{aligned} &(U_0 - u(0), \chi) = 0, \qquad \chi \in \mathfrak{M}_0, \\ &(\partial_t U_j, \chi) + B(t_j; U_j, \chi) \\ &\qquad = \big(f(\cdot, t_j), \chi\big) + \big\langle g, (\cdot, t_j), \chi\big\rangle, \qquad \chi \in \mathfrak{M}_j, j = 1, \ldots, K, \end{aligned} \tag{2.16}$$

where

$$\partial_t U_j = (U_j - U_{j-1})/\Delta t_j. \tag{2.17}$$

It is immediate that for Δt_j sufficiently small (2.16) has a unique solution; it suffices to have $\Delta t_j < \varepsilon_0$, where for $0 \leqslant t \leqslant T$ and $\psi \in H^1(\Omega)$

$$\varepsilon_0^{-1}(\psi, \psi) + B(t; \psi, \psi) \geqslant 0. \tag{2.18}$$

I assume that the Δt_j's are all sufficiently small that (2.16) defines the sequence $\{U_j\}$. Adopt the notation $u_j(x) = u(x, t_j)$. Then

$$(2.19)\qquad \begin{aligned}&(\partial_t u_j, \chi) + B(t_j; u_j, \chi)\\ &\quad = \big(f(\cdot, t_j) + \rho_j, \chi\big) + \big\langle g(\cdot, t_j), \chi\big\rangle, \qquad \chi \in \mathfrak{M}_j,\end{aligned}$$

where

$$(2.20)\qquad \rho_j(x) = \partial_t u_j(x) - \frac{\partial u}{\partial t}(x, t_j).$$

In analogy with the previous section we define a norm $|||\cdot|||_\tau$ for all functions $\psi(x, t)$ defined for $t_j, j = 0,\dots,K$ and $x \in \Omega$ such that $\psi(\cdot, t_j) = \psi_j \in H^1(\Omega)$. The convenient definition is

$$(2.21)\qquad |||\psi|||_\tau^2 = \max_{0\le j\le K} \|\psi_j\|^2 + \sum_{j=1}^{K} \|\psi_j\|^2_{H^1(\Omega)}\Delta t_j + \sum_{j=1}^{K} \|\partial_t\psi_j\|^2_{H^{-1}(\Omega, \mathfrak{M}_j)}\Delta t_j.$$

This norm depends not only on the partition τ of $[0, T]$, but also on the sequence $\{\mathfrak{M}_j\}_{j=1}^K$ of spaces.

The following result gives a close analogue to Theorem 2.1 for this discrete-time case.

THEOREM 2.2. *There exist constants C and $\varepsilon > 0$, dependent on Q and on the functions a, b, and v but independent on u, τ, and $\{\mathfrak{M}_j\}$, such that if u and $\{U_j\}_{j=0}^K$ solve* (2.1) *and* (2.16), *respectively, then*

$$(2.22)\qquad \begin{aligned}|||u - U|||_\tau \le C\Bigg[&\inf\big\{|||u - V|||_\tau : V = \{V_j\}, V_j \in \mathfrak{M}_j\big\}\\ &+ \left(\sum_{j=1}^{K} \|\rho_j\|^2_{H^{-1}(\Omega, \mathfrak{M}_j)}\Delta t_j\right)^{1/2}\Bigg],\end{aligned}$$

provided $\Delta t_j \le \varepsilon$ for $j = 1,\dots,K$.

Proof. Let $V = \{V_j\}_{j=0}^K$ where $V_j \in \mathfrak{M}_j$, and define $\vartheta_j = U_j - V_j$ and $\eta_j = u_j - V_j$. Then, for $j = 1,\dots,K$,

$$(2.23)\qquad (\partial_t\vartheta_j, \chi) + B(t_j; \vartheta_j, \chi) = (\partial_t\eta_j - \rho_j, \chi) + B(t_j; \eta_j, \chi), \qquad \chi \in \mathfrak{M}_j.$$

Use $\chi = \vartheta_j$ in (2.23), and apply the identity

$$(2.24)\qquad (\partial_t\vartheta_j, \vartheta_j) = \frac{1}{2\Delta t_j}\left[\|\vartheta_j\|^2 - \|\vartheta_{j-1}\|^2\right] + \frac{\Delta t_j}{2}\|\partial_t\vartheta_j\|^2$$

to get

$$(2.25)\qquad \tfrac{1}{2}\partial_t\big(\|\vartheta_j\|^2\big) + B(\vartheta_j, \vartheta_j) \le C\Big[\|\partial_t\eta_j - \rho_j\|_{H^{-1}(\Omega, \mathfrak{M}_j)} + \|\eta_j\|_{H^1(\Omega)}\Big]\|\vartheta_j\|_{H^1(\Omega)}.$$

This relationship then gives, via the discrete Gronwall lemma,

$$(2.26)\qquad \max_{0\le j\le K} \|\vartheta_j\|^2 + \sum_{j=1}^{K} \|\vartheta_j\|^2_{H^1(\Omega)}\Delta t_j \le C\big[|||\eta|||_\tau^2 + \|\vartheta_0\|^2\big].$$

Note that it is this step that gives the $\Delta t_j \leq \varepsilon$ constraint. In (2.26) the $\|\vartheta_0\|^2$ term is treated using $\|\vartheta_0\| \leq \|\eta_0\|$ to remove it from (2.26). Then (2.23) is used to get

$$\|\partial_t \vartheta_j\|_{H^{-1}(\Omega,\mathfrak{M}_j)} \leq C\Big[\|\vartheta_j\|_{H^1(\Omega)} + \|\partial_t \eta\|_{H^{-1}(\Omega,\mathfrak{M}_j)} + \|\rho_j\|_{H^{-1}(\Omega,\mathfrak{M}_j)} + \|\eta\|_{H^1(\Omega)}\Big]. \tag{2.27}$$

Next (2.26), with $\|\vartheta_0\|^2$ removed, and (2.27) imply that

$$|||\vartheta|||_\tau^2 \leq C\left[|||\eta|||_\tau^2 + \sum_{j=1}^{K} \|\rho_j\|^2_{H^{-1}(\Omega,\mathfrak{M}_j)}\Delta t_j\right]. \tag{2.28}$$

Finally the triangle inequality and taking the infimum over $V = \{V_j\}$ complete the proof. □

The analysis of the first order correct backward difference scheme is very natural but the fact that the spaces are allowed to change every step restricts the types of argument that can be used. For example, a Crank-Nicolson difference scheme is most naturally treated using a test function V that is the average of elements of $\mathfrak{M}_j$ and $\mathfrak{M}_{j-1}$; this is not possible in this context. Also $H^1(\Omega)$-norm estimates can be derived when the $\mathfrak{M}_j$'s are fixed by using a time-difference test function; this is not possible in general when the $\mathfrak{M}_j$'s vary.

