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On the Uniqueness of Best $L_2[0, 1]$ Approximation by Piecewise Polynomials With Variable Breakpoints*

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Abstract. In this paper a sufficient condition for the uniqueness of best $L_2[0, 1]$ approximation by piecewise polynomials of order k with variable breakpoints is generalized from that of order 2. Other extensions included here are nonuniqueness and eventual uniqueness results.

1. Introduction. Let P_N^k denote the manifold of all piecewise polynomials of order k with N arbitrary but distinct breakpoints in $(0, 1)$. In other words, for each $p \in P_N^k$, there exists a mesh $\boldsymbol{\sigma}^N = (\sigma_1, \ldots, \sigma_N)$, with

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N < \sigma_{N+1} = 1,$$

such that the restriction of p to (σ_i, σ_{i+1}) is a polynomial of order k.

Recently, the authors in [1] found a fairly large class of strictly convex functions such that each member in the class admits one and only one best $L_2[0, 1]$ approximation from S_N^2, the continuous submanifold of P_N^2. They also discovered that a sufficiently smooth strictly convex function eventually, that is for all large N, has unique best $L_2[0, 1]$ approximation from S_N^2. To demonstrate the sharpness of these two results, they actually constructed, for each positive integer N, an infinitely differentiable strictly convex function which does have more than one best $L_2[0, 1]$ approximation from S_N^2.

The main purpose of this paper is to extend the above three results in [1] for P_N^k. However, it is important to point out here that when k is even the extensions hold for the continuous submanifold of P_N^k. In particular, when k equals 2 the continuous submanifold S_N^2 of P_N^2 turns out to be a spline manifold.

Let $\Sigma^N \subset R^N$ be the closed simplex

$$\{\boldsymbol{\sigma}^N = (\sigma_1, \ldots, \sigma_N) \colon 0 = \sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_N \leqslant \sigma_{N+1} = 1\}.$$

We denote the linear manifold of all piecewise polynomials of order k on a given mesh $\boldsymbol{\sigma}^N \in \Sigma^N$ by $P^k(\boldsymbol{\sigma}^N)$. Let $P^k(\Sigma^N)$ be the union of $P^k(\boldsymbol{\sigma}^N)$ for all $\boldsymbol{\sigma}^N \in \Sigma^N$. Clearly, $P_N^k \subset P^k(\Sigma^N)$ consists of all $P^k(\boldsymbol{\sigma}^N)$ where $\boldsymbol{\sigma}^N \in \operatorname{int} \Sigma^N$. If $\hat{f} \in P^k(\hat{\boldsymbol{\sigma}}^N)$ happens to be a best $L_2[0, 1]$ approximation to f from $P^k(\Sigma^N)$, then $\hat{\boldsymbol{\sigma}}^N = (\hat{\boldsymbol{\sigma}}_1, \ldots, \hat{\boldsymbol{\sigma}}_N)$ is said to be an optimal mesh (with respect to f and k) in Σ^N. It is also clear that the restriction of $\hat{f}$ to each nontrivial interval $(\hat{\sigma}_i, \hat{\sigma}_{i+1})$ has to be the unique best L_2 polynomial of order k to approximate f on $(\hat{\sigma}_i, \hat{\sigma}_{i+1})$. Since $f^{(k)}$ is required to be

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continuous and of one sign, say positive, on [0, 1] throughout this paper, $\hat{\boldsymbol{\sigma}}^N$ being optimal (with respect to f and k) cannot occur on the boundary of Σ^N, and hence a best $L_2[0, 1]$ approximation $\hat{f}$ from $P^k(\Sigma^N)$ to f always lies in P^k_N.

In Section 2, certain properties of best L_2 approximation by polynomials are studied, and various formulae for evaluations of an error function at the endpoints of a finite interval are derived. When applied to a function with the kth derivative of one sign on [0, 1], the formulae become extremely useful.

Section 3 contains the uniqueness and nonuniqueness results for best $L_2[0, 1]$ approximation from P^k_N. In order to establish the uniqueness result, we need to construct a map $\mathbf{F}$, which depends on f and k, from Σ^N into R^N, such that $\mathbf{F}$ vanishes at every optimal mesh $\hat{\boldsymbol{\sigma}}^N$ (with respect to f and k) in Σ^N. The formulae derived in Section 2 are crucial for this construction. Let $J(\hat{\boldsymbol{\sigma}}^N; \mathbf{F})$ be the Jacobian matrix of $\mathbf{F}$ at $\boldsymbol{\sigma}^N$, where $\boldsymbol{\sigma}^N$ solves $\mathbf{F}(\cdot) = \mathbf{0} = (0, \ldots, 0)$. A sufficient condition for the uniqueness result to hold is that if $\mathbf{F}(\boldsymbol{\sigma}^N) = \mathbf{0}$, then the determinant of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ is always positive. With this, we can count the number of solutions of $\mathbf{F}(\cdot) = \mathbf{0}$, which is given by the topological degree of $\mathbf{F}$. The magic number is one.

The eventual uniqueness result for best $L_2[0, 1]$ approximation from P^k_N is discussed in Section 4. Since the hypothesis on $f^{(k)}$ in this result is considerably weaker than that in the uniqueness result, we have to work somehow harder to establish that for N sufficiently large the determinant of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ is positive whenever $\boldsymbol{\sigma}^N$ solves $\mathbf{F}(\cdot) = \mathbf{0}$.

In Section 5, we give some numerical results which indicate that the optimal meshes are indeed better than the balanced ones [2], even though they behave quite the same asymptotically.

2. Preliminaries. Let k be a positive integer and P^k the set of all polynomials of order k. Throughout this paper, we always use $\pi(\cdot\,; f, k, [a, b])$ to denote the unique best L_2 approximation from P^k to f on $[a, b]$. For each bounded function f on $[a, b]$, we define

$$\lambda(f; k, [a, b]) = f(a) - \pi(a; f, k, [a, b])$$

and

$$\rho(f; k, [a, b]) = f(b) - \pi(b; f, k, [a, b]).$$

Clearly, λ and ρ are well defined. First, we would like to find out how they act on $(\cdot - t)^{k-1}_+$ for $t \in [a, b]$.

