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Linear Multistep Methods for Stable Differential Equations $y^{\cdot\cdot} = Ay + B(t)y^{\cdot} + c(t)$

By Eckart Gekeler

Abstract. The approximation of $y^{\cdot\cdot} = Ay + B(t)y^{\cdot} + c(t)$ by linear multistep methods is studied. It is supposed that the matrix A is real symmetric and negative semidefinite, that the multistep method has an interval of absolute stability $[-s, 0]$, and that $h^2\|A\| \leqslant s$ where h is the time step. A priori error bounds are derived which show that the exponential multiplication factor is of the form $\exp\{\Gamma_s |||B|||_n(nh)\}$, $|||B|||_n = \max_{0 \leqslant t \leqslant nh}\|B(t)\|$.

1. Introduction. In this work we consider the real initial value problem

$$(1) \qquad y^{\cdot\cdot} = Ay + B(t)y^{\cdot} + c(t), \qquad t > 0, y(0) = z_0, y^{\cdot}(0) = z_1,$$

where A is a symmetric and negative semidefinite matrix. Very similar problems are obtained in two ways:

If a linear hyperbolic initial boundary value problem with damping is discretized in the space direction by a finite element method or, more generally, by a Galerkin procedure, then the resulting semidiscrete system has the form

$$(2) \qquad My^{\cdot\cdot} = -Ky + C(t)y^{\cdot} + f(t);$$

see, e.g., Fried [16, Chapter 9]. M is a constant matrix and K is a constant matrix if the elliptic operator in the partial differential equation does not depend on time t.

In engineering mechanics systems of the same form are obtained in a different way by the finite element approach of linear dynamic problems in matrix structural analysis; see, e.g., Bathe and Wilson [8, Chapters 3, 8] and Przemieniecki [28, Chapters 10, 12, 13]. M, C, and K are then the mass, damping, and stiffness matrix, and f is the external load vector.

M and in many cases also K are real symmetric and positive definite matrices but K is in general very ill-conditioned.

Several methods were proposed for the solution of initial value problems with the system (2). Sometimes the mass matrix M is diagonalized a priori by 'lumping' (cf., e.g., Fried [16, Chapter 3], Strang and Fix [29, Chapter 6]) or Jordan canonical decomposition (modal analysis) (cf., e.g., Bathe and Wilson [8, Chapter 8]). Argyris et al. [2], [3], [4] transform the second order system (2) into a first order system of twice as large dimension by introducing $y^{\cdot}$ as a further variable. The matrix of this system is no longer symmetric. Hence the transformed problem is solved by a special class of absolutely stable single-step multiderivative methods which are known as Obrechkoff methods; see, e.g., Lambert [26, Chapter 7]. For an error analysis of

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these procedures we refer to Gekeler and Johnsen [18]. Runge-Kutta-Nyström methods for systems (2) without damping are studied, e.g., by Fried [16, Chapter 9] and Gekeler [19]. Further one- or two-step multistage methods are used by Baker and Bramble [7] and Baker et al. [6] for the solution of systems (2) obtained by Galerkin procedures and hyperbolic problems.

Special linear two- and three-step methods were applied by Bathe and Wilson [8, Chapters 8, 9] in matrix structural analysis and by Baker [5], Dendy [13], Dupont [15], and Wheeler [30] in connection with the numerical solution of hyperbolic problems. Error estimations of Galerkin multistep procedures of arbitrary order and hyperbolic problems were deduced by Dougalis [14] and Gekeler [17], [21]. However, besides Bathe and Wilson [8], systems (2) without damping are considered in the quoted literature on linear multistep methods and Dendy [13] only admits a time-varying stiffness matrix K.

In the present paper we make no specific assumptions on the damping matrix $B(t)$. But with respect to (2) we tacitly assume that $C(t)$ has the same shape as the mass matrix M (cf. [28, Section 13.10]). Finite element systems with a damping matrix proportional to K or having the form $\alpha M + \beta K$ are considered in [8, Chapters 3, 8] and [28, Section 13.11], too. However, in this case nonlinear implicit methods seem to be more suitable than linear multistep methods.