The second order correct backward difference method is

$$\begin{aligned}(\delta_2 U_j, V) &+ B(t_j; U_j, V)\\ &= \big(f(\cdot, t_j), V\big) + \big\langle g(\cdot, t_j), V\big\rangle, \qquad V \in \mathfrak{M}_j, j \geq 2,\end{aligned} \tag{2.29}$$

where

$$\delta_2 U_j = \partial_t U_j + \frac{\Delta t_j}{\Delta t_j + \Delta t_{j-1}}\big(\partial_t U_j - \partial_t U_{j-1}\big). \tag{2.30}$$

For this scheme only the constant stepsize case will be considered here. Also the choice of U_0 and U_1 will not be specified.

Note that u satisfies

$$(\delta_2 u_j, \chi) + B(t_j; u_j, \chi) = \big(f(\cdot, t_j) + \tilde{\rho}_j, \chi\big) + \big\langle g(\cdot, t_j), \chi\big\rangle, \qquad \chi \in \mathfrak{M}_j, \tag{2.31}$$

where

$$\tilde{\rho}_j(x) = \delta_2 u_j(x) - \frac{\partial u}{\partial t}(x, t_j). \tag{2.32}$$

For smooth functions u the time truncation $\tilde{\rho}_j$ will be $O(\Delta t_j(\Delta t_j + \Delta t_{j-1}))$. Define the analogue to $|||\cdot|||_\tau$ for this case:

$$|||c|||_{\tau,2}^2 = \max_{0 \leq j \leq K} \|\psi_j\|^2 + \sum_{j=2}^{K} \|\psi_j\|^2_{H^1(\Omega)}\Delta t_j + \sum_{j=2}^{K} \|\delta_2 \psi_j\|^2_{H^{-1}(\Omega,\mathfrak{M}_j)}\Delta t_j.$$

THEOREM 2.3. *Suppose that* $\Delta t_j = \Delta t = T/K$ *for* $j \geq 1$. *There exist constants* C *and* $\varepsilon > 0$ *such that for* u *and* $\{U_j\}$ *solutions of* (2.1) *and* (2.29), *respectively*,

$$|||u - U|||_{\tau,2} \leq C\Bigg[\inf\{|||u - V|||_{\tau,2}: V = \{V_j\}, V_j \in \mathfrak{M}_j\} \tag{2.33}$$
$$+ \left(\sum_{j=2}^{K} \|\tilde{\rho}_j\|^2_{H^{-1}(\Omega, \mathfrak{M}_j)}\Delta t + \|U_0 - u_0\|^2 + \|U_1 - u_1\|^2\right)^{1/2}\Bigg],$$

provided $\Delta t \leq \varepsilon$.

Proof. As before let $\vartheta_j = U_j - V_j$ and $\eta_j = u_j - V_j$, where $V_j \in \mathfrak{M}_j$. Note that the following identity is true:

$$(\delta_2\vartheta_j, \vartheta_j)\Delta t = 2(\vartheta_j - \vartheta_{j-1}, \vartheta_j) - \tfrac{1}{2}(\vartheta_j - \vartheta_{j-2}, \vartheta_j)$$
$$= \|\vartheta_j\|^2 - \|\vartheta_{j-1}\|^2 + \|\vartheta_j - \vartheta_{j-1}\|^2$$
$$- \tfrac{1}{4}\left[\|\vartheta_j\|^2 - \|\vartheta_{j-2}\|^2 + \|\vartheta_j - \vartheta_{j-2}\|^2\right].$$

This can now be summed on j to give, for $m \geq 3$,

$$\sum_{j=2}^{m} (\delta_2\vartheta_j, \vartheta_j)\Delta t = \|\vartheta_m\|^2 - \|\vartheta_1\|^2 + \sum_{2}^{m} \|\vartheta_j - \vartheta_{j-1}\|^2$$
$$- \tfrac{1}{4}\Bigg[\|\vartheta_m\|^2 + \|\vartheta_{m-1}\|^2 - \|\vartheta_1\|^2 - \|\vartheta_0\|^2 \tag{2.34}$$
$$+ \sum_{2}^{m} \|\vartheta_j - \vartheta_{j-2}\|^2\Bigg].$$

Since

$$\|\vartheta_j - \vartheta_{j-2}\|^2 \leq 2\left[\|\vartheta_j - \vartheta_{j-1}\|^2 + \|\vartheta_{j-1} - \vartheta_{j-2}\|^2\right],$$

it follows that

$$\sum_{j=2}^{m} (\delta_2\vartheta_j, \vartheta_j)\Delta t \geq \tfrac{3}{4}\|\vartheta_m\|^2 - \tfrac{1}{4}\|\vartheta_{m-1}\|^2 + \tfrac{1}{2}\|\vartheta_m - \vartheta_{m-1}\|^2$$
$$- \tfrac{3}{4}\|\vartheta_1\|^2 + \tfrac{1}{4}\|\vartheta_0\|^2 - \tfrac{1}{2}\|\vartheta_1 - \vartheta_0\|^2.$$

Next use

$$\tfrac{1}{2}\|\vartheta_m\|^2 - \tfrac{1}{4}\|\vartheta_{m-1}\|^2 + \tfrac{1}{2}\|\vartheta_m - \vartheta_{m-1}\|^2 = \|\vartheta_m - \tfrac{1}{2}\vartheta_{m-1}\|^2$$

to get that

$$\sum_{j=2}^{m} (\delta_2\vartheta_j, \vartheta_j)\Delta t \geq \tfrac{1}{4}\|\vartheta_m\|^2 - \left[\tfrac{7}{4}\|\vartheta_1\|^2 + \tfrac{5}{4}\|\vartheta_0\|^2\right]. \tag{2.35}$$

The inequality (2.35), together with the same estimates used for Theorem 2.2, implies that

$$\max_{2\leqslant j\leqslant K} \|\vartheta_j\|^2 + \sum_{j=2}^{K} \|\vartheta_j\|^2_{H^1(\Omega)}\Delta t$$

$$\leqslant C\Bigg[\|\vartheta_0\|^2 + \|\vartheta_1\|^2 + \sum_{j=2}^{K} \|\eta_j\|^2_{H^1(\Omega)}\Delta t \tag{2.36}$$

$$+ \sum_{j=2}^{K}\Big(\|\delta_2\eta_j\|^2_{H^{-1}(\Omega,\mathfrak{M}_j)} + \|\tilde{\rho}_j\|^2_{H^{-1}(\Omega,\mathfrak{M}_j)}\Big)\Delta t\Bigg].$$

Estimate the $\delta_2\vartheta_j$ term exactly as the $\partial_t\vartheta_j$ was bounded, and use the triangle inequality to complete the proof. □

3. Some Applications and Extensions. The results and arguments of the previous section are used here to get asymptotic error estimates and give an indication of the situation for some nonlinear equations. Attention is restricted to the continuous-time case in this section.

The results of Section 2 can be used to give asymptotic error bounds provided that additional structure is imposed on the finite-dimensional spaces. The constraints needed on the rate of change of the spaces seem quite reasonable. In the case of nonlinear equations with smooth solutions the results of Section 2.2 can be proved without major revision. A mild restriction is needed on Ω to be able to treat nonlinear boundary conditions.

3.1. *Some Asymptotic Error Bounds.* If the function spaces used in a Galerkin process are changed in a very wild manner, one might guess that the approximate solutions could converge extremely slowly, if at all. In fact, Section 4 of this paper gives an example for which they converge to the wrong function as the number of parameters is increased. The results of the previous section allow us to make some positive statements about asymptotic convergence in the presence of mesh changes.