LEMMA 2.1. *If $t \in [a, b]$, then*

$$\lambda\big((\cdot - t)^{k-1}_+; k, [a, b]\big) = (-1)^k (b - t)^k (t - a)^{k-1} / (b - a)^k \tag{2.1}$$

and

$$\rho\big((\cdot - t)^{k-1}_+; k, [a, b]\big) = (b - t)^{k-1} (t - a)^k / (b - a)^k. \tag{2.2}$$

Proof. Let $a_0(t) + a_1(t)(x - a) + \cdots + a_{k-1}(t)(x - a)^{k-1}$ represent the best L_2 approximant $\pi(x; (\cdot - t)^{k-1}_+, k, [a, b])$. It is well known that the k-tuple

$(a_0, a_1,\ldots,a_{k-1})$ is the unique solution of the following system of linear equations

$$\begin{pmatrix} (b-a) & (b-a)^2/2 & \cdots & (b-a)^k/k \\ (b-a)^2/2 & (b-a)^3/3 & \cdots & (b-a)^{k+1}/(k+1) \\ \vdots & \vdots & & \vdots \\ (b-a)^k/k & (b-a)^{k+1}/(k+1) & \cdots & (b-a)^{2k-1}/(2k-1) \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{k-1} \end{pmatrix}, \tag{2.3}$$

where

$$r_i = \int_t^b (x-a)^i (x-t)^{k-1}\,dx, \qquad i = 0,1,\ldots,k-1.$$

Let H_k denote the coefficient matrix of (2.3) and H_k^j the matrix obtained from H_k with its $(j+1)$st column replaced by the right side of (2.3). By Cramer's rule, we have

$$a_j = \det H_k^j/\det H_k, \qquad j = 0,1,\ldots,k-1.$$

Since for each $i = 0,1,\ldots,k-1$, $r_i = (b-t)^k p_i(t)$, where $p_i \in P^k$ is a polynomials of order k in t, we obtain, after expanding $\det H_k^j$ by minors of the $(j+1)$st column, that

$$a_j = (b-t)^k q_j(t), \qquad q_j \in P^k. \tag{2.4}$$

Consequently,

$$\lambda\big((\cdot - t)_+^{k-1}; k, [a,b]\big) = -a_0(t) \tag{2.5}$$

and

$$\begin{aligned} &\rho\big((\cdot - t)_+^{k-1}; k, [a,b]\big) \\ &\quad = (b-t)^{k-1} - \big[a_0(t) + (b-a)a_1(t) + \cdots + (b-a)^{k-1}a_{k-1}(t)\big] \end{aligned} \tag{2.6}$$

both are polynomials of order $2k$ in t.

Since $\pi(\cdot; \alpha f + \beta g, k, [a,b]) = \alpha\pi(\cdot; f, k, [a,b]) + \beta\pi(\cdot; g, k, [a,b])$ and $(\cdot - t)_+^{k-1} = [|\cdot - t|^{k-1} + (\cdot - t)^{k-1}]/2$, we have

$$\begin{aligned} &(\cdot - t)_+^{k-1} - \pi\big(\cdot; (\cdot - t)_+^{k-1}, k, [a,b]\big) \\ &\quad = \big[|\cdot - t|^{k-1} - \pi\big(\cdot; |\cdot - t|^{k-1}, k, [a,b]\big)\big]/2 \\ &\quad = (t - \cdot)_+^{k-1} - \pi\big(\cdot; (t - \cdot)_+^{k-1}, k, [a,b]\big). \end{aligned} \tag{2.7}$$

A similar result like (2.5) and (2.6) for $(t - \cdot)_+^{k-1}$ leads us to

$$\lambda\big((\cdot - t)_+^{k-1}; k, [a,b]\big) = C_1(b-t)^k(t-a)^{k-1}$$

and

$$\rho\big((\cdot - t)_+^{k-1}; k, [a,b]\big) = C_2(b-t)^{k-1}(t-a)^k.$$

If we differentiate the above equations $k-1$ times with respect to t and then take appropriate limits, we conclude that $C_1 = (-1)^k/(b-a)^k$ and $C_2 = 1/(b-a)^k$.

The next result is a consequence of the Peano Kernel Theorem [3].

THEOREM 2.2. *Let $f \in C^k[a,b]$. Then*

$$\lambda(f; k, [a,b]) = \frac{(-1)^k (b-a)^k}{(k-1)!} \int_0^1 (1-\tau)^k \tau^{k-1} f^{(k)}(\tau(b-a)+a)\, d\tau \tag{2.8}$$

and

$$\rho(f; k, [a,b]) = \frac{(b-a)^k}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^k f^{(k)}(\tau(b-a)+a)\, d\tau. \tag{2.9}$$

Proof. Since the linear functionals λ and ρ annihilate every element in P^k, a simple application of the Peano Kernel Theorem [3] shows that

$$\lambda(f; k, [a,b]) = \frac{1}{(k-1)!} \int_a^b \lambda\big((\cdot - t)_+^{k-1}; k, [a,b]\big) f^{(k)}(t)\, dt \tag{2.10}$$

and

$$\rho(f; k, [a,b]) = \frac{1}{(k-1)!} \int_a^b \rho\big((\cdot - t)_+^{k-1}; k, [a,b]\big) f^{(k)}(t)\, dt. \tag{2.11}$$

Subsituting (2.1) and (2.2) in (2.10) and (2.11), respectively, one gets

$$\lambda(f; k, [a,b]) = \frac{(-1)^k}{(b-a)^k (k-1)!} \int_a^b (b-t)^k (t-a)^{k-1} f^{(k)}(t)\, dt \tag{2.12}$$

and

$$\rho(f; k, [a,b]) = \frac{1}{(b-a)^k (k-1)!} \int_a^b (b-t)^{k-1} (t-a)^k f^{(k)}(t)\, dt. \tag{2.13}$$

Finally, (2.8) follows from (2.12) by a change of variable and so does (2.9) from (2.13).

3. Unique Best $L_2[0,1]$ Approximation From P_N^k. Let $\Sigma^N \subset R^N$ be the closed simplex as defined in Section 1. To be consistent, we like to use $\pi(\cdot; f, k, \boldsymbol{\sigma}^N)$ to denote the unique best $L_2[0,1]$ approximation from the linear manifold $P^k(\boldsymbol{\sigma}^N)$ to f. We now consider the problem of minimizing $\|f - \pi(\cdot; f, k, \boldsymbol{\sigma}^N)\|_2$ over all $\boldsymbol{\sigma}^N \in \Sigma^N$. Since $\Sigma^N \subset R^N$ is closed and bounded, there exists a mesh $\hat{\boldsymbol{\sigma}}^N$, depending on f and k, in Σ^N such that

$$\|f - \pi(\cdot; f, k, \hat{\boldsymbol{\sigma}}^N)\|_2 = \inf_{\boldsymbol{\sigma}^N \in \Sigma^N} \|f - \pi(\cdot; f, k, \boldsymbol{\sigma}^N)\|_2.$$

Every such $\hat{\boldsymbol{\sigma}}^N$ is called an optimal mesh (with respect to f and k) in Σ^N.

