MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 160
OCTOBER 1982, PAGES 599-615

Complex Chebyshev Polynomials on Circular Sectors With Degree Six or Less

By U. Grothkopf and G. Opfer

Abstract. Let T_n^α denote the nth Chebyshev polynomial on the circular sector $S^\alpha = \{z: |z| \leq 1, |\arg z| \leq \alpha\}$. This paper contains numerical values of $\|T_n^\alpha\|_\infty$ and the corresponding coefficients of T_n^α for $n = 1(1)6$ and $\alpha = 0°(5°)180°$. Also all critical angles for T_n^α, $n = 1(1)6$ are listed, where an angle is called critical when the number of absolute maxima of $|T_n^\alpha|$ changes at that angle. All figures are given to six places. The positions (and hence the number) of extremal points of T_n^α, $n = 1(1)6$ are presented graphically. The method consists of a combination of semi-infinite linear programming, finite linear programming, and Newton's method.

1. Introduction. If D is any nonempty compact set in $\mathbf{C}$ and $n \in \mathbf{N}$, then the polynomial

$$T_n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \cdots + a_0^{(n)} \tag{1.1}$$

which is uniquely defined by

$$\|T_n\|_\infty \leq \|z^n + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_0\|_\infty, \quad \text{for all } b_0, b_1, \ldots, b_{n-1} \in \mathbf{C}, \tag{1.2}$$

is called the nth *Chebyshev polynomial* (or *T-polynomial*) *with respect to D*. The norm $\|\ \|_\infty$ is the ordinary uniform norm in $C(D)$ over $\mathbf{C}$. We will be concerned with T-polynomials with respect to circular sectors

$$S^\alpha = \{z \in \mathbf{C}: |z| \leq 1, |\arg z| \leq \alpha\}, \qquad \alpha \in [0, \pi]. \tag{1.3}$$

Circular sectors are convenient for subdividing the plane. There is an investigation by Coleman and Monaghan [2] in which certain complex Bessel functions are approximated on circular sectors. Another idea of approximating complex Bessel functions on circular sectors is presented by Elliott [3].

T-polynomials with respect to S^α will be denoted by T_n^α. Properties of T_n^α are summarized in

THEOREM 1.1. (1) $m \geq n \Rightarrow \|T_m^\alpha\|_\infty \leq \|T_n^\alpha\|_\infty$ *for all* $\alpha \in [0, \pi]$.
(2) $0 \leq \alpha \leq \beta \leq \pi \Rightarrow \|T_n^\alpha\|_\infty \leq \|T_n^\beta\|_\infty$ *for all* $n \in \mathbf{N}$.
(3) $\|T_n^\alpha\|_\infty \leq 1$, *where equality occurs if and only if* $T_n^\alpha = z^n$.
(4) *The coefficients* $a_j^{(n)}, j = 0, 1, \ldots, n-1$, *of* T_n^α *are all real.*
(5) $T_n^\alpha(z) = z^n \Leftrightarrow \alpha \geq n\pi/(n+1)$.

Proof. Geiger and Opfer [4]. □

Received October 8, 1981; revised January 4, 1982.
1980 *Mathematics Subject Classification*. Primary 41A10, 41A50, 65A05, 65E05.

In that paper one also finds T_1^α explicitly for all $\alpha \in [0, \pi]$, T_2^α for all $\alpha \geqslant 15°$, and T_3^α for $35° \leqslant \alpha \leqslant 60°$. Numerical results for T_n^α, $n = 1, 2, 3, 4$, and $\alpha = 0°(10°)90°$, are given by Opfer [9]. That paper also contains numerical expressions for certain T-polynomials with respect to squares and rectangles.

It should be stressed that by (5) of Theorem 1.1 all T-polynomials T_n^α are known when $\alpha \geqslant n\pi/(n+1)$. Therefore the following tables do not contain entries for $\alpha > n\pi/(n+1)$. In Table 1 we present explicit expressions for some selected T-polynomials T_n^α, $n = 1, 2, 3$, which are computed from the expressions given by Geiger and Opfer [4]. As is well known, the ordinary T-polynomials T_n^0 obey a three-term recurrence relation which takes the form

(1.4) $$T_n^0(z) = (z - 1/2)T_{n-1}^0(z) - (1/16)T_{n-2}^0(z), \qquad n = 3, 4, \ldots,$$

(Abramowitz and Stegun [1, Formula 22.7.8]). From the explicit expressions for $T_j^{\pi/3}, j = 1, 2, 3$, given by Geiger and Opfer [4], one finds

(1.5) $$T_3^{\pi/3}(z) = (z - 1/2)T_2^{\pi/3}(z) - \left(\sqrt{3}/2 - 1\right)T_1^{\pi/3}(z), \qquad z \in S^{\pi/3}.$$

However, this formula does not carry over to larger n, which can be seen by inserting $z = 0$. Thus, as could be expected, the complex T-polynomials in general do not follow a three-term recurrence relation.