In order to introduce linear multistep methods let

$$\rho(\zeta) = \sum_{\kappa=0}^{k} \alpha_\kappa \zeta^\kappa, \qquad \alpha_k > 0, \qquad \sigma(\zeta) = \sum_{\kappa=0}^{k} \beta_\kappa \zeta^\kappa, \qquad \beta_k \geqslant 0,$$

be two real polynomials without common roots including zero, and assume that for every coefficient $\beta_\kappa \neq 0$ of $\sigma(\zeta)$ a further real polynomial is given,

$$\tau_\kappa(\zeta) = \sum_{\mu=0}^{k} \gamma_\mu^{(\kappa)} \zeta^\mu.$$

Let h be a small increment of time t, $y_n = y(nh)$, and let the translation operator Θ be defined by $(\Theta y)(t) = y(t+h)$. Then a linear k-step method $\langle \rho, \sigma, \tau \rangle$ for the system (2) is defined by

$$(3) \quad M\rho(\Theta)v_n + h^2 K\sigma(\Theta)v_n = h \sum_{\kappa=0}^{k} \beta_\kappa C_{n+\kappa} \tau_\kappa(\Theta) v_n + h^2 \sigma(\Theta) f_n, \qquad n = 0, 1, \ldots.$$

For the error estimation we multiply (3) from left by $M^{-1/2}$ and replace $M^{1/2}v_n$ by v_n again. Then

$$(4) \quad \rho(\Theta)v_n = h^2 A\sigma(\Theta)v_n + h \sum_{\kappa=0}^{k} \beta_\kappa B_{n+\kappa} \tau_\kappa(\Theta) v_n + h^2 \sigma(\Theta) c_n, \qquad n = 0, 1, \ldots,$$

where $A = -M^{-1/2}KM^{-1/2}$ etc. The scheme (4) is a linear multistep method $\langle \rho, \sigma, \tau \rangle$ for the explicit problem (1) and the transformation of (3) has the additional effect that estimations of $M^{1/2}v_n$ are derived which are desired in finite element analysis [14], [21], [29].

The method needs a special start procedure for the computation of the initial values $v_0, \ldots, v_{k-1}$. In the sequel we always suppose that these vectors are given. For their computation, the special class of A-stable single step methods of Obrechkoff type considered in [18] may be recommended.

The truncation errors of $\langle \rho, \sigma, \tau \rangle$ are

$$d_{\langle \rho, \sigma \rangle}(h, w)(t) \equiv \rho(\Theta)w(t) - h^2\sigma(\Theta)w^{\cdot\cdot}(t)$$

and

$$d_{\langle \tau_\kappa \rangle}(h, w)(t) \equiv \tau_\kappa(\Theta)w(t) - h\Theta^\kappa w^{\cdot}(t).$$

Definition 1. The method $\langle \rho, \sigma, \tau \rangle$ is consistent if there exists a positive integer q called the order of $\langle \rho, \sigma, \tau \rangle$ such that for all $w \in C^{q+2}(\mathbf{R}; \mathbf{R}^m)$,

$$\| d_{\langle \rho, \sigma \rangle}(h, w)(t) \| \leq \chi h^{q+2}, \qquad \| d_{\langle \tau_\kappa \rangle}(h, w)(t) \| \leq \chi_\kappa h^{q+1},$$

where χ and χ_κ do not depend on h.

By [26, pp. 30, 254] a method $\langle \rho, \sigma, \tau \rangle$ is consistent if and only if

$$\rho(1) = \rho'(1) = 0, \qquad \rho''(1) = 2\sigma(1) \tag{5}$$

and

$$\tau_\kappa(1) = 0, \qquad \tau_\kappa'(1) = 1 \quad \text{for all } \tau_\kappa. \tag{6}$$

Now let $\overline{\mathbf{C}}$ be the complex plane extended by the point ∞ in the usual way. The characteristic polynomial of the method $\langle \rho, \sigma, \tau \rangle$ is then

$$\pi(\zeta, \eta) \equiv \rho(\zeta) - \eta\sigma(\zeta), \qquad \eta \in \mathbf{C}, \qquad \pi(\zeta, \infty) \equiv \sigma(\zeta).$$

Definition 2. The stability region S of the method $\langle \rho, \sigma, \tau \rangle$ consists of those $\eta \in \overline{\mathbf{C}}$ for which all roots $\zeta_\kappa(\eta)$ of $\pi(\zeta, \eta)$ satisfy $|\zeta_\kappa(\eta)| \leq 1$ and all roots of modulus one have multiplicity not greater than two.