Since $f \notin P^k(\Sigma^N)$, it is not difficult to see that an optimal mesh $\hat{\boldsymbol{\sigma}}^N$ (with respect to f and k) has to lie in the interior of Σ^N. Based on this fact, the following lemma says $\pi(\cdot, f, k, \hat{\boldsymbol{\sigma}}^N)$ is continuous at $\hat{\sigma}_i$ if k is even and discontinuous at $\hat{\sigma}_i$, but symmetric with respect to $f(\hat{\sigma}_i)$ if k is odd.

LEMMA 3.1. *Suppose $f^{(k)}$ is continuous and positive on* [0, 1]. *Then*

$$(3.1)\quad \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i]) = \begin{cases} \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}]), & \text{if } k \text{ even}, \\ 2f(\hat{\sigma}_i) - \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}]), & \text{if } k \text{ odd}, \end{cases}$$

where $\hat{\boldsymbol{\sigma}}^N = (\hat{\sigma}_1, \ldots, \hat{\sigma}_N) \in \operatorname{int} \Sigma^N$ is an optimal mesh (*with respect to f and k*).

Proof. For $i = 1, \ldots, N$, define

$$E_i(t) = \int_{\hat{\sigma}_{i-1}}^{t} |f - \pi(\cdot; f, k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i])|^2 + \int_t^{\hat{\sigma}_{i+1}} |f - \pi(\cdot; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}])|^2, \qquad t \in (\hat{\sigma}_{i-1}, \hat{\sigma}_{i+1}),$$

where a natural extension of $\pi(\cdot; f, k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i])$ or $\pi(\cdot; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}])$ has been built in to make the corresponding integral meaningful.

Since $f^{(k)} > 0$ on [0, 1], $\hat{\sigma}_i$ is a local minimum of $E_i(t)$ and hence $E_i'(t)$ vanishes at $\hat{\sigma}_i$. Consequently,

$$(3.2)\quad |f(\hat{\sigma}_i) - \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i])| = |f(\hat{\sigma}_i) - \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}])|.$$

Using Theorem 2.2 and the definitions of λ and ρ, we have

$$\begin{aligned} f(\hat{\sigma}_i) - \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i]) &= \rho(f; k, [\hat{\sigma}_{i-1}, \hat{\sigma}_i]) \\ (3.3)\qquad &= \frac{(\hat{\sigma}_i - \hat{\sigma}_{i-1})^k}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^k f^{(k)}(\tau(\hat{\sigma}_i - \hat{\sigma}_{i-1}) + \hat{\sigma}_{i-1})\, d\tau > 0 \end{aligned}$$

and

$$\begin{aligned} f(\hat{\sigma}_i) - \pi(\hat{\sigma}_i; f, k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}]) &= \lambda(f; k, [\hat{\sigma}_i, \hat{\sigma}_{i+1}]) \\ (3.4)\qquad &= \frac{(-1)^k (\hat{\sigma}_{i+1} - \hat{\sigma}_i)^k}{(k-1)!} \int_0^1 (1-\tau)^k \tau^{k-1} f^{(k)}(\tau(\hat{\sigma}_{i+1} - \hat{\sigma}_i) + \hat{\sigma}_i)\, d\tau \\ &\begin{cases} > 0 & \text{if } k \text{ is even}, \\ < 0 & \text{if } k \text{ is odd}. \end{cases} \end{aligned}$$

If we substitute (3.3) and (3.4) in (3.2), then we obtain (3.1).

This lemma is the key to allowing us to construct a mapping $\mathbf{F}$, depending on f and k, from Σ^N into R^N so that $\mathbf{F}$ vanishes at every optimal mesh $\hat{\boldsymbol{\sigma}}^N$ (with respect to f and k) in Σ^N. To do so, consider $\mathbf{F}\colon \Sigma^N \to R^N$ such that

$$\mathbf{F}(\boldsymbol{\sigma}^N; f, k) = (F_1(\boldsymbol{\sigma}^N; f, k), \ldots, F_N(\boldsymbol{\sigma}^N; f, k)),$$

where for $i = 1, \ldots, N$,

$$\begin{aligned} &F_i(\boldsymbol{\sigma}^N; f, k) \\ (3.5)\qquad &= \begin{cases} \pi(\sigma_i; f, k, [\sigma_i, \sigma_{i+1}]) - \pi(\sigma_i; f, k, [\sigma_{i-1}, \sigma_i]), & \text{if } k \text{ is even}, \\ 2f(\sigma_i) - \pi(\sigma_i; f, k, [\sigma_i, \sigma_{i+1}]) - \pi(\sigma_i; f, k, [\sigma_{i-1}, \sigma_i]), & \text{if } k \text{ is odd}. \end{cases} \end{aligned}$$

As we mentioned earlier, $\mathbf{F}(\hat{\boldsymbol{\sigma}}^N; f, k) = \mathbf{0} = (0, \ldots, 0)$ if $\hat{\boldsymbol{\sigma}}^N \in \operatorname{int} \Sigma^N$ is optimal (with respect to f and k). Letting $\Delta\sigma_j = \sigma_{j+1} - \sigma_j$, we claim that (3.5) becomes

$$
\begin{aligned}
F_i(\boldsymbol{\sigma}^N; f, k) &= \frac{(\Delta\sigma_{i-1})^k}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^k f^{(k)}(\tau\Delta\sigma_{i-1} + \sigma_{i-1})\, d\tau \\
&\quad - \frac{(\Delta\sigma_i)^k}{(k-1)!} \int_0^1 (1-\tau)^k \tau^{k-1} f^{(k)}(\tau\Delta\sigma_i + \sigma_i)\, d\tau, \qquad 1 \leqslant i \leqslant N,
\end{aligned}
\tag{3.6}
$$

no matter what k is. To verify this, one simply rewrites (3.5) and uses Theorem 2.2.

Suppose that f satisfies the hypotheses of Lemma 3.1. The mapping $\mathbf{F}\colon \Sigma^N \to R^N$ defined by (3.6) via (3.5) thus has the following properties:

(a) $\mathbf{F}$ is continuous on Σ^N and differentiable in the interior of Σ^N;

(b) $\mathbf{F}$ does not vanish on the boundary of Σ^N;

(c) $\mathbf{F}(\hat{\boldsymbol{\sigma}}^N) = \mathbf{0}$ if $\hat{\boldsymbol{\sigma}}^N$ is optimal (with respect to f and k) in Σ^N;

(d) $\mathbf{F}$ has at least one such zero in Σ^N.