TABLE 1

Explicit expressions for coefficients in $T_n^\alpha(z) = z^n + a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \cdots + a_0^{(n)}$, $n = 1, 2, 3$; $\alpha = 0°, 30°, 45°, 60°, 90°$

α	$a_0^{(1)}$	$a_0^{(2)}$	$a_1^{(2)}$	$a_0^{(3)}$	$a_1^{(3)}$	$a_2^{(3)}$
0°	$-1/2$	$1/8$	-1	$-1/32$	$9/16$	$-3/2$
30°	$-\sqrt{3}/3$	$1/3$	$-2\sqrt{3}/3$			
45°	$-\sqrt{2}/2$	$\sqrt{2}-1$	-1	$-2(8 - 3\sqrt{2}/2$ $-\sqrt{\sqrt{2}+2}(5-2\sqrt{2}))/17$	$-a_0^{(3)}(\sqrt{2}+3)/2$ $+\sqrt{2}/2$	$a_0^{(3)}(\sqrt{2}-1)/2$ $-\sqrt{2}/2 - 1$
60°	$-1/2$	$1/2$	-1	$(\sqrt{3}-3)/4$	$(4-\sqrt{3})/2$	$-3/2$
90°	0	$\sqrt{2}-1$	$\sqrt{2}-2$			

Associated with each T-polynomial is the set of *extremal points*

(1.6) $$E_n^\alpha = \left\{z \in S^\alpha \colon |T_n^\alpha(z)| = \|T_n^\alpha\|_\infty\right\},$$

which is also referred to as set of *critical points*. Since polynomials are holomorphic functions, the set E_n^α is a nonempty compact subset of the boundary ∂S^α of S^α. The symmetry of the T-polynomials (i.e., $T(\bar{z}) = \overline{T(z)}$) implies that $z \in E_n^\alpha \Rightarrow \bar{z} \in E_n^\alpha$. Therefore it is reasonable to introduce the *essential set of critical points* by

(1.7) $$\text{ess } E_n^\alpha = \{z \in E_n^\alpha \colon \text{Im } z \geqslant 0\}.$$

Definition 1.1. Let $n \in \mathbf{N}$ be fixed. We call $\alpha \in [0, \pi]$ a *critical angle* of the corresponding T-polynomial T_n^α if there is an $\varepsilon_0 > 0$ such that

$$\#\, \text{ess}\, E_n^\alpha \neq \#\, \text{ess}\, E_n^{\alpha+\varepsilon} \quad \text{or} \quad \#\, \text{ess}\, E_n^{\alpha-\varepsilon} \neq \#\, \text{ess}\, E_n^\alpha$$

for all ε with $0 < \varepsilon < \varepsilon_0$. The set of all critical angles will be denoted by A_n.

From Theorem 1.1, part (5), we know that the largest value in A_n always is $\alpha = n\pi/(n+1)$, since for $\alpha \geqslant n\pi/(n+1)$ the number of extremal points becomes infinite, whereas for $\alpha < n\pi/(n+1)$ this number turns out to be finite. At the end of Section 2 we shall see that $0 \notin A_n$, thus A_n is a subset of $]0, \pi[$. We may assume that the critical angles are ordered. To mention some examples

$$A_1 = \{45°, 90°\}, \tag{1.8}$$

$$A_2 = \{\sim 12°, 45°, 72°, 120°\}, \tag{1.9}$$

where $\sim 12°$ indicates that this value is close to 12° but not precisely known (Geiger and Opfer [4]). A more precise value is given in Table 8.

We shall see later that the critical angles can be detected and computed by the use of the corresponding dual program. If, however, two extremal points coalesce or one extremal point splits at $z = 0$ or $z = 1$ for a certain angle α, then this angle (which is *not* critical according to our definition) cannot be detected by this method.

If

$$A_n = \left\{\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{k_n}^{(n)} = n\pi/(n+1)\right\}, \tag{1.10}$$

then, by the definition of the critical angles, the numbers

$$q_j^{(n)} = \#\, \text{ess}\, E_n^\beta, \qquad \beta \in]\alpha_{j-1}^{(n)}, \alpha_j^{(n)}[, \quad j = 1, 2, \ldots, k_n \quad \left(\alpha_0^{(n)} = 0\right) \tag{1.11}$$

are well defined. These numbers will be called *critical multiplicities*. We shall write them in the form

$$N_n = \left(q_1^{(n)}, q_2^{(n)}, \ldots, q_{k_n}^{(n)}\right). \tag{1.12}$$

It is clear that $q_1^{(n)} = n + 1$ since $n + 1$ is the number of extremal points in the real case; cf. Meinardus [8, p. 20]. We have, e.g.,

$$N_1 = (2, 1), \tag{1.13}$$

$$N_2 = (3, 2, 3, 2), \tag{1.14}$$

(Geiger and Opfer [4]). We could call any soluble uniform approximation problem *nice* if it consists of the determination of n real parameters and has the property that the error curve of a best approximation has $n + 1$ extremal points. Under certain conditions such a problem could be dealt with by a Newton-type algorithm which, e.g., was suggested by Hettich [7]. In case there are less than $n + 1$ extremal points (the usual case in nonlinear real approximation), another more difficult approach is needed, which again is described in the mentioned paper by Hettich. With this terminology we see that our problems are not necessarily nice.

2. Description of the Numerical Method. The main idea is the reformulation of the original problem $\|T_n\|_\infty = \min$ as a semi-infinite linear optimization problem. This can be done by using the equation

$$(2.1)\qquad \begin{aligned} D(r) &= \{z \in \mathbf{C}: |z| \leq r\} \\ &= \{z \in \mathbf{C}: \operatorname{Re} \bar{\gamma} z \leq r, \text{ for all } \gamma \in \mathbf{C} \text{ with } |\gamma| = 1\}, \end{aligned}$$

which means that the disk $D(r)$ is represented as the intersection of its supporting halfplanes.