The easiest result that follows from Theorem 2.1 is that the convergence rate due to a fixed underlying function space is not degraded because of the extra freedom that is added and/or removed from the computational spaces, no matter how uncontrolled these changes are. One way of expressing this is as follows: Suppose that $\tilde{\mathfrak{M}}$ is a finite-dimensional subspace of $H^1(\Omega)$ and that for each $t \in [0, T]$ $\tilde{\mathfrak{M}} \subset \mathfrak{M}(t)$. Let W: $[0, T] \to \tilde{\mathfrak{M}}$ be the continuous-time Galerkin approximation to u with $\mathfrak{M}(t) \equiv \tilde{\mathfrak{M}}$. Then

$$|||U - u||| \leqslant C|||W - u|||, \tag{3.1}$$

where U is the solution of (2.6) and C is the constant in Theorem 2.1.

While this particular result does not show any improvement over a fixed-mesh Galerkin procedure, it does indicate that one can safely play with the use of time-varying spaces to try to improve accuracy provided a good degree of error control is left in a fixed subspace.

As discussed in the introduction, one situation in which time-varying meshes seem useful is that in which the solution has a sharp front that propagates across the region. In this type of problem one reasonable rule to use is "add freedom before it is needed and remove it after it is no longer needed." With this approach the spaces $\mathfrak{M}(t)$ are fixed on the intervals $J_j = [T_{j-1}, T_j)$; let $\mathfrak{M}_j$ be the $\mathfrak{M}(t)$ on J_j. It seems fundamental to this technique that the solution at time T_j can be approximated well in the space $\tilde{\mathfrak{M}}_j = \mathfrak{M}_j \cap \mathfrak{M}_{j+1}$. Roughly speaking, if the spaces are chosen so that the extra freedom is kept slightly longer than it is really needed to approximate the solution by some element of the space, then the parabolic Galerkin approximation will be a faithful representation of the solution.

To make the foregoing remarks more precise suppose that the approximation properties of the spaces $\mathfrak{M}_j$ are known to satisfy the following conditions. There are constants C_1 and r, r is a positive integer, such that for each $j = 1, \ldots, M$, there is a map $V_j: J_j \to \mathfrak{M}_j$ such that

$$(3.2) \quad \max_{t \in J_j} \left[\|u - V_j\|_{H^1(\Omega)} + h^{-2} \|u - V_j\|_{H^{-1}(\Omega)} + \|u_t - V_{j,t}\|_{H^{-1}(\Omega)} \right] \leq C_1 h^r.$$

Here h should be thought of as a small parameter that measures the number of unknowns in the spaces rather than the maximum mesh spacing. Suppose in addition that

$$(3.3) \qquad T_j - T_{j-1} > 2h^2$$

and that on $\tilde{J}_j = [T_j - h^2, T_j + h^2]$ there is a mapping $\tilde{V}_j$ into $\tilde{\mathfrak{M}}_j$ satisfying (3.2) on $\tilde{J}_j$ instead of J_j.

Under the above assumptions the solution U of (2.6) satisfies

$$(3.4) \qquad |||U - u||| \leq C_2 C_1 h^r,$$

where C_2 depends only on T and the C of Theorem 2.1.

To verify this result one needs to construct a function $V \in \mathfrak{M}$ for which the $|||\cdot|||$ norm of the error is bounded by Ch^r. This is easily done by going linearly from V_j to $\tilde{V}_j$ on the interval $[T_j - h^2, T_j]$ and linearly from $\tilde{V}_j$ to V_{j+1} on $[T_j, T_j + h^2]$. Since for each time $V(t)$ is either a $V_j(t)$ or a convex combination of a $V_j(t)$ and a $\tilde{V}_k(t)$, both of which approximate u well, it is clear that the $L^2(\Omega)$ and $H^1(\Omega)$ parts of $|||u - V|||$ are bounded by Ch^r. The time-derivative part of $|||u - V|||$ is only slightly more complicated. On $[T_j - h^2, T_j]$, for example,

$$\begin{aligned} \|u_t - V_t\|_{H^{-1}(\Omega, \mathfrak{M}_j)} &\leq \|u_t - V_t\|_{H^{-1}(\Omega)} \\ (3.5) \qquad &\leq \max\left\{\|u_t - V_{j,t}\|_{H^{-1}(\Omega)}, \|u_t - \tilde{V}_{j,t}\|_{H^{-1}(\Omega)}\right\} + h^{-2}\|V_j - \tilde{V}_j\|_{H^{-1}(\Omega)} \\ &\leq Ch^r. \end{aligned}$$

The result (3.4) is very much like P. Jamet's result in [12], but (3.4) is derived under somewhat different hypotheses.

The special case of piecewise linear functions might seem not to be allowed for by (3.2) since the interpolant does not get $H^{-1}(\Omega)$ accuracy that is of a better order than its $L^2(\Omega)$ accuracy. This apparent difficulty can be overcome by constructing the functions V_j on macro elements in which one (interior) degree of freedom is used to match the average value over the macro. Then, provided the mesh is locally quasi-uniform, (3.2) is reasonable.

3.2. *Nonlinear Equations*. The error estimates of Section 2.2 can be carried over to the context of certain nonlinear equations with smooth solutions in very much the same way they were proved in [8].

Suppose that (2.1) is replaced by

$$(3.6)\quad u_t + \nabla \cdot (a(x,t,u)\nabla u) = f(x,t,u,\nabla u), \qquad a(x,t,u)\frac{\partial u}{\partial \nu} = g(x,t,u),$$

where a, f, and g are smooth functions of their arguments. Suppose that a is uniformly bounded above and below by positive constants and that a, f, and g are uniformly Lipschitz with respect to u. Also suppose that f is uniformly Lipschitz with respect to ∇u. Let

$$(3.7)\qquad B(t,\varphi;\psi,\xi) = \int_\Omega a(x,t,\varphi)\nabla\psi \cdot \nabla\xi\, dx.$$

For this problem the continuous-time Galerkin solution is defined to be $U \in \mathfrak{M}$ satisfying

$$(3.8)\qquad \begin{aligned} &(u(0) - U(0), \chi) = 0, \qquad \chi \in \mathfrak{M}(0),\\ &(U_t, \chi) + B(t,U;U,\chi)\\ &\qquad = (f(\cdot,t,U,\nabla U),\chi) + \langle g(\cdot,t,U),\chi\rangle, \qquad \chi \in \mathfrak{M}(t). \end{aligned}$$

The space $\mathfrak{M}$ is the space introduced in Section 2.1.

In order to treat the nonlinearity on the boundary, the argument below uses a trace inequality. Suppose that Ω is such that for each $\varepsilon > 0$ there is a $C(\varepsilon)$ such that for $\psi \in H^1(\Omega)$

$$(3.9)\qquad \|\psi\|^2_{L^2(\partial\Omega)} \leq \varepsilon\|\psi\|^2_{H^1(\Omega)} + C(\varepsilon)\|\psi\|^2.$$

We assume that $\partial\Omega$ is locally Lipschitz and that Ω is bounded; this implies (3.9).