Let $J(\boldsymbol{\sigma}^N; \mathbf{F})$ denote the Jacobian matrix $[\partial F_i/\partial\sigma_j]$ of $\mathbf{F}(\cdot\,; f, k)$ at $\boldsymbol{\sigma}^N \in \operatorname{int}\Sigma^N$. With an additional restriction on f, the following lemma guarantees that the determinant of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ is positive whenever $\boldsymbol{\sigma}^N$ is a solution of $\mathbf{F}(\cdot\,; f, k) = \mathbf{0}$.

LEMMA 3.2. *Let $f \in C^k[0,1]$ with $f^{(k)} > 0$ on $[0,1]$. Suppose that $\log f^{(k)}$ is concave in $(0,1)$. Then $\det J(\boldsymbol{\sigma}^N; \mathbf{F}) > 0$ if $\mathbf{F}(\boldsymbol{\sigma}^N; f, k) = \mathbf{0}$.*

Proof. Since $\mathbf{F}$ does not vanish on the boundary of Σ^N, we only have to consider $\boldsymbol{\sigma}^N \in \operatorname{int}\Sigma^N$. Moreover, the Jacobian matrix $J(\cdot\,; \mathbf{F}) = [\partial F_i/\partial\sigma_j]$ of $\mathbf{F}$ is tridiagonal, and its nonzero entries are

$$
\begin{aligned}
\frac{\partial F_i}{\partial\sigma_{i-1}} &= -\frac{k(\Delta\sigma_{i-1})^{k-1}}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^k f^{(k)}(\tau\Delta\sigma_{i-1} + \sigma_{i-1})\, d\tau \\
&\quad + \frac{(\Delta\sigma_{i-1})^k}{(k-1)!} \int_0^1 (1-\tau)^k \tau^k f^{(k+1)}(\tau\Delta\sigma_{i-1} + \sigma_{i-1})\, d\tau, \qquad 2 \leqslant i \leqslant N,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\frac{\partial F_i}{\partial\sigma_i} &= \frac{k(\Delta\sigma_{i-1})^{k-1}}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^k f^{(k)}(\tau\Delta\sigma_{i-1} + \sigma_{i-1})\, d\tau \\
&\quad + \frac{(\Delta\sigma_{i-1})^k}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \tau^{k+1} f^{(k+1)}(\tau\Delta\sigma_{i-1} + \sigma_{i-1})\, d\tau \\
&\quad + \frac{k(\Delta\sigma_i)^{k-1}}{(k-1)!} \int_0^1 (1-\tau)^k \tau^{k-1} f^{(k)}(\tau\Delta\sigma_i + \sigma_i)\, d\tau \\
&\quad - \frac{(\Delta\sigma_i)^k}{(k-1)!} \int_0^1 (1-\tau)^{k+1} \tau^{k-1} f^{(k+1)}(\tau\Delta\sigma_i + \sigma_i)\, d\tau, \qquad 1 \leqslant i \leqslant N,
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\frac{\partial F_i}{\partial\sigma_{i+1}} &= \frac{k(\Delta\sigma_i)^{k-1}}{(k-1)!} \int_0^1 (1-\tau)^k \tau^{k-1} f^{(k)}(\tau\Delta\sigma_i + \sigma_i)\, d\tau \\
&\quad - \frac{(\Delta\sigma_i)^k}{(k-1)!} \int_0^1 (1-\tau)^k \tau^k f^{(k+1)}(\tau\Delta\sigma_i + \sigma_i)\, d\tau, \qquad 1 \leqslant i \leqslant N-1.
\end{aligned}
\tag{3.9}
$$

Note that $\partial F_i/\partial\sigma_{i-1}$ and $\partial F_i/\partial\sigma_{i+1}$ are negative since (3.7) and (3.9) can be written as

$$\frac{\partial F_i}{\partial\sigma_{i-1}} = \frac{k(\Delta\sigma_{i-1})^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k-1}\tau^k f^{(k)}(\tau\Delta\sigma_{i-1}+\sigma_{i-1})\,d\tau$$
$$-\frac{k(\Delta\sigma_{i-1})^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k-1}\tau^{k-1}(1-2\tau) f^{(k)}(\tau\Delta\sigma_{i-1}+\sigma_{i-1})\,d\tau$$
$$= -\frac{k(\Delta\sigma_{i-1})^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k}\tau^{k-1} f^{(k)}(\tau\Delta\sigma_{i-1}+\sigma_{i-1})\,d\tau, \qquad 2 \leqslant i \leqslant N,$$

and

$$\frac{\partial F_i}{\partial\sigma_{i+1}} = -\frac{k(\Delta\sigma_i)^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k}\tau^{k-1} f^{(k)}(\tau\Delta\sigma_i+\sigma_i)\,d\tau$$
$$+\frac{k(\Delta\sigma_i)^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k-1}\tau^{k-1}(1-2\tau) f^{(k)}(\tau\Delta\sigma_i+\sigma_i)\,d\tau$$
$$= -\frac{k(\Delta\sigma_i)^{k-1}}{(k-1)!}\int_0^1 (1-\tau)^{k-1}\tau^{k} f^{(k)}(\tau\Delta\sigma_i+\sigma_i)\,d\tau, \qquad 1 \leqslant i \leqslant N-1.$$

$(a_0, a_1, \ldots, a_{k-1})$ is the unique solution of the following system of linear equations

$$(2.3)\qquad \begin{pmatrix} (b-a) & (b-a)^2/2 & \cdots & (b-a)^k/k \\ (b-a)^2/2 & (b-a)^3/3 & \cdots & (b-a)^{k+1}/(k+1) \\ \vdots & \vdots & & \vdots \\ (b-a)^k/k & (b-a)^{k+1}/(k+1) & \cdots & (b-a)^{2k-1}/(2k-1) \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{k-1} \end{pmatrix},$$

where

$$r_i = \int_t^b (x-a)^i (x-t)^{k-1}\,dx, \qquad i = 0, 1, \ldots, k-1.$$

Let H_k denote the coefficient matrix of (2.3) and H_k^j the matrix obtained from H with its $(j+1)$st column replaced by the right side of (2.3). By Cramer's rule, w have

$$a_j = \det H_k^j/\det H_k, \qquad j = 0, 1, \ldots, k-1.$$

Since for each $i = 0, 1, \ldots, k-1$, $r_i = (b-t)^k p_i(t)$, where $p_i \in P^k$ is a polynomial of order k in t, we obtain, after expanding $\det H_k^j$ by minors of the $(j+1)$s column, that