It follows by continuity that S is closed in $\overline{\mathbf{C}}$ if at most double roots of modulus one lie on the boundary of S.

Definition 3. The method $\langle \rho, \sigma, \tau \rangle$ is strongly D-stable in $[-s, 0] \subset S$ if

(i) all roots $\zeta_\kappa(0) \neq 1$ of modulus one of $\pi(\zeta, 0) = \rho(\zeta)$ are simple roots of $\pi(\zeta, 0)$,

(ii) all roots of modulus one of $\pi(\zeta, \eta)$ are simple roots of $\pi(\zeta, \eta)$ for $\eta \in [-s, 0)$.

The roots $\zeta_\kappa(\eta)$ of $\pi(\zeta, \eta)$ are the branches of the algebraic variety $\zeta(\eta)$ defined by $\pi(\zeta(\eta), \eta) = 0$. This algebraic variety has the unique finite pole in $\eta = \alpha_k/\beta_k > 0$ if $\beta_k \neq 0$, which cannot lie in S. By (5) there are exactly two branches, $\zeta_1(\eta)$ and $\bar{\zeta}_1(\eta)$, which coalesce to the value one for $\eta = 0$ if the method is consistent. Below we need the polynomial

$$\pi_1(\zeta, \eta) = \pi(\zeta, \eta)/(\zeta - \zeta_1(\eta)). \tag{7}$$

Independently of possible real branching points of $\zeta(\eta)$ this polynomial can be chosen on the entire real axis, with exception of $\eta = \alpha_k/\beta_k$, as a fixed polynomial of degree $k - 1$ in ζ with coefficients that are continuous in η.

Let $\|x\|$ be the Euclid norm of $x \in \mathbf{C}^m$, let $\|A\| = \max_{x \neq 0} \|Ax\|/\|x\|$ be the associated matrix norm (spectral norm), and let $|||B|||_n = \max_{0 \leq t \leq nh} \|B(t)\|$. It is the goal of the present paper to prove the following theorem:

THEOREM. (i) *Let the (m, m)-matrix A in* (1) *be real symmetric and negative semidefinite. Let the solution y of* (1) *be $(q+2)$-times continuously differentiable.*

(ii) *Let the method $\langle \rho, \sigma, \tau \rangle$ be consistent of order q with the stability region S, and let it be strongly D-stable in $[-s, 0] \subset S$, $s > 0$.*

(iii) *Let $h^2 \|A\| \leqslant s$.*

(iv) *Let $h \||B\||_n \leqslant \Omega$ where the constant Ω is defined in* (14).

Then the error $e_n = y_n - v_n$, $n = k, k+1, \ldots$, satisfies

$$\|e_n\| \leqslant \sum_{\kappa=0}^{k-2} \|e_\kappa\| + \Gamma_s nh \exp\{\Gamma_s^* \||B\||_n nh\}$$
$$\times \left[\|A\|^{1/2} \sum_{\kappa=0}^{k-2} \|e_\kappa\| + h^{-1} \sum_{\kappa=1}^{k-1} \|e_\kappa - e_{\kappa-1}\| \right.$$
$$\left. + h^q \int_0^{nh} \left(\|y^{(q+2)}(\tau)\| + \||B\||_n \|y^{(q+1)}(\tau)\| \right) d\tau \right].$$

Remarks. (i) Γ_s and Γ_s^* depend only on the data of the method $\langle \rho, \sigma, \tau \rangle$ if $s = \infty$. For instance, we have $s = \infty$ in the implicit 3-step method of order $q = 2$ with the polynomial

$$\pi(\zeta, \eta) = 2\zeta^3 - 5\zeta^2 + 4\zeta - 1 - \eta\zeta^3.$$

However, Dahlquist [12] has proved that $s < \infty$ if $q > 2$.