The analogue of Theorem 2.1 holds in this case provided we restrict the infimum to V's in $\mathfrak{M}$ that have bounded gradients.

For $p \geq 2$ and $L \geq 0$ let

$$(3.10)\qquad \mathfrak{M}_{p,L} = \{V \in \mathfrak{M}: \|V(t)\|_{W^{1,p}(\Omega)} \leq L,\ 0 \leq t \leq T\}.$$

THEOREM 3.1. *Suppose that, for some $p > d$ (recall that $\Omega \subset R^d$) with $p \geq 2$, each $\mathfrak{M}(t)$ is a subspace of $W^{1,p}(\Omega)$. Let $L > 0$ be given. Then there exists a constant C such that*

$$(3.11)\qquad |||u - U||| \leq C\inf\{|||u - V|||: V \in \mathfrak{M}_{p,L}\}.$$

Proof. Let V be in $\mathfrak{M}_{p,L}$, and define $\vartheta = U - V$ and $\eta = u - V$. Then the analogue of (2.10) is

$$(3.12)\qquad \begin{aligned} &(\vartheta_t,\chi) + B(t,U;\vartheta,\chi)\\ &\quad = (\eta_t,\chi) + B(t,u;\eta,\chi) + [B(t,u;V,\chi) - B(t,U;V,\chi)]\\ &\qquad + (f(\cdot,t,U,\nabla U) - f(\cdot,t,u,\nabla u),\chi)\\ &\qquad + \langle g(\cdot,t,U) - g(\cdot,t,u),\chi\rangle, \qquad \chi \in \mathfrak{M}(t). \end{aligned}$$

As before use $\chi = \vartheta$. The first two terms on the right-hand side of (3.12) are bounded as in the proof of Theorem 2.1. The difference of the f's is treated easily using the Lipschitz continuity of f to get

$$(3.13)\quad (f(\cdot,t,U,\nabla U) - f(\cdot,t,u,\nabla u),\vartheta) \leq \varepsilon\|\vartheta\|^2_{H^1} + C[\|\vartheta\|^2 + \|\eta\|^2_{H^1}].$$

The boundary term in (3.12) is bounded using the trace inequality (3.9) to get

$$\begin{aligned}\langle g(\cdot,t,U)-g(\cdot,t,u),\vartheta\rangle &\leq C\big[\|\vartheta\|_{L^2(\partial\Omega)}+\|\eta\|_{L^2(\partial\Omega)}\big]\|\vartheta\|_{L^2(\partial\Omega)}\\ &\leq \varepsilon\|\vartheta\|^2_{H^1(\Omega)}+C\big[\|\vartheta\|^2+\|\eta\|^2_{H^1(\Omega)}\big].\end{aligned} \tag{3.14}$$

Let q be defined by

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{2}=1.$$

Then

$$\frac{1}{q}=\frac{1}{2}-\frac{1}{p}>\frac{1}{2}-\frac{1}{d}. \tag{3.15}$$

Hence for $\varepsilon>0$ there is $C(\varepsilon)$ such that for $\psi\in H^1(\Omega)$

$$\|\psi\|_{L^q(\Omega)}\leq\varepsilon\|\psi\|_{H^1(\Omega)}+C(\varepsilon)\|\psi\|. \tag{3.16}$$

Now use the Lipschitz continuity of $a(x,t,u)$ with respect to u to get that

$$\begin{aligned}B(t,u;V,\vartheta)-B(t,U;V,\vartheta) &\leq C\|u-U\|_{L^q(\Omega)}\|V\|_{W^{1,p}(\Omega)}\|\vartheta\|_{H^1}\\ &\leq \varepsilon\|\vartheta\|^2_{H^1}+C(\varepsilon,L)\big[\|\vartheta\|^2+\|\eta\|^2_{H^1}\big].\end{aligned} \tag{3.17}$$

The inequalities (3.13), (3.14), and (3.17) used with (3.12) with $\chi=\vartheta$ then give (for suitable choices of ε) the inequality (2.11). The remainder of the proof is almost exactly like that of Theorem 2.1. □

4. A Counterexample. To illustrate that changing the mesh in a completely uncontrolled way can cause convergence to the wrong answer a single example suffices.

For simplicity this counterexample is constructed for a periodic problem in one space dimension. The first-order-correct backward-difference formulation will be used but the same result can be demonstrated for the continuous-time Galerkin process.

Let $u(x,t)$ for $t\geq 0$ be the solution of the problem

$$\begin{aligned}&\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0, \qquad x\in R,\ t>0,\\ &u(x,0)=\sin(2\pi x), \qquad x\in R,\\ &u(\cdot,t)\quad \text{1-periodic in } R.\end{aligned} \tag{4.1}$$

For $j=0,1,2,\ldots,$ let $\mathfrak{M}_j$ be the 1-periodic piecewise linear functions over the mesh $x_i=(i+\frac{1}{2}j)\Delta x$, $i=0,\pm1,\pm2,\ldots,$ where $\Delta x=1/N$. Let $t_j=j\Delta t$ for $\Delta t>0$ given. Let $U_j\in\mathfrak{M}_j$ be such that

$$(\partial_t U_j,\chi)+\left(\frac{\partial}{\partial x}U_j,\frac{\partial\chi}{\partial x}\right)=0, \qquad \chi\in\mathfrak{M}_j, \tag{4.2}$$

where the inner product is taken over (0, 1), say. Just as in Section 2.3, take U_0 to be the $L^2(\Omega)$-projection of the initial data.

First note that the average value of each U_j is zero since $\chi\equiv1$ is a possible test function and $(U_0,1)=0$. With $\chi=U_j$ (4.2) gives that

$$\frac{1}{2\Delta t}\big[\|U_j\|^2-\|U_{j-1}\|^2\big]+\frac{1}{2\Delta t}\|U_j-U_{j-1}\|^2+\left\|\frac{\partial}{\partial x}U_j\right\|^2=0,$$

where the L^2 norm is taken over (0, 1). This gives that

$$\|U_j\|^2 - \|U_{j-1}\|^2 + \|U_j - U_{j-1}\|^2 \leq 0. \tag{4.3}$$

If $\tilde{\mathfrak{M}}_j$ is the space of possibly discontinuous 1-periodic piecewise linear functions over the same mesh as the space $\mathfrak{M}_j$, then

$$\|U_j - U_{j-1}\| \geq \|U_j - W_{j-1}\|, \tag{4.4}$$

where W_{j-1} is the L^2-projection of U_j into $\tilde{\mathfrak{M}}_{j-1}$. For the points x_i in the mesh for $\mathfrak{M}_j$ let

$$\partial_x^2 U_j(x_i) = (\Delta x)^{-2}\big(U(x_i + \Delta x) - 2U(x_i) + U(x_i - \Delta x)\big).$$

A computation shows that

$$\|U_j - W_{j-1}\|^2 = \frac{1}{192}(\Delta x)^4 \sum_{i=1}^{N} \big(\partial_x^2 U_j(x_i)\big)^2 \Delta x. \tag{4.5}$$

Because U_j has average value zero and is 1-periodic, another calculation shows that

$$\|U_j\|^2 \leq \frac{1}{768} \sum_{i=1}^{N} \big(\partial_x^2 U_j(x_i)\big)^2 \Delta x. \tag{4.6}$$