Since $f^{(k+1)} = f^{(k)}(f^{(k+1)}/f^{(k)})$, the concavity of $\log f^{(k)}$ does imply that

$$\sum_{j=1}^{N} \partial F_i/\partial\sigma_j = d_i(c_{i-1} - c_i) \geqslant 0, \qquad 2 \leqslant i \leqslant N-1.$$

For $i = 1$ and N, we actually have $\Sigma_{j=1}^{N} \partial F_i/\partial\sigma_j > 0$, because one can argue as above after noticing that $\partial F_1/\partial\sigma_0$ and $\partial F_N/\partial\sigma_{N+1}$ are well defined by the right-hand side of (3.7) and (3.9), respectively. Hence, $J(\boldsymbol{\sigma}^N; \mathbf{F})$ is diagonally dominant. Furthermore, the first and last rows of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ are strictly dominated by the corresponding diagonal elements. By Gerschgorin's Theorem [4], all eigenvalues of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ lie in the open right half plane of the complex plane. Since $J(\boldsymbol{\sigma}^N; \mathbf{F})$ has only real entries, the complex eigenvalues come in conjugate pairs, so that the product of all eigenvalues of $J(\boldsymbol{\sigma}^N; \mathbf{F})$ is positive. This shows that $\det J(\boldsymbol{\sigma}^N; \mathbf{F}) > 0$, whenever $\boldsymbol{\sigma}^N \in \operatorname{int} \Sigma^N$ solves $\mathbf{F}(\cdot\,; f, k) = \mathbf{0}$.

We are now ready to establish the uniqueness theorem.

THEOREM 3.3. *Let $f \in C^k[0,1]$ with $f^{(k)} > 0$ on $[0,1]$. Suppose that $\log f^{(k)}$ is concave in $(0,1)$. Then, for each positive integer N, f has a unique best $L_2[0,1]$ approximation from P_N^k.*

Proof. Let $g(x) = x^k$. Then there exists an optimal $\tilde{\boldsymbol{\sigma}}^N = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N) \in \operatorname{int} \Sigma^N$ such that $\pi(\cdot\,; g, k, \tilde{\boldsymbol{\sigma}}^N) \in P_N^k$ is a best $L_2[0,1]$ approximation to g. Moreover, $\mathbf{F}(\tilde{\boldsymbol{\sigma}}^N; g, k) = \mathbf{0}$. Since $g^{(k)}$ is constant, by (3.6) we conclude that $\tilde{\sigma}_i = i/(N+1)$, $1 \leqslant i \leqslant N$, and that $\tilde{\boldsymbol{\sigma}}^N$ is the only mesh in Σ^N satisfying $\mathbf{F}(\cdot\,; g, k) = \mathbf{0}$. Thus, $\pi(\cdot\,; g, k, \tilde{\boldsymbol{\sigma}}^N)$ is the unique best $L_2[0,1]$ approximant from P_N^k to g. Similarly, there exists an optimal $\hat{\boldsymbol{\sigma}}^N \in \operatorname{int} \Sigma^N$ such that $\pi(\cdot\,; f, k, \hat{\boldsymbol{\sigma}}^N) \in P_N^k$ is a best $L_2[0,1]$ approximant to f. Also, $\mathbf{F}(\hat{\boldsymbol{\sigma}}^N; f, k) = \mathbf{0}$. Now, if we can show that $\hat{\boldsymbol{\sigma}}^N$ is the only solution of $\mathbf{F}(\cdot\,; f, k) = \mathbf{0}$ in Σ^N, then $\pi(\cdot\,; f, k, \hat{\boldsymbol{\sigma}}^N)$ would certainly be the unique best $L_2[0,1]$ approximant from P_N^k to f.

In [6], the topological degree of a smooth map $\mathbf{G}$ from a bounded open set $D \subset R^N$ into R^N, where $\mathbf{G}$ is defined and continuous on the boundary of D, is given by

$$\det(\mathbf{p}, \mathbf{G}, D) = \sum_{\mathbf{G}(\mathbf{x}) = \mathbf{p}} \operatorname{sign} \det J(\mathbf{x}; \mathbf{G}),$$

where the sum is taken over all solutions $\mathbf{x} \in D$ of $\mathbf{G}(\mathbf{x}) = \mathbf{p}$, as long as $J(\mathbf{x}; \mathbf{G})$, the Jacobian matrix of $\mathbf{G}$ at $\mathbf{x}$, is nonsingular and $\mathbf{p}$ is not the image of a boundary point of D under $\mathbf{G}$. It is known that the degree is invariant under homotopy provided that the functions in the homotopy do not introduce solutions on the boundary of D.

Again, let $g(x) = x^k$. For $0 \leqslant \alpha \leqslant 1$, put

$$\mathbf{F}^{\alpha}(\cdot) = \mathbf{F}(\cdot\,; (1-\alpha)g + \alpha f, k).$$

Clearly, $\alpha \to \mathbf{F}^{\alpha}$ is a homotopy. Then by (3.6) it is easy to see that $\mathbf{F}^{\alpha}$ does not vanish on the boundary of Σ^N. Since the degree is invariant under homotopy, we have that

$$\deg(\mathbf{0}, \mathbf{F}^1, \Sigma^N) = \deg(\mathbf{0}, \mathbf{F}^0, \Sigma^N). \tag{3.10}$$

From earlier discussion we know that $\mathbf{F}^0(\cdot) = \mathbf{F}(\cdot\,; g, k) = \mathbf{0}$ has exactly one solution in Σ^N. Therefore, $\deg(\mathbf{0}, \mathbf{F}^0, \Sigma^N) = 1$ by Lemma 3.2. It then follows from

(3.10) that $\deg(\mathbf{0}, \mathbf{F}^1, \Sigma^N) = 1$. Note that f satisfies the hypotheses of Lemma 3.2. Therefore, $\det J(\boldsymbol{\sigma}^N; \mathbf{F}^1) > 0$ whenever $\boldsymbol{\sigma}^N$ is a solution of $\mathbf{F}^1(\cdot) = \mathbf{F}(\cdot\,; f, k) = \mathbf{0}$. Consequently, $\mathbf{F}^1(\cdot) = \mathbf{F}(\cdot\,; f, k) = \mathbf{0}$ has one and only one solution in Σ^N, because $\deg(\mathbf{0}, \mathbf{F}^1, \Sigma^N) = 1$ counts the number of solutions of $\mathbf{F}^1(\cdot) = \mathbf{F}(\cdot\,; f, k) = \mathbf{0}$ in Σ^N.