More precisely we use the following

LEMMA 2.1. *Let $D \subset \mathbf{C}$ be a nonempty set, $f: D \to \mathbf{C}$ a function, and r a nonnegative real number.*

Define

$$M_1 = \{z: |f(z)| \leq r\},$$
$$M_2 = \{z: \operatorname{Re} \bar{\gamma} f(z) \leq r \text{ for all } \gamma \in \mathbf{C} \text{ with } |\gamma| = 1\}.$$

Then

$$M_1 = M_2.$$

Proof. Let $z \in M_1$. Then, for all $\gamma \in \mathbf{C}$ with $|\gamma| = 1$, we have $\operatorname{Re} \bar{\gamma} f(z) \leq |\bar{\gamma} f(z)| = |f(z)| \leq r$, and hence $z \in M_2$. Conversely, let $z \in M_2$. If $f(z) = 0$, then $z \in M_1$. Assume therefore that $f(z) \neq 0$. In this case $\gamma = f(z)/|f(z)|$ is of modulus one, and hence $\operatorname{Re} \bar{\gamma} f(z) = |f(z)| \leq r$, hence $z \in M_1$. □

Applied to the problem under investigation we obtain from Lemma 2.1 the semi-infinite linear optimization problem

$$(2.2)\qquad \begin{cases} \operatorname{Re} \bar{\gamma} T_n(z) \leq a_n & \text{for all } z \in \partial S^\alpha \text{ and all } \gamma \in \mathbf{C} \text{ with } |\gamma| = 1, \\ a_n = \min. \end{cases}$$

If we set $\gamma = e^{i\phi}$, $\phi \in [0, 2\pi[$ and $T_n(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, then (2.2) becomes

$$(2.3)\qquad \begin{cases} -\sum_{j=0}^{n-1} a_j g_j(z, \phi) + a_n \geq g_n(z, \phi), \\ \qquad\qquad \text{for all } z \in \partial S^\alpha \text{ and for all } \phi \in [0, 2\pi[, \\ a_n = \min, \end{cases}$$

where

$$(2.4)\qquad g_j(z, \phi) = \operatorname{Re}(z^j)\cos\phi + \operatorname{Im}(z^j)\sin\phi, \qquad j = 0, \ldots, n.$$

For ease of computation, z was represented in polar coordinates. The semi-infinite problem was solved approximately in a first step by admitting only finitely many $\phi \in [0, 2\pi[$ and finitely many $z \in \partial S^\alpha$. In this case (2.3) is an ordinary finite linear optimization problem which can be solved by a suitable version of the Simplex algorithm; cf. Glashoff and Gustafson [5, Chapter IV]. The result of this computation is an approximation for

(1) the coefficients $a_0, a_1, \ldots, a_{n-1}$ and the norm a_n and

(2) for the error function

$$(2.5)\qquad f(z, \phi) = -\sum_{j=0}^{n-1} a_j g_j(z, \phi) + a_n - g_n(z, \phi).$$

In particular, one obtains the number q and the positions of the zeros of f:

$$f(z_j, \phi_j) = 0, \qquad j = 1,2,\ldots,q.$$

From solving the dual problem, one obtains the "masses"

$$w_1, w_2, \ldots, w_q, \quad w_j > 0, \quad \sum_{j=1}^{q} w_j = 1.$$

Now, if we want to improve on these approximations, we regard the $n + 1 + 4q$ quantities $(z_j \in \mathbf{C}, j = 1,2,\ldots,q)$

$$a_0, a_1, \ldots, a_n, \quad z_1, z_2, \ldots, z_q, \quad \phi_1, \phi_2, \ldots, \phi_q, \quad w_1, w_2, \ldots, w_q$$

as unknowns in the following problem; cf. Glashoff and Gustafson [5, Section 16].

$$(2.6) \qquad f(z_j, \phi_j) = 0, \qquad j = 1,2,\ldots,q,$$

$$(2.7) \qquad \sum_{k=1}^{q} w_k g_j(z_k, \phi_k) = 0, \qquad j = 0,1,\ldots,n-1,$$

$$(2.8) \qquad \sum_{k=1}^{q} w_k = 1,$$

$$(2.9) \qquad (z_j, \phi_j) \text{ are local minima of } f, \qquad j = 1,2,\ldots,q.$$

Condition (2.9) always yields $3q$ equations either by setting partial derivatives to be zero or by boundary positions of (z_j, ϕ_j). Thus (2.6)–(2.9) results in a system of $n + 1 + 4q$ equations for the same number of real unknowns.

This system was solved using Newton's method, where the already computed approximations were taken as starting values. Since the T_n^α are symmetric (i.e., $T_n^\alpha(\bar{z}) = \overline{T_n^\alpha(z)}$), one can restrict the domain to

$$\{z \in \partial S^\alpha \colon \operatorname{Im} z \geqslant 0\},$$

and consequently we obtain

$$q = q(\alpha) = \# \operatorname{ess} E_n^\alpha.$$

If $\alpha \in A_n$, i.e., α is a critical angle and

$$(2.10) \qquad q(\alpha + \varepsilon) = q(\alpha - \varepsilon) \pm 1,$$

for all sufficiently small and positive ε, then in (2.9) the number of local minima is reduced by one. That means that a certain "mass" w_{k_0} vanishes. Since for $\alpha = 0$ (real case) we always have $n + 1$ positive masses, $\alpha = 0$ cannot be a critical angle.

Therefore we can introduce α in (2.6)–(2.9) as a new unknown and simultaneously replace w_{k_0} by zero, which leaves the total number of unknowns unchanged.

It should be noticed that a slightly different idea was used by Glashoff and Roleff [6] to obtain the same semi-infinite linear analogue of the Chebyshev approximation problem. However, the idea used here, namely to replace the disks in $\mathbf{C}$ by the union of its supporting halfplanes, immediately carries over to approximation problems with vector valued functions.

There is another investigation by Streit and Nuttall [10], who treat the linear complex approximation problem, in which D is a priori discrete, by introducing a new variable as we did in (2.1).

3. Numerical Figures. In Tables 2–7 we list the coefficients $a_j^{(n)}, j = 0, 1, \ldots, n - 1$, of T_n^α for $n = 1(1)6$ and $\alpha = 0°(5°)n \cdot 180°/(n + 1)$. In Table 8 we list critical angles and corresponding multiplicities of T_n^α for $n = 1(1)6$.

