REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

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17[9.35].—WILLIAM B. JONES & W. J. THRON, "Continued fractions, analytic theory and applications," *Encyclopedia of Mathematics and its Applications*, Addison-Wesley, Reading, Mass., 1980, xxviii + 428 pp., 24 cm. Price $37.50.

The history of continued fractions (C.F.) is an old one since it can be viewed as beginning with Euclid's algorithm (c. 330 B.C.—c. 275 B.C.) for the g.c.d. In modern notation the algorithm can be put in the form of a three-term recurrence relation which in turn can be expressed as a C.F. True, Euclid did not present the algorithm in this fashion though he used geometrical considerations concerning the relative length of two segments. The algorithm is useful to simplify ratios as exemplified in the work of Archimedes (287 B.C.—212 B.C.), whose approximation $22/7$ for π was often used before the advent of electric and hand computers. It was Theon of Alexandria (c. 365 A.D.) who used the start of the C.F.

$$(a^2+b)^{1/2} = a + \frac{b}{2a+}\,\frac{b}{2a+}\cdots$$

to find the side of a square of a given area. In this connection, in 1572, R. Bombelli in essence gave the approximate expansion

$$13^{1/2} = 3 + \frac{4}{6+}\,\frac{4}{6+}\cdots.$$

In 1613, P. Cataldi gave the analogous form for $18^{1/2}$. Indeed he gave the first 15 convergents and proved that they are alternately greater and smaller than $18^{1/2}$ and that they converge to $18^{1/2}$.

Another problem of ancient times which leads to the early development of C.F. is named after Diophantus (c. 250 A.D.) who found a rational solution of $ax + by = c$ where a, b and c are given positive integers. The problem was completely solved by the Indian mathematician Āryabhata (476–550). The solution was rediscovered by a number of workers including Lagrange (1736–1813).

In 1655, John Wallis (1616–1703) published his now famous infinite product expansion for $4/\pi$. He reports that Lord William Brouncker (1620–1684) gave the expansion which in contemporary notation reads

$$\frac{4}{\pi} = 1 + \frac{1|}{|2} + \frac{9|}{|2} + \cdots + \frac{n^2|}{|2} + \cdots.$$

Wallis uses the words (in Latin) 'continued fraction' to describe the expression and thus the genesis of the words continued fractions. From this time on, the study of C.F. moves in essentially two directions. One has to do with the theory of numbers, the other is in their use in the representation and approximation of functions.

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Contributions to number theory by use of C.F. in the period 1700–1900 were made by many well-known names in the history of mathematics. In particular, Euler (1707–1783) made many important discoveries. He proved that every rational number can be developed into a terminating C.F. and that a periodic C.F. is the root of a quadratic equation. He discussed Euclid's algorithm and the simplification of fractions. The proof that every prime of the form $4n + 1$ is the sum of two squares is due to Euler. Lagrange gave the solution of the Pell equation $u^2 - Dv^2 = 1$, where D is a positive integer, while the complete solution is due to Legendre (1752–1833). Other results were obtained by Gauss (1777–1855) and Liouville (1809–1882).

Contributions to the representation and approximation of functions by use of C.F. in the period 1700–1900 also contains a number of notable figures. Euler derived the formal divergent series

$$\sum_{k=0}^{\infty} (-1)^k k!/x^k$$

for the integral

$$x \int_0^{\infty} e^{-t}\, dt/(x+t).$$

He converted the series into a C.F. and used it to 'evaluate' the divergent series for $x = 1$. Lambert (1728–1777) expressed $\tan x$, $\arctan x$, $\ln(1+x)$ and $(e^x - 1)/(e^x + 1)$ as C.F. In 1776, Lagrange wrote a paper which developed a technique for converting a power series into a C.F. In particular cases, these were reduced to ordinary fractions. He noticed that the latter as a power series agreed with the original power series as far as is possible. We have here the genesis of Padé approximants. Lagrange did no further work on the subject. In a letter to d'Alembert, he makes reference to the volume containing the 1776 paper and states "there is naturally something of myself in this, but nothing that merits your attention..."—a diametrically opposite view of the current interest in Padé approximations. Laplace (1749–1827) expressed the error function