Thus,

$$\|U_j - W_{j-1}\|^2 \geq 4(\Delta x)^4 \|U_j\|^2. \tag{4.7}$$

This relation, (4.4), and (4.3) imply that

$$\|U_j\|^2 \leq \frac{1}{1 + 4(\Delta x)^4} \|U_{j-1}\|^2. \tag{4.8}$$

Hence

$$\|U_j\|^2 \leq \tfrac{1}{2}\big[1 + 4(\Delta x)^4\big]^{-j}. \tag{4.9}$$

Now fix any $t > 0$. Then

$$\max_{j\Delta t > t} \|U_j\|^2 \leq \tfrac{1}{2}\big[1 + 4(\Delta x)^4\big]^{-t/\Delta t}. \tag{4.10}$$

If Δt and Δx tend to zero in such a way that $(\Delta x)^4/\Delta t \to \infty$, then for any $t > 0$

$$\max_{j\Delta t > t} \|U_j\|^2 \to 0. \tag{4.11}$$

This scheme for changing the mesh is clearly not a good one to use. It does point out that changing meshes results in dissipation. In this case the mesh change seems to look something like the operator

$$\big((\Delta x)^4/\Delta t\big)\left(\frac{\partial}{\partial x}\right)^4.$$

5. First Order Equations. For an equation of the form

$$u_t + v(x, t)u_x + b(x, t)u = f(x, t), \qquad 0 < x < 1, t < 0, \tag{5.1}$$

where $v(x, t) > 0$, with initial-boundary conditions

$$u(x, 0) = u_0(x), \qquad 0 \leq x \leq 1, \qquad u(0, t) = g_0(t), \qquad t \geq 0, \tag{5.2}$$

there is a natural Galerkin method (and several that are less than natural). The purpose of this section is to indicate that the corresponding natural method based on changing function spaces has the same types of properties as its fixed function space prototype.

Let the problem (5.1)–(5.2) be transformed so that $g_0 \equiv 0$; this will in general involve changing f and the meaning of u. Now suppose that (5.1)–(5.2) has a weak formulation as follows: Find u: $[0, T] \to \mathcal{H}$, where $\mathcal{H}$ is the subspace of $H^1(0,1)$ consisting of all $\psi(x)$ with $\psi(0)$, such that

(5.3) $$(u_t, \chi) + B(u, \chi) = (f, \chi), \qquad \chi \in \mathcal{H}, t > 0, \qquad u(\cdot, t) = u_0.$$

Here

$$B(u, \chi) = \int_0^1 [\nu u_x \chi + bu\chi]\, dx.$$

There are two important properties of the bilinear form B:

(5.4a) $$B(\varphi, \varphi) \leqslant C \|\varphi\|^2, \qquad \varphi \in \mathcal{H},$$

(5.4b) $$B(\varphi, \psi) \leqslant C \|\varphi\|_{H^1(0,1)} \|\psi\|, \qquad \varphi \in H^1(0,1), \psi \in L^2(0,1).$$

With these two properties the analysis for changing function spaces is very much like that expounded by B. Swartz and B. Wendroff [22].

Suppose now that $\Omega \subset R^d$ is a domain and that $\mathcal{H}$ is a given subspace of $H^1(\Omega)$. Suppose that u: $[0, T] \to \mathcal{H}$ is continuously differentiable. (This is more than minimal but we cannot get useful convergence results for minimally smooth functions here.) And suppose that $B(t; \cdot, \cdot)$ is a bilinear form on $\mathcal{H} \times \mathcal{H}$ such that (5.4a) and (5.4b) hold for each $t \in [0, T]$ and the constant C can be taken to be independent of t. Suppose that u satisfies

(5.5) $$(u_t, \chi) + B(u, \chi) = (f, \chi), \qquad \chi \in \mathcal{H},$$

where f is a continuous map of $[0, T]$ into $L^2(\Omega)$. Let

(5.6) $$u_0(x) = u(x, 0), \qquad x \in \Omega.$$

Now suppose that the spaces $\mathfrak{M}(t)$ of Section (2.1) are subspaces of $\mathcal{H}$. Define $U \in \mathfrak{M}$ by

(5.7) $$\begin{aligned} (U_t, \chi) + B(U, \chi) &= (f, \chi), \qquad \chi \in \mathfrak{M}(t), t \in [0, T], \\ (U(0) - u_0, \chi) &= 0, \qquad \chi \in \mathfrak{M}(0). \end{aligned}$$

In this case we get an error estimate that is analogous to Theorem 2.1 but which is not symmetric; at this level of generality this is as it should be [9]. Let, for all sufficiently regular maps ψ of $[0, T]$ into $H^1(\Omega)$,

(5.8) $$[[\psi]] = \|\psi_t\|_{L^1(0,T;\, L^2(\Omega))} + \|\psi\|_{L^1(0,T;\, H^1(\Omega))}.$$

THEOREM 5.1. *There exists a constant C such that for u and U solutions of* (5.5) *and* (5.7), *respectively*,

(5.9) $$\|u - U\|_{L^\infty(0,T;\, L^2(\Omega))} \leqslant C \inf\{[[u - V]] : V \in \mathfrak{M}\}.$$

Proof. Take $V \in \mathfrak{M}$, and let $\vartheta = U - V$ and $\eta = u - V$. Then

(5.10) $$(\vartheta_t, \chi) + B(\vartheta, \chi) = (\eta_t, \chi) + B(\eta, \chi), \qquad \chi \in \mathfrak{M}(t), 0 < t < T.$$

Take $\chi = \vartheta$ to see that

$$\left(\frac{d}{dt}\|\vartheta\|\right)\|\vartheta\| \leq -B(\vartheta, \vartheta) + (\eta_t, \vartheta) + B(\eta, \vartheta)$$
$$\leq C\left[\|\vartheta\| + \|\eta_t\| + \|\eta\|_{H^1(\Omega)}\right]\|\vartheta\|.$$

Hence for $\|\vartheta\| \neq 0$

$$\frac{d}{dt}\|\vartheta\| \leq C\left[\|\vartheta\| + \|\eta_t\| + \|\eta\|_{H^1(\Omega)}\right]. \tag{5.11}$$

Integrate this over each J_j, use (2.12), (2.15), and Gronwall's lemma to get

$$\|\vartheta\|_{L^\infty(0,T;\, L^2(\Omega))} \leq C[[\eta]]. \tag{5.12}$$

The conclusion now follows from the triangle inequality and the fact that

$$\|\eta\|_{L^\infty(0,T;\, L^2(\Omega))} \leq C[[\eta]].$$

6. A Moving Finite Element. K. Miller has defined a class of methods for systematically moving the mesh associated with a finite element function space. These procedures, which he calls MFE's or moving finite element methods, seem experimentally to be quite effective for problems with sharp fronts [17], [18], [11], [2].

This section presents some preliminary steps toward understanding these techniques. In particular, the results of this section are concerned with the one space-dimensional problem in which the underlying function space consists of piecewise polynomial functions.