Figure n, $n = 1, 2, \ldots, 6$, contains the positions of the (essential) extremal points of T_n^α for $\alpha = 0°, 5°, 10°, \ldots$. The vertical axis represents α, the horizontal axis represents the distance s from zero measured on the arc ∂S^α. Thus a dot with coordinates (s, α) in Figure n means that T_n^α has an extremal point on ∂S^α which has the shortest distance s from $z = 0$ measured on ∂S^α, $n = 1, 2, \ldots, 6$. Figure 7 contains an enlarged detail of Figure 6.

Figure 8 shows the norm $\|T_4^\alpha\|$ in dependence of α and the corresponding coefficients also as they depend on α.

From Figure 8 and the other given values of $\|T_n^\alpha\|$ it could be conjectured that $\|T_n^\alpha\|_\infty$ is differentiable with respect to α. However, it can be shown that the $a_j^{(n)}$ are *not* differentiable with respect to α. It suffices to look at the explicit expressions in Geiger and Opfer [4] for $n = 1, 2$.

For the case $n = 4$ and $\alpha = \pi/4$, we present in Figures 9 and 10 the corresponding lemniscate and the error curve. The lemniscate is defined by $\{z \in \mathbf{C}: |T_4^{\pi/4}(z)| = \|T_4^{\pi/4}\|_\infty\}$ and the error curve by $\{w \in \mathbf{C}: w = T_4^{\pi/4}(z), z \in \partial S^{\pi/4}\}$.

TABLE 2

Norms and coefficients of T_1^α

α	$\|T_1\|$	$-a_0$
0	0.500000	0.500000
5	0.501910	0.501910
10	0.507713	0.507713
15	0.517638	0.517638
20	0.532089	0.532089
25	0.551689	0.551689
30	0.577350	0.577350
35	0.610387	0.610387
40	0.652704	0.652704
45	0.707107	0.707107
50	0.766044	0.642788
55	0.819152	0.573576
60	0.866025	0.500000
65	0.906308	0.422618
70	0.939693	0.342020
75	0.965926	0.258819
80	0.984808	0.173648
85	0.996195	0.087156
90	1.000000	0.000000

TABLE 3

Norms and coefficients of T_2^α

α	$\|T_2\|$	a_0	$-a_1$
0	0.125000	0.125000	1.000000
5	0.131004	0.131004	1.019809
10	0.153659	0.153659	1.091273
15	0.205605	0.205605	1.164525
20	0.254855	0.254855	1.179178
25	0.297071	0.297071	1.175545
30	0.333333	0.333333	1.154701
35	0.364505	0.364505	1.117737
40	0.391279	0.391279	1.065781
45	0.414214	0.414214	1.000000
50	0.437527	0.437527	1.000000
55	0.465733	0.465733	1.000000
60	0.500000	0.500000	1.000000
65	0.541936	0.541936	1.000000
70	0.593810	0.593810	1.000000
75	0.655775	0.586707	0.930931
80	0.716881	0.532089	0.815207
85	0.774798	0.474555	0.699757
90	0.828427	0.414214	0.585786
95	0.876672	0.351180	0.474508
100	0.918447	0.285575	0.367128
105	0.952684	0.217523	0.264839
110	0.978346	0.147153	0.168807
115	0.994435	0.074599	0.080164
120	1.000000	0.000000	0.000000

TABLE 4

Norms and coefficients of T_3^α

α	$\|T_3\|$	$-a_0$	a_1	$-a_2$
0	0.031250	0.031250	0.562500	1.500000
5	0.038243	0.038243	0.638574	1.591890
10	0.056051	0.056051	0.788605	1.713151
15	0.069168	0.069168	0.868537	1.753742
20	0.081289	0.081289	0.919433	1.756856
25	0.096515	0.096515	0.987671	1.794640
30	0.118714	0.118714	1.079206	1.841777
35	0.150067	0.150067	1.167181	1.867046
40	0.184793	0.184793	1.184793	1.815207
45	0.219545	0.219545	1.191667	1.752576
50	0.253540	0.253540	1.186069	1.678990
55	0.286168	0.286168	1.166968	1.594632
60	0.316987	0.316987	1.133975	1.500000
65	0.347734	0.347734	1.132974	1.440616
70	0.383038	0.383038	1.138724	1.390008
75	0.424426	0.424426	1.141694	1.339559
80	0.473814	0.473814	1.142121	1.290515
85	0.533875	0.533479	1.139700	1.243705
90	0.600566	0.485868	1.000000	1.086434
95	0.666767	0.437027	0.865487	0.932030
100	0.731114	0.387020	0.736674	0.782117
105	0.792163	0.335872	0.613881	0.638341
110	0.848395	0.283564	0.497240	0.502367
115	0.898232	0.230029	0.386719	0.375881
120	0.940035	0.175140	0.282130	0.260599
125	0.972113	0.118706	0.183145	0.158284
130	0.992711	0.060451	0.089302	0.070769
135	1.000000	0.000000	0.000000	0.000000