(ii) If the method $\langle \rho, \sigma, \tau \rangle$ is strongly D-stable in $\eta = 0$ and has a stability region S containing a left-side neighborhood of $\eta = 0$, then it has a stability interval $[-s, 0]$ satisfying assumption (ii) because the algebraic variety $\zeta(\eta)$ has only a finite number of branching points where some roots $\zeta_\kappa(\eta)$ of $\pi(\zeta, \eta)$ coalesce.

(iii) $\pi_1(\zeta, \eta)$ can be considered as the characteristic polynomial of a not necessarily consistent linear multistep method for a differential equation of first order. Assumption (ii) then is the weakest condition such that $\zeta_1(\eta) \neq \bar{\zeta}_1(\eta)$ for $\eta \in [-s, 0)$ and that the numerical approximation of $y^{\cdot} = \lambda y$, $\lambda < 0$, defined by $\pi_1(\Theta, h\lambda) v_n = 0$, $n = 0, 1, \ldots$, remains bounded in modulus for arbitrary but fixed $h\lambda \in [-s, 0]$.

(iv) Of course, the differential system (1) can be transformed into a first order system of twice as large dimension and then be approximated by linear multistep methods. However, the matrix of this system is diagonalizable only under very restrictive assumptions on the damping matrix $B(t)$; see, e.g., [22].

(v) The initial error is multiplied by $\|A\|^{1/2}$ in the above result. This phenomenon was already observed by Dupont [15] for two-step methods. The difference quotient of the initial error corresponds to the initial condition $y^{\cdot}(0) = z_1$ in the analytic initial value problem (1).

(vi) Of course, the estimates are only reasonable if $\||B\||_n$ is uniformly bounded or if the time interval is bounded.

2. Auxiliary Results. In the sequel Γ denotes a generic positive constant depending only on the data of the method $\langle \rho, \sigma, \tau \rangle$ and not necessarily the same in two different contexts. Further dependencies are indicated by subscripts.

The following lemma is a slight modification of a result due to Dahlquist [11, Chapter 4]; see also Lambert [26, Section 3.3].

LEMMA 1. *If a linear k-step method* $\langle \rho, \sigma, \tau \rangle$ *for the problem* (1) *is consistent of order q, then the composite truncation error*

$$d_{\langle \rho,\sigma,\tau \rangle}(h, w)(t) \equiv d_{\langle \rho,\sigma \rangle}(h, w)(t) - h \sum_{\kappa=0}^{k} \beta_\kappa B(t + \kappa h) d_{\langle \tau_\kappa \rangle}(h, w)(t)$$

satisfies, for all $w \in C^{q+2}(\mathbf{R}; \mathbf{R}^m)$,

$$\| d_{\langle \rho,\sigma,\tau \rangle}(h, w)(t) \| \leq \Gamma h^{q+1} \int_t^{t+kh} \left(\| w^{(q+2)}(\tau) \| + |||B|||_n \| w^{(q+1)}(\tau) \| \right) d\tau.$$

The next estimation concerns the principal root $\zeta_1(\eta)$ of $\pi(\zeta, \eta)$.

LEMMA 2. *Let assumption* (ii) *be fulfilled, and let* $\eta \in [-s, 0]$. *Then*
(i) $| 1 - \zeta_1(\eta) | \leq \Gamma | \eta |^{1/2}$,
(ii) $\lim_{\eta \to 0} | (\zeta_1(\eta) - 1)/(\zeta_1(\eta) - \overline{\zeta_1(\eta)}) | < \infty$.