6.1. *One-Dimensional Description*. Take Ω to be the interval (0, 1). Suppose for simplicity that $a(x, t)$ of (2.1) does not depend on t. The space $\mathfrak{M}(t)$ will be all continuous piecewise polynomial functions of degree at most r defined over a mesh $\{s_i(t)\}_{i=0}^N$, where

$$s_i(t) = S(ih, t) \tag{6.1}$$

and $h = 1/N$. The function $S(y, t)$ will be defined as part of the MFE, but it will be a continuous piecewise linear function over a mesh $\{ih\}_{i=0}^N$.

Let β be a continuously differentiable function from R^+ (the positive reals) to R^+. Suppose that

$$\lim_{x \to 0} \beta(x) = +\infty. \tag{6.2}$$

The function β is a penalty term used to control the mesh, and for the sake of generality we only give the assumptions we use in the proof.

The formal motivation for the MFE method treated here is to choose the time derivatives of the approximate solution $U(t) \in \mathfrak{M}(t)$ and S so as to minimize

$$\|U_t + \mathfrak{L}U - f\|^2 + \|S_{yt} + \beta'(S_y)\|^2$$

at each time. Here $\mathfrak{L}$ is the spatial operator associated with (2.1). This must be purely formal because of the fact that $\mathfrak{L}U$ is not usually in $L^2(\Omega)$ when U is a continuous piecewise polynomial function. The function β' is, of course, the derivative of β with respect to its argument.

For each $t \in [0, T]$

$$S(y, t) = y + \tilde{S}(y, t),$$

where $\tilde{S}(\cdot, t) \in \mathfrak{M}_0$. The space $\mathfrak{M}_0$ is the collection of all continuous piecewise linear functions over the uniform partition $\{ih\}$ which vanish at $y = 0$ and $y = 1$. For each t for which $S(\cdot, t)$ is one-to-one, S^{-1} will denote the inverse of S as map of $[0, 1]$ to R.

The bilinear form $B(t; \cdot, \cdot)$ from Section 2 is to be extended to functions that are only piecewise smooth. For such functions φ and ψ take

$$B(t; \varphi, \psi) = \sum_{i=1}^{N} \int_{s_{i-1}}^{s_i} (a\varphi_x\psi_x + v\varphi_x\psi + b\varphi\psi)\, dx. \tag{6.3}$$

The notation used earlier for the jump in a function

$$\psi|_{s_j} = \psi(s_j + 0) - \psi(s_j - 0) \tag{6.4}$$

will be used in this section too.

The initial conditions for U and S will be

$$(U(0) - u(0), \chi) = 0, \qquad \chi \in \mathfrak{M}(0), \qquad S(y,0) = y, \qquad y \in [0,1]. \tag{6.5}$$

(The particular choice of $S(\cdot, 0)$ being the identity is convenient, but any "tame" strictly monotone increasing map of [0, 1] onto [0, 1] would do as well.)

The evolution of U and S is governed by the following set of orthogonalities:

$$(U_t, \psi) + B(t; U, \psi) = (f(\cdot, t), \psi) + \langle g(\cdot, t), \psi\rangle, \qquad \psi \in \mathfrak{M}(t), \tag{6.6a}$$

$$\begin{aligned}(U_t, -U_x\tilde{\chi}) + B(t; U, -U_x\tilde{\chi}) - \frac{1}{2}\sum_{j=1}^{N-1} a(s_j)(U_x)^2|_{s_j}\tilde{\chi}(s_j)&\\ + (S_{yt} + \beta'(S_y), \chi_y) = (f(\cdot, t), -U_x\tilde{\chi}),& \qquad \chi \in \mathfrak{M}_0,\end{aligned} \tag{6.6b}$$

where $\chi(y) = \tilde{\chi}(S(y, t))$.

One way to motivate (6.6) is to note that in the case of a fixed mesh, say $S(y, t) \equiv y$, the usual Galerkin orthogonalities are expressed by (6.6a). (K. Miller was, to my knowledge, the first to observe that these can be formally derived from a minimization at each time.) The use of the S_{yt}-terms in the minimization is to prevent singularity of the evolution equations in certain cases. If the approximate solution is a single polynomial of degree $\leqslant r$ on two adjacent subintervals, then the solution is not changed if the interior boundary between the two subintervals is moved; this would give singular equations without the $S_{yt} + \beta'(S_y)$-term. The S_{yt}-term is called a viscosity term by Miller and Miller since it keeps the adjacent knots moving at about the same speed. The $\beta'(S_y)$-term is called a spring-force-term and it keeps the knots from coalescing.

6.2. *A Fundamental Stability Result.* It is easily seen that the relations (6.6) are equivalent to a system of ordinary differential equations that has a solution locally in time, but it is not obvious that a solution exists for all $t \in [0, T]$.

THEOREM 6.1. *The solution* (U, S) *of* (6.6) *exists for all* $t \in [0, T]$, *and, at each* t, $S(\cdot, t)$ *is a strictly monotone map of* [0, 1] *onto* [0, 1]. *Further there is a constant C such that*

$$\|U(\cdot, t)\|^2_{H^1(\Omega)} + \int_0^1 \beta(S_y(y, t))\, dy \leqslant C, \qquad 0 \leqslant t \leqslant T, \tag{6.7}$$

and

$$\|S_{yt}\|^2_{L^2(0,T;\, L^2(\Omega))} + \|U_t\|^2_{L^2(0,T;\, L^2(\Omega))} \leqslant C. \tag{6.8}$$

Proof. Suppose that $S(\cdot, t)$ is a strictly monotone increasing function of $[0, 1]$ onto $[0, 1]$ for $t \in [0, t_1]$. Then either $t_1 = T$ or the system (6.6) has a solution that exists on some interval $[0, t_2)$, where $t_2 > t_1$, and if t_2 is taken sufficiently close to t_1, the mapping $S(\cdot, t)$ will be strictly monotone for $t \leqslant t_2$.

For $j = 1, \ldots, N$ and $k = 0, \ldots, r$, let

$$U_{j,k} = U\big(s_{j-1}(t) + \big(s_j(t) - s_{j-1}(t)\big)k/r, t\big).$$

Take $U_t^V \in \mathfrak{M}(t)$ to have the values $U'_{j,k}$ at the points $s_{jk} = s_{j-1} + (s_j - s_{j-1})k/r$. Set

$$U_t = U_t^V + U_t^H. \tag{6.9}$$

Then it is an easy exercise to see that on each interval (s_{j-1}, s_j)

$$U_t^H(x, t) = -U_x(x, t)S_t\big(S^{-1}(x, t), t\big). \tag{6.10}$$

Take χ in (6.6a) to be U_t^V and take χ in (6.6b) to be S_t. Then add these two relations to get that

$$\begin{aligned}(U_t, U_t) + B(t; U, U_t) - \frac{1}{2}\sum_{j=1}^{N-1} a(s_j)(U_x)^2\big|_{s_j} s'_j + \big(S_{yt} + \beta'(S_y), S_{yt}\big)& \\ = \big(f(\cdot, t), U_t\big) + \big\langle g(\cdot, t), U_t\big\rangle.&\end{aligned} \tag{6.11}$$