TABLE 5

Norms and coefficients of T_4^{α}

α	$\|T_4\|$	a_0	$-a_1$	a_2	$-a_3$
0	0.007813	0.007813	0.250000	1.250000	2.000000
5	0.011768	0.011768	0.334312	1.497180	2.171285
10	0.016025	0.016025	0.406893	1.666305	2.261571
15	0.021196	0.021196	0.481030	1.820855	2.339825
20	0.029862	0.029862	0.587999	2.022021	2.434022
25	0.040224	0.040224	0.680104	2.131215	2.451111
30	0.051229	0.051229	0.752572	2.187826	2.435254
35	0.063908	0.063908	0.819091	2.234083	2.416428
40	0.080351	0.080351	0.893141	2.281320	2.397556
45	0.100991	0.100991	0.975487	2.327736	2.376468
50	0.126349	0.126349	1.064864	2.371226	2.350874
55	0.156643	0.156643	1.124365	2.350008	2.283050
60	0.189080	0.189080	1.130279	2.239444	2.159529
65	0.222250	0.222250	1.124812	2.117838	2.027420
70	0.255366	0.255366	1.106222	1.985523	1.887337
75	0.289325	0.289325	1.119523	1.904134	1.784612
80	0.328277	0.328277	1.141383	1.835218	1.693835
85	0.373965	0.373965	1.158573	1.761699	1.603125
90	0.428592	0.428592	1.171307	1.685598	1.514292
95	0.494426	0.452739	1.117092	1.546398	1.387619
100	0.564502	0.407633	0.960605	1.311457	1.193983
105	0.635204	0.362426	0.815251	1.094966	1.006937
110	0.704919	0.317200	0.680982	0.897115	0.828414
115	0.771884	0.271971	0.557416	0.717687	0.660359
120	0.834198	0.226682	0.443868	0.556132	0.504748
125	0.889833	0.181182	0.339378	0.411646	0.363617
130	0.936625	0.135217	0.242730	0.283244	0.239107
135	0.972252	0.088395	0.152441	0.169843	0.133545
140	0.994190	0.040150	0.066728	0.070330	0.049562
144	1.000000	0.000000	0.000000	0.000000	0.000000

TABLE 6

Norms and coefficients of T_5^α

α	$\|T_5\|$	$-a_0$	a_1	$-a_2$	a_3	$-a_4$
0	0.001953	0.001953	0.097656	0.781250	2.187500	2.500000
5	0.003213	0.003213	0.142876	1.024616	2.594298	2.707439
10	0.004748	0.004748	0.188142	1.228294	2.883099	2.833451
15	0.007473	0.007473	0.255481	1.489231	3.206689	2.957994
20	0.010744	0.010744	0.318329	1.682778	3.391165	3.005232
25	0.015180	0.015180	0.388620	1.875026	3.560447	3.045014
30	0.021309	0.021309	0.468525	2.069014	3.711247	3.072378
35	0.029361	0.029361	0.551634	2.238666	3.812858	3.073847
40	0.039309	0.039309	0.628213	2.335576	3.802042	3.019182
45	0.051064	0.051064	0.698276	2.402171	3.757297	2.951273
50	0.066077	0.066077	0.771287	2.461514	3.703657	2.881276
55	0.084697	0.084697	0.848132	2.510686	3.636350	2.804401
60	0.107174	0.107174	0.926878	2.546993	3.553607	2.719145
65	0.133780	0.133780	1.004058	2.563508	3.448826	2.621816
70	0.164491	0.164491	1.063459	2.528615	3.292224	2.498086
75	0.197594	0.197594	1.061541	2.365997	3.013667	2.314023
80	0.231734	0.231734	1.061760	2.220842	2.762001	2.139631
85	0.269830	0.269830	1.100522	2.150424	2.593373	2.009517
90	0.314564	0.314564	1.133474	2.069578	2.422388	1.879428
95	0.369548	0.351858	1.117373	1.927059	2.205527	1.730968
100	0.433496	0.385900	1.093419	1.781440	1.997268	1.580030
105	0.505215	0.368630	0.970140	1.526789	1.703335	1.376849
110	0.579587	0.325852	0.812021	1.247267	1.392867	1.156252
115	0.654515	0.283782	0.669715	1.002162	1.116759	0.947157
120	0.727999	0.242487	0.542372	0.789051	0.873747	0.751733
125	0.797834	0.201940	0.428759	0.605012	0.662069	0.572166
130	0.861619	0.162009	0.327306	0.446779	0.479644	0.410708
135	0.916753	0.122422	0.236130	0.310861	0.324246	0.269762
140	0.960400	0.082736	0.153027	0.193628	0.193698	0.152020
145	0.989384	0.042270	0.075390	0.091329	0.086082	0.060707
150	1.000000	0.000000	0.000000	0.000000	0.000000	0.000000

TABLE 7

Norms and coefficients of T_6^{α}

α	$\lVert T_6 \rVert$	a_0	$-a_1$	a_2	$-a_3$	a_4	$-a_5$
0	0.000488	0.000488	0.035156	0.410156	1.750000	3.375000	3.000000
5	0.000863	0.000863	0.055401	0.580982	2.245246	3.952626	3.232961
10	0.001519	0.001519	0.084967	0.792396	2.772877	4.489658	3.424211
15	0.002476	0.002476	0.119685	0.998054	3.203455	4.856957	3.531907
20	0.003952	0.003952	0.163885	1.225248	3.629212	5.187653	3.620357
25	0.006136	0.006136	0.216814	1.458043	4.008844	5.441044	3.674288
30	0.009114	0.009114	0.273819	1.667010	4.283386	5.567765	3.677570
35	0.013260	0.013260	0.337548	1.868722	4.512703	5.645296	3.663768
40	0.018926	0.018926	0.406882	2.056621	4.687027	5.666076	3.628787
45	0.026426	0.026426	0.478634	2.215631	4.784859	5.614600	3.566738
50	0.035992	0.035992	0.552113	2.335873	4.787592	5.474119	3.470288
55	0.048025	0.048025	0.624410	2.427599	4.738073	5.294625	3.361169
60	0.063304	0.063304	0.700347	2.501906	4.653468	5.088778	3.242030
65	0.082195	0.082195	0.776454	2.553958	4.533589	4.859114	3.112244
70	0.104964	0.104964	0.850912	2.581697	4.378866	4.606105	2.970692
75	0.131770	0.131770	0.920547	2.580055	4.185853	4.327451	2.815616
80	0.162498	0.162498	0.979864	2.541298	3.949453	4.020371	2.644956
85	0.196460	0.196460	1.004473	2.415445	3.612544	3.642093	2.440520
90	0.234697	0.234573	1.045453	2.325346	3.340048	3.323342	2.263062
95	0.282738	0.264664	1.035628	2.156657	2.999414	2.966671	2.070212
100	0.338601	0.291988	1.016859	1.987198	2.677464	2.634553	1.880814
105	0.402565	0.317544	0.990755	1.816628	2.371833	2.321824	1.690844
110	0.475243	0.328145	0.927085	1.604263	2.039645	1.994463	1.484899
115	0.552890	0.287229	0.762565	1.277515	1.610150	1.596075	1.235213
120	0.632036	0.247534	0.617762	1.000597	1.247534	1.249799	1.000597
125	0.710395	0.209154	0.491277	0.768299	0.944912	0.952544	0.783413
130	0.785399	0.172084	0.381298	0.574997	0.694994	0.700631	0.586022
135	0.854224	0.136192	0.285656	0.414867	0.490406	0.490089	0.410862
140	0.913782	0.101180	0.201857	0.282037	0.323963	0.316960	0.260575
145	0.960655	0.066535	0.127057	0.170644	0.188869	0.177650	0.138248
150	0.990912	0.031430	0.057917	0.074737	0.078819	0.069330	0.047849
~154	1.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