Proof. By Ahlfors [1, p. 226] we can write $\zeta_1(\eta)$ in a neighborhood of the branching point $\eta = 0$ as a convergent series in powers of $\eta^{1/2}$. Consequently

$$\zeta_1(\eta) = 1 + \chi_1 i | \eta |^{1/2} \left(1 + \phi\left(i | \eta |^{1/2}\right)\right), \tag{8}$$

where $\chi_1^2 = 2\sigma(1)/\rho''(1) = 1$. $\chi_1 = \pm 1$ is the growth parameter of $\zeta_1(\eta)$ depending on the branch chosen for ζ_1; and ϕ is a holomorphic function in a neighborhood of zero with $\phi(0) = 0$. But $\phi(\eta^{1/2}) = [(\zeta_1(\eta) - 1)/\eta^{1/2}] - 1$ is then bounded in $[-s, 0]$ because $| \zeta_1(\eta) | \leq 1$ in this interval. Hence the first assertion follows with $\Gamma = \sup_{-s \leq \eta \leq 0} | 1 + \phi(\eta^{1/2}) | < \infty$. The second assertion follows from (8).

Now let $F(\eta)$ be the Frobenius matrix associated with the reduced polynomial $\pi_1(\zeta, \eta) \equiv \sum_{\kappa=0}^{k-1} \theta_\kappa(\eta) \zeta^\kappa$ defined in (7),

$$F(\eta) = \begin{bmatrix} 0 & 1 & & \\ & & & \mathbf{0} \\ \mathbf{0} & & 0 & 1 \\ -\theta_0(\eta)/\theta_{k-1}(\eta) & \cdots & -\theta_{k-2}(\eta)/\theta_{k-1}(\eta) & \end{bmatrix}.$$

In the following lemmas $\operatorname{spr}(A)$ denotes the spectral radius of the matrix A, and we write $A \leq B$ for two hermitean matrices if and only if $B - A$ is positive semidefinite.

LEMMA 3. *Let assumption* (ii) *be fulfilled. Then*

$$\sup_{-s \leq \eta \leq 0} \sup_{n \in \mathbf{N}} \| F(\eta)^n \| < \infty.$$

Proof. For $\eta \in [-s, 0]$ the polynomial $\pi_1(\zeta, \eta)$ has in a neighborhood of $\eta = 0$ the simple root $\bar{\zeta}_1(\eta)$ with $\lim_{\eta \to 0} | \bar{\zeta}_1'(\eta) | = \infty$ but depending continously on η. By assumption all roots $\zeta_\kappa(\eta)$ of $\pi_1(\zeta, \eta)$ satisfy $| \zeta_\kappa(\eta) | \leq 1$, and all unimodular roots of $\pi_1(\zeta, \eta)$ are simple roots of $\pi_1(\zeta, \eta)$ for $\eta \in [-s, 0]$. Therefore Lemma 2 of [20] applies literally to $F(\eta)$.

Lemma 3 was proved in a different way by Crouzeix and Raviart [10, Theorem 8.1]; see also Crouzeix [9] and LeRoux [27]. The proof in [10] does not depend on the special form of the Frobenius matrix F and needs only continuity with respect to the argument η.

We now quote the well-known Matrix Theorem of Kreiss [25] in a somewhat simplified form using a modification of Widlund [31].

LEMMA 4 (KREISS). *If the assertion of Lemma* 3 *is true, then there exists to every matrix* $F(\eta)$ *a hermitean matrix* $H(\eta)$ *and a constant* Γ_s *such that*

$$F(\eta)^H H(\eta) F(\eta) \leqslant (1 + \operatorname{spr}(F(\eta)))H(\eta)/2, \qquad 0 < \Gamma_s^{-1} I \leqslant H(\eta) \leqslant \Gamma_s I.$$

Now let Ξ be the finite set of the branching points of $\zeta_1(\eta)$ in $[-s, 0)$, i.e., the set of $\eta \in [-s, 0)$ where $\zeta_1(\eta)$ coincides with some other branch $\zeta_\kappa(\eta)$ of the algebraic variety $\zeta(\eta)$. Then $|\zeta_1(\eta)| < 1$ $\forall \eta \in \Xi$ by assumption (ii) and Definition 3(ii), and, by continuity of $\zeta_1(\eta)$ for $\eta \neq \alpha_k/\beta_k$, there is a closed set Ω^* which is the intersection of an open neighborhood of Ξ and $[-s, 0]$ such that $|\zeta_1(\eta)| < 1$ $\forall \eta \in \Omega^*$. We write

$$\Omega = [-s, 0] \setminus \Omega^*$$

and prove the following auxiliary result.