The condition that $\log f^{(k)}$ is concave in $(0,1)$ is a technical one; but some concavity condition like this is necessary for the uniqueness result to hold in Theorem 3.3. In fact, for each positive integer N, there is a function $f \in C^\infty[0,1]$ with $f^{(k)} > 0$ on $[0,1]$, such that f has more than one best $L_2[0,1]$ approximation from P_N^k. To construct such an f, we need the following lemmas.

LEMMA 3.4. *Let* $p(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1}$, $\varepsilon > 0$ *and* b *all be given. For each* $l > 0$, *let*

$$a_{0,l} + a_{1,l}x + \cdots + a_{k-1,l}x^{k-1} = \pi(x; f_l, k, [0, l+\varepsilon]),$$

where

$$f_l(x) = \begin{cases} p(x), & \text{if } x \in [0, l), \\ b(x-l)^{k-1} + \sum\limits_{i=0}^{k-2} p^{(i)}(l)(x-l)^i/i!, & \text{if } x \in [l, l+\varepsilon]. \end{cases}$$

Then, $\max_{0 \leqslant j \leqslant k-1} |a_{j,l} - a_j| \to 0$ *as* $l \to \infty$.

Proof. Suppose that the conclusion does not hold. Then for some j, $0 \leqslant j \leqslant k-1$, there is a subsequence $\{a_{j,l_j}\}$ bounded away from a_j, i.e., there exists a $\delta > 0$ such that for all l_i

$$|a_{j,l_i} - a_j| > \delta.$$

Then

$$\int_0^{l_i} |p - \pi(\cdot\,; f_{l_i}, k, [0, l_i + \varepsilon])|^2 \to \infty$$

as $l_i \to \infty$. However,

$$\begin{aligned}(b - a_{k-1}/(k-1)!)^2 \int_0^\varepsilon x^{2(k-1)} &= \int_0^{l_i+\varepsilon} |f_{l_i} - p|^2 \\ &\geqslant \int_0^{l_i+\varepsilon} |f_{l_i} - \pi(\cdot\,; f_{l_i}, k, [0, l_i+\varepsilon])|^2 \\ &\geqslant \int_0^{l_i} |f_{l_i} - \pi(\cdot\,; f_{l_i}, k, [0, l_i+\varepsilon])|^2 \\ &= \int_0^{l_i} |p - \pi(\cdot\,; f_{l_i}, k, [0, l_i+\varepsilon])|^2 \to \infty\end{aligned}$$

contradicting the fact that $(b - a_{k-1}/(k-1)!)^2 \int_0^\varepsilon x^{2(k-1)}$ is finite.

The next lemma shows how to construct a generalized convex spline function s of order k in $C^{k-2}[0, ml+1]$ with interior knots at jl, $j = 1, 2, \ldots, m$. Moreover, the error between s and its best $L_2[0, ml+1]$ approximation from the set of all piecewise polynomials of order k on $[0, ml+1]$ with arbitrary $m-1$ breakpoints is bounded below by a constant independent of l.

LEMMA 3.5. *Let m be a positive integer, and let $0 \leqslant \alpha_0 < \alpha_1 < \cdots < \alpha_m \leqslant 1$. Suppose*

$$s(x)\,|_{[0,l)} = \alpha_0 x^{k-1}/(k-1)! = s_0(x)$$

and

$$\begin{aligned} s(x)\,|_{[jl,(j+1)l)} &= \alpha_j (x-jl)^{k-1}/(k-1)! + \sum_{i=0}^{k-2} s_{j-1}^{(i)}(jl)(x-jl)^i/i! \\ &= s_j(x), \qquad j = 1,2,\ldots,m. \end{aligned}$$

Then, $s \in C^{k-2}[0, ml+1]$ and

$$\int_{ml}^{ml+1} |s_m(x) - s_{m-1}(x)|^2 dx = (\alpha_m - \alpha_{m-1})^2 \int_0^1 [y^{k-1}/(k-1)!]^2\, dy.$$

Proof. The first part is clear. Since

$$s_{m-1}(x) = \alpha_{m-1}(x-ml)^{k-1}/(k-1)! + \sum_{i=0}^{k-2} s_{m-1}^{(i)}(ml)(x-ml)^i/i!,$$

we have

$$\begin{aligned} \int_{ml}^{ml+1} |s_m - s_{m-1}|^2 &= \int_{ml}^{ml+1} |(\alpha_m - \alpha_{m-1})(x-ml)^{k-1}/(k-1)!|^2\, dx \\ &= (\alpha_m - \alpha_{m-1})^2 \int_0^1 [y^{k-1}/(k-1)!]^2\, dy. \end{aligned}$$

LEMMA 3.6. *Let s be a continuous piecewise polynomial of order k on $[a, b]$ with breakpoints at $\tau_1, \ldots, \tau_n$. Suppose*

$$s^{(k-1)}(\tau_j+) - s^{(k-1)}(\tau_j-) = \alpha_j - \alpha_{j-1} > 0.$$

for $j = 1, \ldots, n$. Then, for each $\varepsilon > 0$, there exists an $f \in C^\infty[a, b]$ such that $f^{(k)} > 0$ on $[a, b]$ and $\|s - f\|_\infty < \varepsilon$.

Proof. Let $\{h_i\}$ be a sequence of functions of the form

$$h_i(x) = ih(ix), \qquad i = 1,2,\ldots,$$

where $h \in C_0^\infty(-\infty, \infty)$, $h \geqslant 0$, and $\int_{-\infty}^{\infty} h(x)\, dx = 1$. From [5], we know that, for each i, $s * h_i \in C^\infty(-\infty, \infty)$ and

$$\begin{aligned} (s * h_i)^{(k)} = s^{(k)} * h_i &= \left(\sum_{j=1}^{n} \left[s^{(k-1)}(\tau_j+) - s^{(k-1)}(\tau_j-) \right] \delta_{\tau_j} \right) * h_i \\ &= \sum_{j=1}^{n} \left[s^{(k-1)}(\tau_j+) - s^{(k-1)}(\tau_j-) \right] (\delta_{\tau_j} * h_i) \\ &= \sum_{j=1}^{n} [\alpha_j - \alpha_{j-1}] h_i(\cdot - \tau_j) > 0 \quad \text{on } [a, b], \end{aligned}$$

where δ_{τ_j} is the Dirac measure at τ_j. Since $\{s * h_i\}$ converges uniformly to s on every compact subset of $(-\infty, \infty)$, by choosing $f = s * h_i$ for i sufficiently large we can get $\|s - f\|_\infty < \varepsilon$.