TABLE 8

Critical angles α_j (in degree) and multiplicities (in parentheses) of T_n^α for $n = 1(1)6$

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
	(2)	(3)	(4)	(5)	(6)	(7)
α_1	45.000000	12.006236	5.292324	2.966729	1.895544	1.315134
	(1)	(2)	(3)	(4)	(5)	(6)
α_2	90.000000	45.000000	19.708591	11.035546	7.054588	4.897497
	(∞)	(3)	(4)	(5)	(6)	(7)
α_3		72.000000	33.532925	19.275445	12.557292	8.828178
		(2)	(3)	(4)	(5)	(6)
α_4		120.000000	84.982583	52.940364	36.738556	25.727565
		(∞)	(2)	(3)	(4)	(5)
α_5			135.000000	71.205990	41.987818	28.817323
			(∞)	(4)	(5)	(6)
α_6				93.168809	69.109827	47.344683
				(3)	(4)	(5)
α_7				144.000000	101.658947	82.631133
				(∞)	(3)	(4)
α_8					150.000000	84.749366
					(∞)	(5)
α_9						88.496168
						(6)
α_{10}						89.944043
						(5)
α_{11}						108.824858
						(4)
α_{12}						154.285714
						(∞)

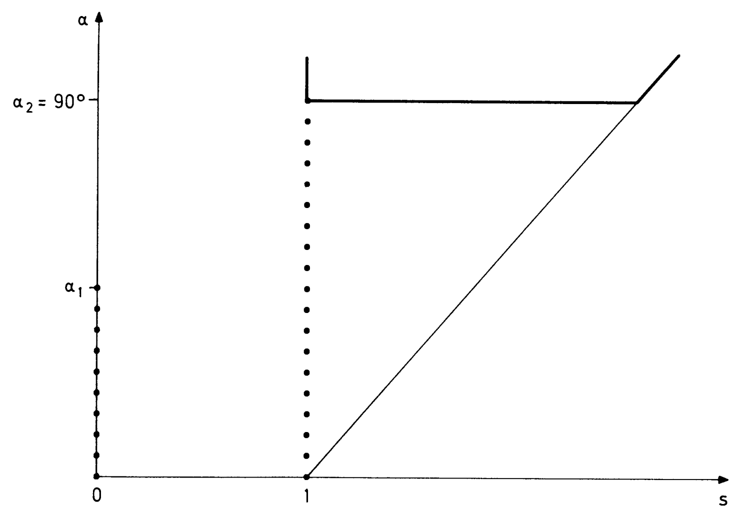

FIGURE 1
Positions of extremal points of T_1^α

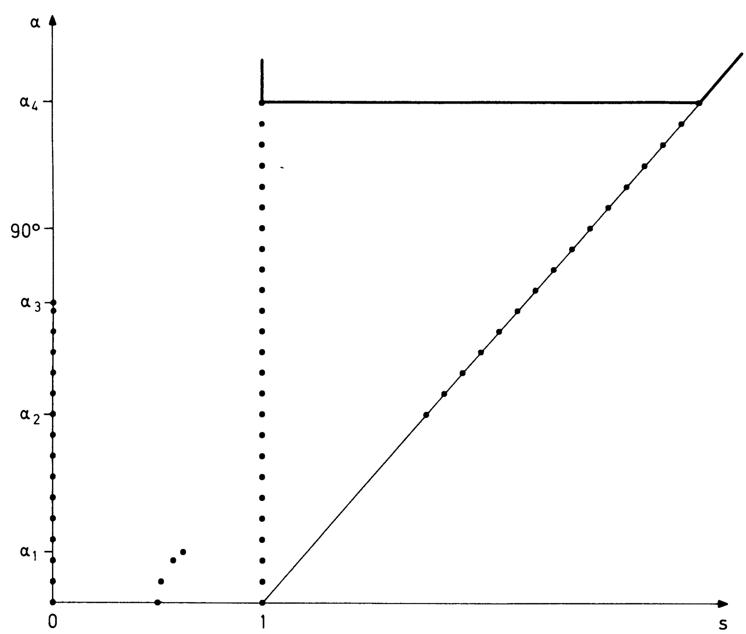

FIGURE 2
Positions of extremal points of T_2^α

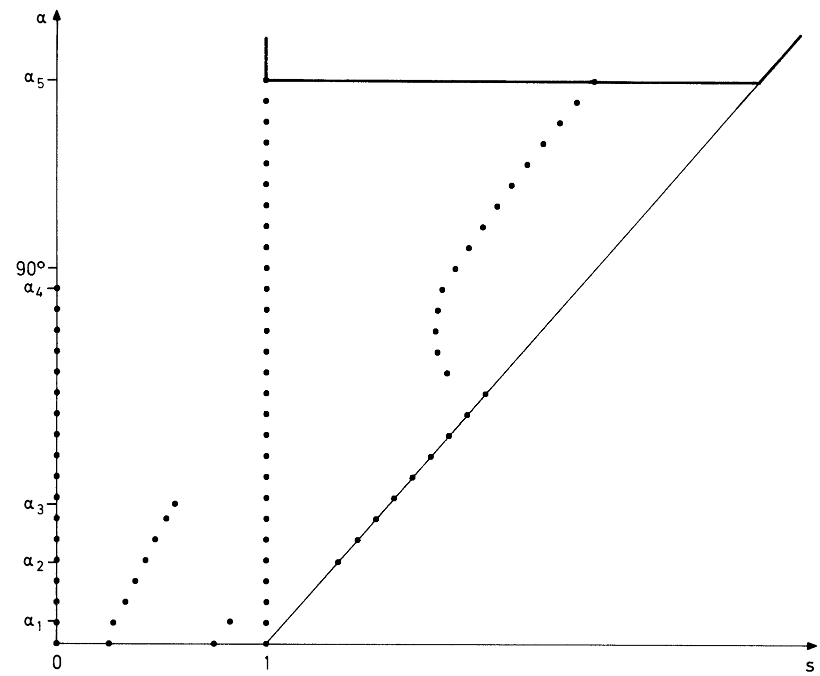

FIGURE 3