LEMMA 5. *Let assumption* (ii) *be fulfilled, and let*

$$Z(\eta) = (\zeta_1(\eta) I - F(\eta))^{-1}(I - F(\eta)), \qquad 0 \neq \eta \in \Omega,\ Z(0) = I.$$

Then

$$\sup_{\eta \in \Omega} \|Z(\eta)\| < \infty.$$

Proof. $(\zeta_1(\eta) I - F(\eta))^{-1}$ is continuous in $\bar{\Omega} \setminus \{0\}$ by Kato [24, Theorem 2.1.5] since no eigenvalues of $F(\eta)$ coincide with $\zeta_1(\eta)$ in this set by definition of Ω. Hence we must show that $Z(\eta)$ is bounded in a neighborhood of $\eta = 0$. Let $U(\eta)$ be a unitary matrix such that

$$R(\eta) \equiv \{r_{ij}(\eta)\}_{i,j=1}^{k-1} = U(\eta) F(\eta) U(\eta)^H$$

is an upper triangular matrix with $r_{11}(\eta) = \bar{\zeta}_1(\eta)$. Then we have, near $\eta = 0$,

$$|r_{ij}(\eta)| \leqslant \|R(\eta)\| = \|F(\eta)\| < \Gamma.$$

Let $e = (1, 0, \ldots, 0)^T$ be a column vector, and let

$$Z^*(\eta) = (\zeta_1(\eta) I - R(\eta))(I - \zeta_1(\eta) e e^T)^{-1}.$$

We obtain

$$\begin{aligned} U(\eta) Z(\eta) U(\eta)^H - I &= (1 - \zeta_1(\eta))(\zeta_1(\eta) I - R(\eta))^{-1} \\ &= (1 - \zeta_1(\eta))(I - \zeta_1(\eta) e e^T)^{-1} Z^*(\eta)^{-1}, \end{aligned}$$

and $\|(1 - \zeta_1(\eta))(I - \zeta_1(\eta) e e^T)^{-1}\|$ is bounded near $\eta = 0$. In order to prove that $\|Z^*(\eta)^{-1}\|$ is bounded, we omit the argument η and write for this upper triangular matrix

$$Z^* = \operatorname{diag}(Z^*) + R^* = \operatorname{diag}(Z^*)(I + \operatorname{diag}(Z^*)^{-1} R^*),$$

where $\operatorname{diag}(Z^*)$ is the diagonal of Z^*. Then $(R^*)^k \equiv 0$, $\|R^*\| = \|R - \operatorname{diag}(R)\|$ is bounded, and $\|\operatorname{diag}(Z^*)^{-1}\|$ is bounded by (5), Lemma 2(ii), and Definition 3 near $\eta = 0$. Thus a von Neumann series expansion of Z^{*-1} proves the assertion.

LEMMA 6. *Let assumption* (ii) *be fulfilled, and let*

$$G(\eta)=\begin{bmatrix} F(\eta) & 0 \\ F(\eta)-I & \zeta_1(\eta)I \end{bmatrix}.$$

Then

$$\sup_{-s\leqslant\eta\leqslant 0}\ \sup_{n\in\mathbf{N}} \|G(\eta)^n\|<\infty .$$

Proof. We obtain

$$G(\eta)^n=\begin{bmatrix} F(\eta)^n & 0 \\ \left(F(\eta)^n-\zeta_1(\eta)^n I\right)Z(\eta) & \zeta_1(\eta)^n I \end{bmatrix},\qquad \eta\in[-s,0],$$

and we have by assumption and Lemma 3

$$\sup_{-s\leqslant\eta\leqslant 0}\ \sup_{n\in\mathbf{N}} \left(\|F(\eta)^n\|+|\zeta_1(\eta)|^n\right)\leqslant 1+\Gamma_s .$$