Now we have all the tools to construct a smooth generalized convex function which has more than one best $L_2[0, 1]$ approximation from P_N^k.

THEOREM 3.7. *Let N be any positive integer. There exists a function $f \in C^\infty[0,1]$, with $f^{(k)} > 0$ on $[0,1]$, such that f has more than one best $L_2[0,1]$ approximation from P_N^k, where k is a positive integer greater than 1.*

The proof of Theorem 3.7 is almost exactly the same as the proof of Theorem 3 in [1] except for the construction of f_l, when k is odd, on the left half of the interval in question. Hence we will not repeat it here. Nevertheless, we like to point out that Lemma 3.4 and Lemma 3.5 will lead to a contradiction for some f_l if uniqueness is assumed, and Lemma 3.6 can be used to show the existence of a smooth generalized convex function which approximates f_l arbitrarily close.

4. Eventual Uniqueness Result. Let $f^{(k)}$ be continuous and positive on $[0,1]$. We want to prove in this section that for N sufficiently large the uniqueness result for best $L_2[0,1]$ approximation from P_N^k to f eventually holds for extra smooth f but without assuming the concavity of $\log f^{(k)}$ in $(0,1)$. If we can show that, for N sufficiently large,

$$\det J(\boldsymbol{\sigma}^N; \mathbf{F}) > 0 \tag{4.1}$$

whenever $\boldsymbol{\sigma}^N$ solves $\mathbf{F}(\cdot\,; f, k) = \mathbf{0}$, then the same topological degree argument used in the proof of Theorem 3.3 will lead us to the conclusion. Before we spend the rest of this section to establish (4.1), let us state the eventual uniqueness result first.

THEOREM 4.1. *Let $f \in C^{k+3}[0,1]$ with $f^{(k)} > 0$ on $[0,1]$. Then there exists a positive N_0 such that, for each integer $N > N_0$, f has a unique best $L_2[0,1]$ approximation from P_N^k.*

The proposition we state below is independent of k, and its proof can be found in [1].

PROPOSITION 4.2. *Let $A = [a_{ij}]$ be a tridiagonal $N \times N$ real matrix with positive diagonal entries. Suppose*

$$a_{n,n-1}a_{n-1,n} \leqslant a_{n,n}a_{n-1,n-1}\big(1 + \pi^2/(4N^2)\big)/4,$$

for $n = 2,\ldots,N$. Then $\det A$ *is positive.*

In the rest of this section, we will simply state some results which can be directly derived from those in [1]. Before we do that, the same notations used there have to be introduced. For $\boldsymbol{\sigma}^N \in \Sigma^N$, let

$$h_i = \Delta\sigma_i = \sigma_{i+1} - \sigma_i, \qquad i = 0,\ldots,N,$$

and let

$$\Delta = \max_{0 \leqslant i \leqslant N} h_i \quad \text{and} \quad \delta = \min_{0 \leqslant i \leqslant N} h_i.$$

Now, we rewrite Eq. (3.6) in the following form

$$\begin{aligned} F_i(\boldsymbol{\sigma}^N; f, k) = {} & h_{i-1}^k \int_0^1 w(\tau; k) g(\sigma_i - \tau h_{i-1})\, d\tau \\ & - h_i^k \int_0^1 w(\tau; k) g(\sigma_i + \tau h_i)\, d\tau, \qquad i = 1,\ldots,N, \end{aligned} \tag{4.2}$$

where $w(\tau; k) = (1-\tau)^k \tau^{k-1}/(k-1)!$ and $g = f^{(k)}$.

LEMMA 4.2. *Let $f \in C^{k+1}[0,1]$ with $f^{(k)} > 0$ on $[0,1]$. Then there is a constant $c > 0$, depending on f and k but not on N, such that if $\mathbf{F}(\boldsymbol{\sigma}^N; f, k) = \mathbf{0}$, $\Delta/\delta \leqslant c$.*

COROLLARY 4.3. $\Delta \leqslant c/(N+1)$ *and* $\delta \geqslant 1/(c(N+1))$.

LEMMA 4.4. *Suppose that $f \in C^{k+3}[0,1]$ with $f^{(k)} > 0$ on $[0,1]$. Let $\boldsymbol{\sigma}^N \in \Sigma^N$ solve $\mathbf{F}(\cdot\,; f, k) = \mathbf{0}$, and let $[\alpha_{ij}] = J(\boldsymbol{\sigma}^N; \mathbf{F})$. Then, for N sufficiently large,*

$$\alpha_{i,i+1}\alpha_{i+1,i} = \alpha_{i,i}\alpha_{i+1,i+1}(1 + O(\Delta^3))/4, \qquad i = 1,\ldots,N-1. \tag{4.3}$$

Proof. Let $g_i = g(\sigma_i)$, $g_i' = g'(\sigma_i)$, and $g_i'' = g''(\sigma_i)$. Set $F_i = 0$ in (4.2), and expand g about σ_i. Then we obtain

$$\begin{aligned} h_{i-1}^k\big(Ag_i - h_{i-1}Bg_i' + h_{i-1}^2Cg_i'' + O(\Delta^3)\big)& \\ = h_i^k\big(Ag_i + h_iBg_i' + h_i^2Cg_i'' + O(\Delta^3)\big)&, \end{aligned} \tag{4.4}$$

where

$$A = \int_0^1 w(\tau; k)\,d\tau, \quad B = \int_0^1 \tau w(\tau; k)\,d\tau \quad \text{and} \quad C = \frac{1}{2}\int_0^1 \tau^2 w(\tau; k)\,d\tau.$$

First, we want to show that

$$h_{i-1} = h_i + O(\Delta^2) \tag{4.5}$$

independent of k. To see this, take the ratio form of (4.4) and obtain

$$\begin{aligned} \frac{h_{i-1}}{h_i} &= \left(\frac{Ag_i + h_iBg_i' + h_i^2Cg_i'' + O(\Delta^3)}{Ag_i - h_{i-1}Bg_i' + h_{i-1}^2Cg_i'' + O(\Delta^3)}\right)^{1/k} \\ &= \left(1 + (h_i + h_{i-1})\frac{Bg_i'}{Ag_i} + O(\Delta^2)\right)^{1/k} \\ &= 1 + \frac{h_i + h_{i-1}}{k}\frac{Bg_i'}{Ag_i} + O(\Delta^2). \end{aligned}$$

Hence, (4.5) follows.