Now write

$$B(t; \varphi, \psi) = B_1(\varphi, \psi) + B_0(t; \varphi, \psi), \tag{6.12}$$

where

$$B_1(\varphi, \psi) = \sum_{j=1}^{N} B_{1,j}(\varphi, \psi) \tag{6.13}$$

and

$$B_{1,j} = \int_{s_{j-1}}^{s_j} a(x)(\varphi_x\psi_x + \varphi\psi)\, dx. \tag{6.14}$$

Note that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}B_{1,j}(U, U) = B_{1,j}(U, U_t) + \tfrac{1}{2}a(s_j)\big(U_x(s_j)\big)^2 s'_j + \tfrac{1}{2}\big(U(s_j)\big)^2 s'_j& \\ - \tfrac{1}{2}a(s_{j-1})\big(U_x(s_{j-1})\big)^2 s'_{j-1} - \tfrac{1}{2}\big(U(s_{j-1})\big)^2 s'_{j-1}.&\end{aligned} \tag{6.15}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}B_1(U, U) = B_1(U, U_t) - \frac{1}{2}\sum_{j=1}^{N-1} a(s_j)(U_x)^2\big|_{s_j} s'_j. \tag{6.16}$$

Thus it follows that

$$\begin{aligned}\|U_t\|^2 + \|S_{yt}\|^2 + \frac{d}{dt}\big[\tfrac{1}{2}B_1(U, U) + \big(\beta(S_y), 1\big)\big]& \\ = \big(f(\cdot, t), U_t\big) + \big\langle g(\cdot, t), U_t\big\rangle - B_0(t; U, U_t)& \\ \leqslant \tfrac{1}{2}\|U_t\|^2 + C\big[1 + \|U\|_1^2\big] + \frac{d}{dt}\big\langle g(\cdot, t), U\big\rangle.&\end{aligned} \tag{6.17}$$

In deriving (6.17) we used the fact that g_t and f are bounded.

The inequality (6.17) together with Gronwall's lemma and the fact that $a(x) \geq \underline{a} > 0$ imply there is a constant C such that on any interval $[0, t_2]$ for which the solution exists

$$\max_{[0, t_2]} \big(\beta(S_y), 1\big) \leq C.$$

This implies that on no subinterval $((j-1)h, jh)$ is S_y equal to zero. Hence, since S_y varies continuously in time on each subinterval, we see that S_y is, in fact, uniformly positive. This then implies that if $t_2 < T$, the interval of definition of (U, S) can be extended. The bounds claimed also follow from the estimate that showed that S_y stays positive. □

The primary reason for allowing the mesh to change with time is to cluster much of the flexibility of the space $\mathfrak{M}(t)$ in those areas where u is rough. Thus it is desirable for $S_y(\cdot, t)$ to be quite small in a part of its domain if the solution is very sharply changing. Hence to allow for this possibility β should probably be chosen so that it is small until S_y becomes extremely small.

The penalty term $S_{yt} + \beta'(S_y)$ has been included to give a nondegenerate problem, but it can also be used to control the movement of the mesh.

Suppose now that, instead of being defined on R^+, β is only defined on an interval $(\underline{\beta}, \bar{\beta})$, where $\underline{\beta} \geq 0$. Take β to be a continuously differentiable nonnegative function on $(\underline{\beta}, \bar{\beta})$ that goes to $+\infty$ as its argument goes to $\underline{\beta}$ or to $\bar{\beta}$. Then, we need to have $\underline{\beta} < 1$ and $\bar{\beta} > 1$ so that the initial value of $(\beta(S_y), 1)$ is finite.

For the remainder of this section the conditions that

$$\lim_{x \to \underline{\beta}} \beta(x) = +\infty, \qquad \lim_{x \to \bar{\beta}} \beta(x) = +\infty, \tag{6.18}$$

will be assumed to hold. Under these new conditions on β Theorem 6.1 remains valid. Note that for β as above $s_j - s_{j-1} \leq h\bar{\beta}$. If, for example, $\bar{\beta} = 2$, then one half the points s_j can be tightly grouped about a single area of roughness while the longest subinterval is no more than twice that of a uniform fixed mesh.

6.3. *Application of the Basic Results*. The stability result, Theorem 6.1, tells us enough about the mesh $\{s_j\}$ that we can derive asymptotic error bounds. The order of convergence is what one would expect from a fixed-mesh procedure.

THEOREM 6.2. *Suppose that u, the solution of* (2.1), *is sufficiently smooth. Then*

$$|||u - U||| \leq Ch^r. \tag{6.19}$$

Proof. The estimation of the $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H^1(\Omega))$ norms is routine, but the calculation of the time-derivative error is not so trivial.

Let $s_{jk}(t) = S((j-1+k/r)h, t)$, for $k = 0, \ldots, r$. Take $W(x, t) \in \mathfrak{M}(t)$ to be the interpolant of u based on the points s_{jk}. Define

$$W_{jk}(t) = W\big(s_{jk}(t), t\big) = u\big(s_{jk}(t), t\big). \tag{6.20}$$

Then

$$\begin{aligned} W'_{jk}(t) &= u_t\big(s_{jk}(t), t\big) + u_x\big(s_{jk}(t), t\big)s'_{jk}(t) \\ &= u_t\big(s_{jk}(t), t\big) + u_x\big(s_{jk}(t), t\big)S_t\big(S^{-1}\big(s_{jk}(t), t\big), t\big). \end{aligned} \tag{6.21}$$

Just as for U_t we can decompose W_t as follows

$$W_t = W_t^V + W_t^H, \tag{6.22}$$

where $W_t^V \in \mathfrak{M}(t)$ has the values W'_{jh} at each s_{jk}. Then, as before, the H-component is in general discontinuous and is given by

$$W_t^H(x, t) = -W_x(x, t)S_t(S^{-1}(x, t), t). \tag{6.23}$$

Letting $\mathcal{I}$ denote the subinterval by subinterval interpolation operator we see that

$$W_t = \mathcal{I}(u_t) + \mathcal{I}(u_x S_t(S^{-1})) - \mathcal{I}(u)_x S_t(S^{-1}) = \mathcal{I}(u_t) + p. \tag{6.24}$$

At the points s_{jk} the polynomial p satisfies

$$p = S_t(S^{-1})[u_x - (\mathcal{I}u)_x]. \tag{6.25}$$

Hence, if $u \in L^\infty(0, T; H^{r+1}(\Omega))$, we see that

$$\|p\|_{L^2(0,T;\, L^2(\Omega))} \leq Ch^r. \tag{6.26}$$

Of course the $\mathcal{I}(u_t)$ term in (6.24) is easily compared to u_t, and the result follows. □

The smoothness required on u is not minimal in the above argument. One should probably use an H^1-type projection of u instead of the interpolant, but that would complicate the argument.

6.4. *Some Remarks on Moving Finite Element Methods*. The estimates of the previous subsection indicate that the MFE method would work as well as a fixed-grid method on a smooth problem, but they do not indicate why the procedure is as effective as it seems to be on sharp front problems.

The function space $\mathfrak{M}_0$ can be replaced by one that has fewer parameters than $N - 1$ for manipulating the mesh. For example, we could move every second meshpoint using the given evolution law and move the other points by an affine relation to their neighbors; this corresponds exactly to replacing $\mathfrak{M}_0$ by the space of piecewise linear functions on a mesh $\{2ih\}_{i=0}^{N/2}$ if N is even.