Therefore we obtain by Lemma 5

$$\sup_{\eta\in\Omega}\ \sup_{n\in\mathbf{N}} \|\left(F(\eta)^n-\zeta_1(\eta)^n I\right)Z(\eta)\|\leqslant(1+\Gamma_s)\|Z(\eta)\|<\infty .$$

Now, Ω^* is a closed set and $|\zeta_1(\eta)|<1\ \forall\eta\in\Omega^*$ by definition; hence, for $\eta\in\Omega^*$,

$$\begin{aligned}\|\left(F(\eta)^n-\zeta_1(\eta)^n\right)Z(\eta)\| &= \left\|\left(\sum_{j=0}^{n-1} F(\eta)^j\zeta_1(\eta)^{n-1-j}\right)\left(F(\eta)-I\right)\right\| \\ &\leqslant \Gamma_s\sum_{j=0}^{\infty}|\zeta_1(\eta)|^j\leqslant\Gamma_s\left(1-|\zeta_1(\eta)|\right)^{-1}\leqslant\Gamma_s^* .\end{aligned}$$

This proves the assertion because $[-s,0]=\Omega\cup\Omega^*$.

COROLLARY 1. *Under the assumptions of the Theorem there exists to every matrix* $G(h^2A)$ *a norm* $\|\cdot\|_G$ *such that*

$$\|G(h^2A)\|_G\leqslant 1,\qquad \|G\|_G=\max_{W\neq 0}\|GW\|_G/\|W\|_G,$$
$$\Gamma_s^{-1/2}\|W\|\leqslant\|W\|_G\leqslant\Gamma_s^{1/2}\|W\|,\quad \forall W\in C^{2\times(k-1)\times m}.$$

Proof. Let $A=X\Lambda X^T$, $X^TX=I$, be the Jordan canonical decomposition of the matrix A. Then we observe that

$$\|G(h^2A)^n\|=\|G(h^2\Lambda)^n\|\leqslant\sup_{-s\leqslant\eta\leqslant 0}\|G(\eta)^n\|,\qquad n\in\mathbf{N}.$$

Hence, by Lemma 6 and Lemma 4, there exists to every matrix $G(h^2A)$ a Kreiss matrix $H_G(h^2A)$ with the property

$$\|H_G(h^2A)^{1/2}G(h^2A)H_G(h^2A)^{-1/2}\|\leqslant 1$$

because $\operatorname{spr}(G(h^2A))\leqslant 1$ for $h^2\|A\|\leqslant s$. The norm $\|W\|_G\equiv\|H_G(h^2A)^{1/2}W\|$ then has the desired properties.

3. Proof of the Theorem. The error $e_n = y_n - v_n$ of the method $\langle \rho, \sigma, \tau \rangle$ for the problem (1) satisfies

$$\begin{aligned}\pi(\Theta, h^2A)e_n &= \pi_1(\Theta, h^2A)\left(\Theta - \zeta_1(h^2A)\right)e_{n-1} \\ &= h \sum_{\kappa=0}^{k} \beta_\kappa B_{n+\kappa}\tau_\kappa(\Theta)e_n + d_{\langle \rho,\sigma,\tau\rangle}(h, y)_n, \qquad n = 1,2,\ldots.\end{aligned} \tag{9}$$

By (6) all polynomials $\tau_\kappa(\zeta)$ of a consistent method have the root $\zeta = 1$. We write

$$\tau_\kappa(\zeta)/(\zeta - 1) = \sum_{\mu=0}^{k-1} \delta_\mu^{(\kappa)}\zeta^\mu,$$

and introduce the vectors of block dimension $k - 1$

$$E_n = \left(e_{n-k+2},\ldots,e_n\right)^T,$$

$$D_n = \left(0,\ldots,0,\left(\alpha_k I - \beta_k h^2A\right)^{-1} d_{\langle \rho,\sigma,\tau\rangle}(h, y)_{n-k}\right)^T.$$