Next, we want to prove (4.3) by induction on even integers and on odd integers. For $k = 2$, (4.3) was proved in [1]. The case $k = 1$ can be established the same way, or even easier, as in [1]. Suppose now the case $k = m - 2$ has been verified. We want to show (4.3) for the case $k = m$. Using (4.5) in (4.4), we have

$$\begin{aligned} h_{i-1}^{m-2}\big(A_ig_i - h_{i-1}B_ig_i' + h_{i-1}^2C_ig_i'' + O(\Delta^3)\big)& \\ = h_i^{m-2}\big(A_ig_i + h_iB_ig_i' + h_i^2C_ig_i'' + O(\Delta^3)\big)&, \end{aligned}$$

where $A_i = h_i^2A$, $B_i = h_i^2B$, and $C_i = h_i^2C$. Since the proof of (4.3) in [1] only involves h_{i-1} and h_i, hence, by the induction hypothesis, (4.3) holds for $k = m$.

5. Numerical Results. Let $\boldsymbol{\sigma}^N$ be a balanced mesh [2] in the following sense

$$\int_{\sigma_{i-1}}^{\sigma_i} |f^{(k)}|^r = (N+1)^{-1}\int_0^1 |f^{(k)}|^r, \qquad i = 1,\ldots,N+1,$$

where $r = (k + 2^{-1})^{-1}$. It was shown in [2] that

$$\lim_{N\to\infty} (N+1)^k \| f - \pi(\cdot\,; f, k, \boldsymbol{\sigma}^N)\|_2$$

$$= \|x^k/k! - \pi(\cdot\,; x^k/k!, k, [0,1])\|_2 \left(\int_0^1 |f^{(k)}|^r\right)^{1/r} = B_{2,k}$$

for all $\boldsymbol{\sigma}^N$ either balanced or optimal.

In each of the following examples, we will compute two sets of asymptotic numbers, namely,

$$C_1 = (N+1)^k \| f - \pi(\cdot\,; f, k, \hat{\boldsymbol{\sigma}}^N)\|_2$$

and

$$C_2 = (N+1)^k \| f - \pi(\cdot\,; f, k, \tilde{\boldsymbol{\sigma}}^N)\|_2,$$

where $\hat{\boldsymbol{\sigma}}^N$ is optimal and $\tilde{\boldsymbol{\sigma}}^N$ is balanced, for $k = 2, 3$, and 4.

Example 5.1. Let $f(x) = x^5/5!$ on [0, 1]. Then

$$B_{2,2} \simeq .00086522, \qquad B_{2,3} \simeq .00032375, \quad \text{and}\, B_{2,4} \simeq .00008043,$$

which are almost exact because $|f^{(k)}|^r$ has an antiderivative in [0, 1]. For each $k = 2, 3$, or 4 in this example, $f^{(k)}$ is also a polynomial, so that the integrals which form the entries of $\mathbf{F}$ and $J(\cdot\,; \mathbf{F})$ can be computed very accurately by using enough Gaussian points and weights. The numerical results of this example are contained in Table 1.

TABLE 1
Numerical results for Example 5.1
$f(x) = x^5/5!$

	$N+1$	C_1	C_2
$k = 2$	2	.00101827	.00104806
$r = 2/5$	4	.00093866	.00096213
$f^{(k)}(x) = x^3/3!$	8	.00090118	.00091526
$B_{2,k} = .00086522$	16	.00088301	.00089067
	32	.00087407	.00087806
	64	.00086963	.00087167
$k = 3$	2	.00038256	.00039306
$r = 2/7$	4	.00035221	.00036059
$f^{(k)}(x) = x^2/2$	8	.00033774	.0003428
$B_{2,k} = .00032375$	16	.00033069	.00033345
	32	.0003272	.00032864
	64	.00032547	.00032621
$k = 4$	2	.00008994	.00009096
$r = 2/9$	4	.00008514	.00008596
$f^{(k)}(x) = x$	8	.00008277	.00008326
$B_{2,k} = .00008043$	16	.00008159	.00008186
	32	.00008101	.00008115
	64	.00008072	.00008079

Example 5.2. Let $f(x) = -4x^{1/2}$ on $[0, 1]$. Then

$$B_{2,2} \simeq .36828478, \quad B_{2,3} \simeq .37896504, \quad \text{and} \quad B_{2,4} \simeq .64722762,$$

which are almost exact by the same reason as in Example 5.1. For each $k = 2, 3$, or 4, $f^{(k)}$ is not only not a polynomial but also unbounded in $(0, 1]$. Fortunately, every integral over (σ_0, σ_1) is under control, since $f^{(k)}$ has to be multiplied by a weight function with a zero at least of order 1 at σ_0. To improve the accuracy of numerical integrations over (σ_i, σ_{i+1}), we apply a fixed Gaussian quadrature formula repeatedly on each equally divided subinterval (σ_i, σ_{i+1}). We give the numerical results of this example in Table 2.

TABLE 2
Numerical results for Example 5.2
$f(x) = -4x^{1/2}$

	$N+1$	C_1	C_2
$k = 2$	2	.25062068	.27659286
$r = 2/5$	4	.29904901	.32179519
$f^{(k)}(x) = x^{-3/2}$	8	.33031606	.34487614
$B_{2,k} = .36828478$	16	.34833578	.35654119
	32	.35805037	.36240349
	64	.36310003	.36534181
$k = 3$	2	.14966696	.21896258
$r = 2/7$	4	.22567238	.2949939
$f^{(k)}(x) = -(3/2)x^{-5/2}$	8	.28745951	.336062
$B_{2,k} = .37896504$	16	.32842602	.35730558
	32	.35232005	.36808667
	64	.36527248	.37351415
$k = 4$	2	.11282576	.26905235
$r = 2/9$	4	.24999202	.43665692
$f^{(k)}(x) = (15/4)x^{-7/2}$	8	.38617489	.53724224
$B_{2,k} = .64722762$	16	.49385335	.59121773
	32	.56344529	.61899226
	64	.60334329	.63305554

Theoretically we know that $\pi(\cdot\,; f, k, \hat{\boldsymbol{\sigma}}^n)$ is a better approximation than $\pi(\cdot\,; f, k, \tilde{\boldsymbol{\sigma}}^n)$, although their errors become indistinguishable as n goes to ∞. this fact has just been demonstrated numerically by the two examples.

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