For problems with an underlying flow, the boundary conditions that say the transformation S takes the boundary to itself seem wrong. This is a point that I expect to study and report on later.

7. Sobolev Equations. Evolution equations that have a second order elliptic operator applied to the time derivative and some other second order operator applied in the spatial variables are frequently called Sobolev equations. Such equations have been studied as models for various important physical phenomena, from unidirectional water waves [3] to flow in a fractured oil reservoir [10].

The mesh modification error estimates of Section 2 carry over in part to the context of Sobolev equations. This will be illustrated by looking at an abstract continuous-time Galerkin method and then specializing it to a particular Sobolev equation.

For simplicity this section deals only with linear equations, but this analysis can be extended, as in Section 3.2, to nonlinear problems that are sufficiently general to include the equations such as the so-called BBM equation presented in [3]. Such equations can have solitary-wave solutions that move without changing shape (each at a speed that is related to its size). Since these "elementary" solutions are of significant interest, moving-mesh solution techniques seem to be very natural for these equations. Although the analysis of this section is an easy extension of that done for the parabolic case, it seems useful to have it as a basis for using methods of the type treated here on the scientifically important problems referred to above.

Let B and B_1 be continuous bilinear forms on a Hilbert space H. Denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and norm on H. Suppose that the form B_1 is coercive and symmetric, and that u: $[0, T] \to H$ is a C^1 function satisfying

$$B_1(u_t, \chi) + B(u, \chi) = (f, \chi), \qquad \chi \in H, \tag{7.1}$$

where f is a continuous map of $[0, T]$ into H.

As in Section 2, let $0 = T_0 < T_1 < \cdots < T_M = T$ partition $[0, T]$, and take $J_j = [T_{j-1}, T_j)$. Suppose that, for each t, $\mathfrak{M}(t)$ is a finite-dimensional subspace of H. In addition assume that for $j = 1,\dots,M$ there are Lipschitz continuous maps $\psi_{j,l}$: $J_j \to H$, $l = 1,\dots,N_j$ such that the set $\{\psi_{j,l}(t)\}_{l=1}^{N_j}$ is a basis for $\mathfrak{M}(t)$ when $t \in J_j$ and such that the matrix $(B_1(\psi_{j,l}, \psi_{jk})) = (b_{lk})$, when continuously extended to $\bar{J}_j$, is nonsingular for each $t \in \bar{J}_j$. In analogy with Section 2 define $\mathfrak{M}$ to be the space of all maps V: $[0, T) \to H$ such that

$$V(t) \in \mathfrak{M}(t), \qquad 0 \leq t < T, \tag{7.2a}$$

$$V \text{ is Lipschitz on } J_j, \qquad j = 1,\dots,M, \tag{7.2b}$$

$$V|_{T_j} = V(T_j) - V(T_j - 0) \text{ satisfies } B_1(V|_{T_j}, \chi) = 0, \quad \chi \in \mathfrak{M}(T_j), j = 1,\dots,M-1. \tag{7.2c}$$

The Galerkin solution is $U \in \mathfrak{M}$ such that

$$\begin{aligned} B_1(U_t, \chi) + B(U, \chi) &= (f, \chi), \qquad \chi \in \mathfrak{M}(t), 0 < t < T, \\ B_1(U(0) - u(0), \chi) &= 0, \qquad \chi \in \mathfrak{M}(0), \end{aligned} \tag{7.3}$$

where the time derivatives in (7.3) are one-sided at $t = T_j$. It is straightforward that U exists. Define for all $\psi \in H$

$$[\psi] = B_1(\psi, \psi)^{1/2}$$

and

$$[\psi]_{f,t} = \sup_{0 \neq \chi \in \mathfrak{M}(t)} B_1(\psi, \chi)/[\chi],$$

where the subscript f is to remind us that the seminorm is defined by duality with respect to the finite-dimensional space $\mathfrak{M}(t)$. Then define for all sufficiently regular ψ: $[0, T) \to H$

$$|||\psi||| = \sup_{0 \leq t < T} [\psi] + \int_0^T [\psi_t]_{f,t}\, dt. \tag{7.4}$$

THEOREM 7.1. *There exists a constant C such that if u solves* (7.1) *and U solves* (7.3), *then*

$$|||U - u||| \leq C \inf\{|||V - u|||: V \in \mathfrak{M}\}. \tag{7.5}$$

Proof. Take $V \in \mathfrak{M}$. Let $\vartheta = U - V$ and $\eta = u - V$. Then

$$B_1(\vartheta_t, \chi) + B(\vartheta, \chi) = B_1(\eta_t, \chi) + B(\eta, \chi), \qquad \chi \in \mathfrak{M}(t). \tag{7.6}$$

Take $\chi = \vartheta$ to get

$$(7.7)\qquad \begin{aligned}[\vartheta]\frac{d}{dt}[\vartheta] &= -B(\vartheta, \vartheta) + B_1(\eta_t, \vartheta) + B(\eta, \vartheta)\\ &\leq C[\vartheta]\{[\vartheta] + [\eta] + [\eta_t]_{f,t}\}.\end{aligned}$$

The relation (7.7) and the fact that

$$(7.8)\qquad [\vartheta(T_j)] \leq [\vartheta(T_j - 0)]$$

give the inequality

$$(7.9)\qquad \max_{0\leq t\leq T} [\vartheta(t)] \leq C\int_0^T \{[\eta] + [\eta_t]_{f,t}\}\, dt + [\vartheta(0)].$$

But then (7.6) implies that

$$(7.10)\qquad |||\vartheta||| \leq C\int_0^T \{[\eta] + [\eta_t]_{f,t}\}\, dt.$$

The triangle inequality then completes the proof. □

Now take Ω to be R^n. Let

$$(7.11)\qquad \begin{aligned}B_1(\varphi, \psi) &= \int_\Omega (\varphi\psi + a_1(x)\nabla\varphi \cdot \nabla\psi)\, dx,\\ B(\varphi, \psi) &= \int_\Omega (a(x)\nabla\varphi \cdot \nabla\psi + v(x) \cdot \nabla\varphi\psi + b(x)\varphi\psi)\, dx.\end{aligned}$$

Let $H = H^1(\Omega)$, and assume that $a_1(x)$ is bounded above and below by positive constants. Also assume that a, v, and b are bounded.

Now suppose that u: $[0, T] \to H^1(R^n)$ satisfies

$$(7.12)\qquad [1 - \nabla \cdot a_1\nabla]u_t - \nabla \cdot a\nabla u + v \cdot \nabla u + bu = 0$$

in the sense that

$$(7.13)\qquad B_1(u_t, \chi) + B(u, \chi) = 0 \quad \text{for } \chi \in H^1(R^n),\ t \in [0, T).$$

Then Theorem 7.1 says that if the space $\mathfrak{M}$ contains anything close to u, then U is close to u.

Note that in this example the $L^2(\Omega)$-projection at the points T_j was replaced by projection with respect to a bilinear form that behaves like the $H^1(\Omega)$-inner product.

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