Then Eq. (9) is equivalent to the two-step scheme

$$\begin{aligned}E_n - \zeta_1(h^2A)E_{n-1} = F(h^2A)&\left(E_{n-1} - \zeta_1(h^2A)E_{n-2}\right) \\ &+ hP_n(E_n - E_{n-1}) + hQ_n(E_{n-1} - E_{n-2}) + D_n, \\ &\qquad n = k, k+1,\ldots.\end{aligned} \tag{10}$$

Here P_n and Q_n are matrices of which only the last rows are nonzero and are chosen such that

$$\text{last element of } P_n(E_n - E_{n-1}) + Q_n(E_{n-1} - E_{n-2})$$
$$= \left(\alpha_k I - \beta_k h^2A\right)^{-1} \sum_{\kappa=0}^{k} \beta_\kappa B_{n-k+\kappa} \sum_{\mu=0}^{k-1} \delta_\mu^{(\kappa)}\left(e_{n-k+\mu+1} - e_{n-k+\mu}\right).$$

Hence there exists a constant Γ_1 such that

$$|||P|||_n + |||Q|||_n \leqslant \Gamma_1 |||B|||_n. \tag{11}$$

We now write briefly ζ for $\zeta_1(h^2A)$, F for $F(h^2A)$, and G for $G(h^2A)$. A substitution of

$$E_n - E_{n-1} = (E_n - \zeta E_{n-1}) - (E_{n-1} - \zeta E_{n-2}) + \zeta(E_{n-1} - E_{n-2}) \tag{12}$$

into (10) and a substitution of (10) into (12) yields

$$W_n = (I - hP_n)^{-1}GW_{n-1} + hL_nW_{n-1} + D_n^*, \qquad n = 1,2,\ldots, \tag{13}$$

where

$$W_n = (E_n - \zeta E_{n-1}, E_n - E_{n-1})^T, \qquad D_n^* = (I - hP_n)^{-1}(D_n, D_n)^T,$$

and

$$L_n = (I - hP_n)^{-1}\begin{bmatrix} -P_n & Q_n + \zeta P_n \\ 0 & Q_n \end{bmatrix}.$$

Supposing that

$$\Gamma_s\Gamma_1 h |||B||| \leqslant \tfrac{1}{2}, \tag{14}$$

we obtain by Corollary 1

$$\|W_n\|_G \leqslant (1 + \Gamma_s\Gamma_1 h |||B|||_n + \Gamma_s h \|L_n\|)\|W_{n-1}\|_G + \|D_n^*\|_G, \qquad n = k, k+1, \ldots.$$

But $\|L_n\| \leqslant \Gamma |||B|||_n$, hence

$$(15) \qquad \|E_n - E_{n-1}\| \leqslant \Gamma_s \exp\{\Gamma_s^* |||B|||_n (nh)\}\left[\|W_{k-1}\| + \sum_{\nu=k}^{n} \|D_\nu\|\right].$$

Finally, we observe that

$$\begin{aligned}\|W_{k-1}\| &\leqslant \|E_{k-1} - E_{k-2}\| + \|E_{k-1} - \zeta E_{k-2}\| \\ &\leqslant 2\|E_{k-1} - E_{k-2}\| + \|(\zeta_1(h^2A) - I)E_{k-2}\|,\end{aligned}$$

therefore we have by Lemma 2

$$\|W_{k-1}\| \leqslant \Gamma(\|E_{k-1} - E_{k-2}\| + h\|A\|^{1/2}\|E_{k-2}\|).$$

A substitution of this bound into (15) yields

$$\begin{aligned}\|E_n\| \leqslant \|E_{k-2}\| &+ n\Gamma_s \exp\{\Gamma_s^* |||B|||_n (nh)\} \\ &\times\left[\|E_{k-1} - E_{k-2}\| + h\|A\|^{1/2}\|E_{k-2}\| + \sum_{\nu=k}^{n} \|d_{\langle\rho,\sigma,\tau\rangle}(h, y)_{\nu-k}\|\right],\end{aligned}$$

and a substitution of the result of Lemma 1 into this estimation proves the Theorem.

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Mathematisches Institut A der Universität
Postfach 560
7000 Stuttgart 1
Federal Republic of Germany

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