$$\int_x^{\infty} e^{-t^2}\, dt$$

as a C.F. Gauss (1777–1855) derived three-term recurrence relations involving the parameters of the hypergeometric function which bears his name, and from these the corresponding C.F. for ${}_2F_1(1, a; c; -z)$ and its confluent forms are readily derived. If $P(x)$, $Q(x)$, and $R(x)$ are polynomials in x, Laguerre (1834–1886) studied the C.F. representations of some particular solutions of the differential equation

$$P(x)y' = Q(x)y + R(x).$$

Frobenius (1849–1917), Padé (1863–1953) and others studied the problem of expanding formal power series into the ratio of two polynomials. Frobenius derived certain recursion relations between the numerators and denominators of Padé fractions.

Perhaps the most important contributions to the theory of C.F. in the 19th century were made by Stieltjes (1856–1894). Indeed he is really the founder of the modern analytic theory of C.F. Stieltjes studied

$$F(x) = x\int_0^\infty \frac{d\alpha(t)}{x+t},$$

its asymptotic expansion, its C.F. representation, and the connection of the latter with the Gaussian quadrature of $F(x)$. If P_n/Q_n is the nth convergent of the C.F., he showed that the sequences $\{Q_n\}$ are orthogonal polynomials over $(0, \infty)$ with respect to $d\alpha(t)$. Stieltjes studied the expansion of arbitrary functions in series of orthogonal polynomials and the so-called moment problem. That is, determine a function $\alpha(t)$ associated with a given sequence $\{c_n\}$ defined by

$$c_n = \int_0^\infty t^n d\alpha(t)\, dt.$$

In the first half of the 20th century research in C.F. deals mostly with the analytic theory. This era is marked by some important volumes which effectively summarize research on the subject of C.F. from its beginnings. First is the 1929 volume by Perron [1] and second is the 1948 volume by Wall [2] (1902–1971). Also pertinent are two volumes (1954, 1957) by Perron [3], [4] which form a revised and improved edition of his 1929 volume. There is also the 1963 translation (by Wynn) of a 1956 work by Khovanskii [5]. Approximate cut off dates for [2]–[5] can be taken as 1950. Since that time two developments have taken place which have spawned much interest in the subject. First is the recognition that Padé approximations (though discussed by Frobenius in 1881 and Padé in 1892) play an important role in physical applications; and, second is the advent of the high speed digital computer. Clearly, an up to date exposition of the subject is long overdue. In view of their many research contributions, Jones and Thron are eminently qualified to remedy this deficiency. It is a credit to their scholarship, erudition and skill that they have produced a most worthy and useful volume—a valuable research tool. As noted from the title, emphasis is on the analytic theory.

The book is intended for all pure and applied workers in mathematics and the sciences. The only requisite knowledge is the rudiments of complex analysis.

There are twelve chapters, two appendices, author and subject indices. The bibliography contains about 370 references, with about 240 of them dating from 1950.

The book is Volume 11 of a series which is to form an encyclopedia of mathematics. The editor is Gian-Carlo Rota, who presents a description of this effort. Of more pertinence to the subject of the present volume is a Foreword by Felix E. Browder, who is the General Editor, Section on Analysis, of the projected encyclopedia and an Introduction by Peter Henrici. The Foreword and Introduction are interesting reviews of the subject of continued fractions.

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1. O. PERRON, *Die Lehre von den Kettenbrüchen*, Teubner, Leipzig, 1929.

2. H. S. WALL, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

3. O. PERRON, *Die Lehre von den Kettenbrüchen*, Band I, Teubner, Stuttgart, 1954.

4. O. PERRON, *Die Lehre von den Kettenbrüchen*, Band II, Teubner, Stuttgart, 1957.

5. A. M. KHOVANSKII, *The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory*, Noordhoff, Groningen, 1963.