Positions of extremal points of T_3^α

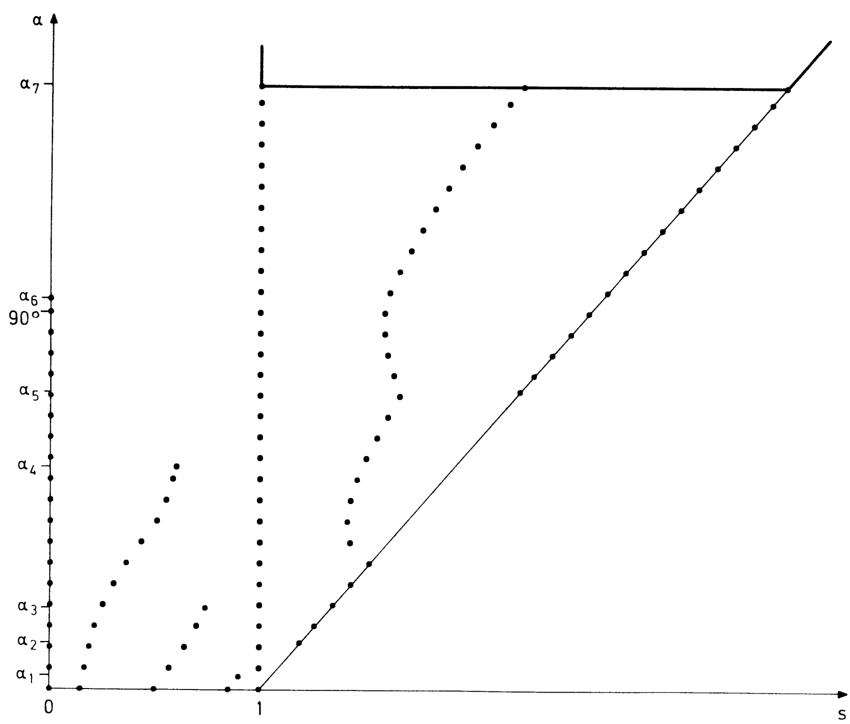

FIGURE 4
Positions of extremal points of T_4^α

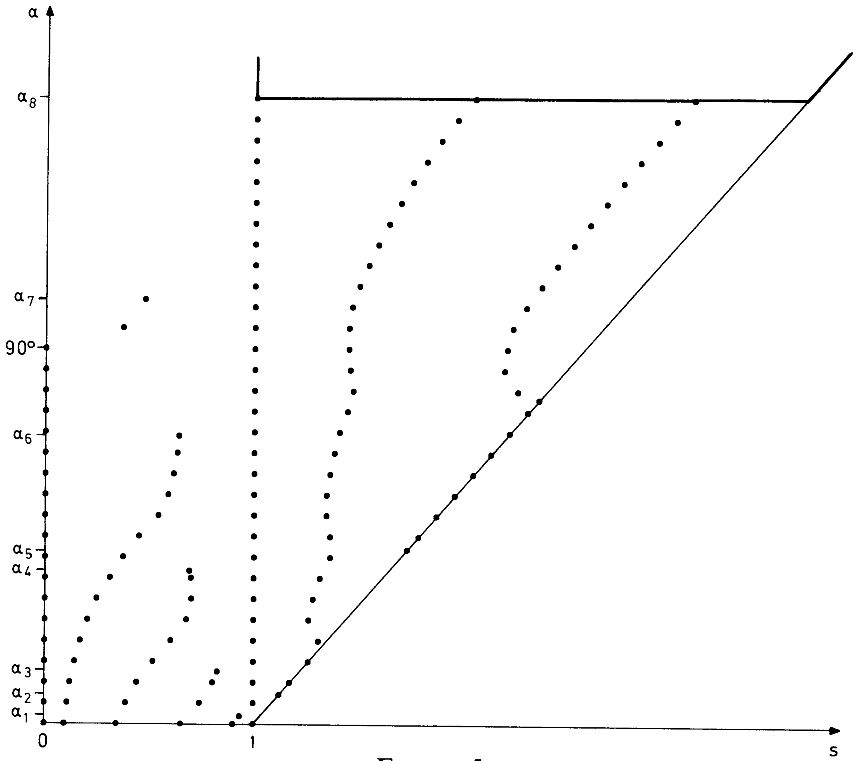

FIGURE 5
Positions of extremal points of T_5^α

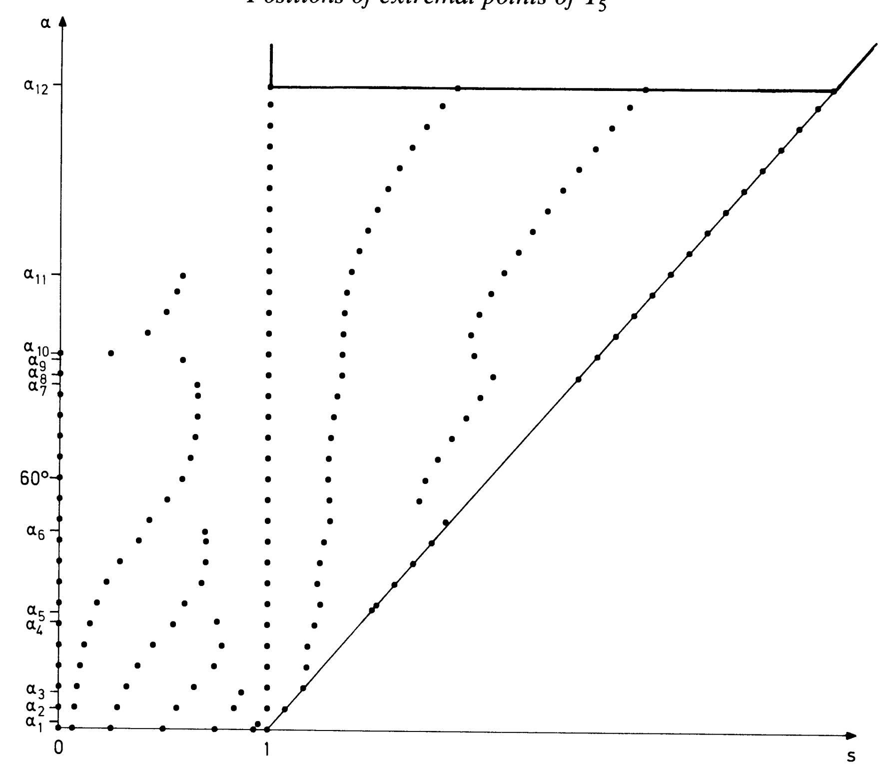

FIGURE 6
Positions of extremal points of T_6^α

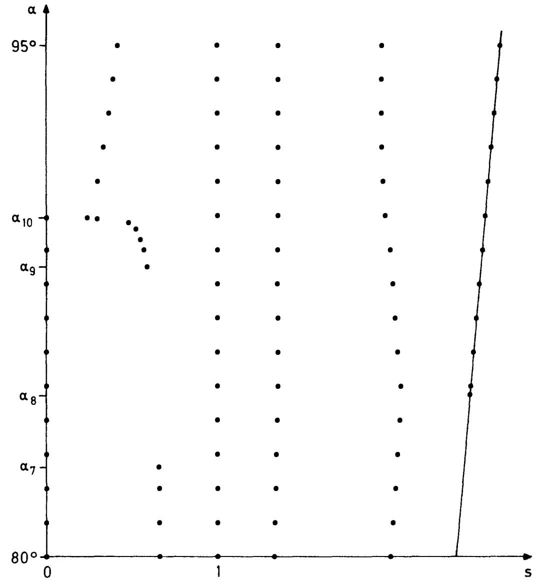

FIGURE 7
Enlarged detail of Figure 6 near $\alpha = 90°$

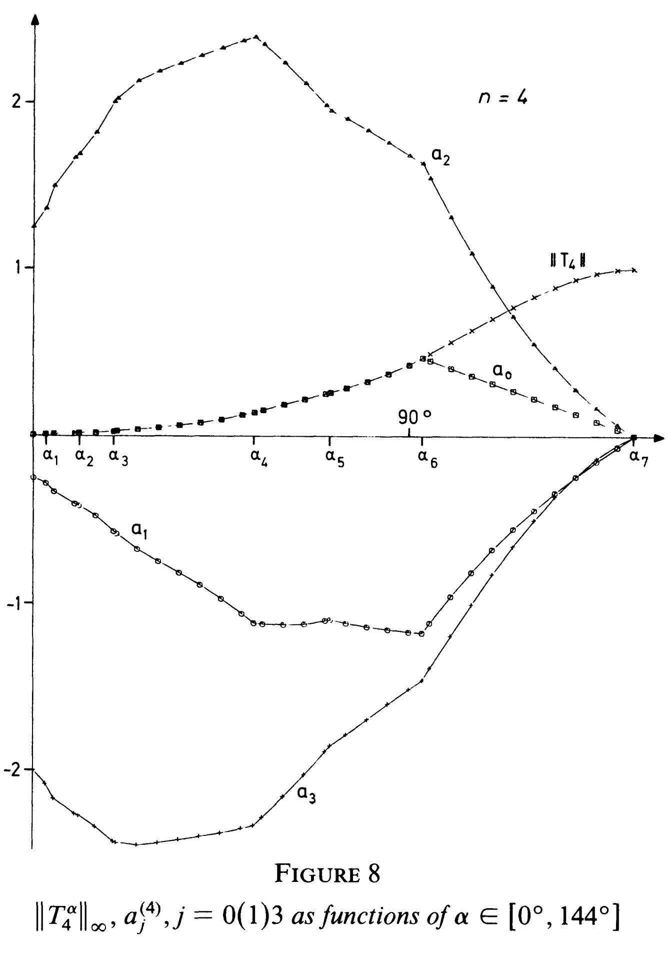

FIGURE 8
$\|T_4^\alpha\|_\infty$, $a_j^{(4)}, j = 0(1)3$ *as functions of* $\alpha \in [0°, 144°]$

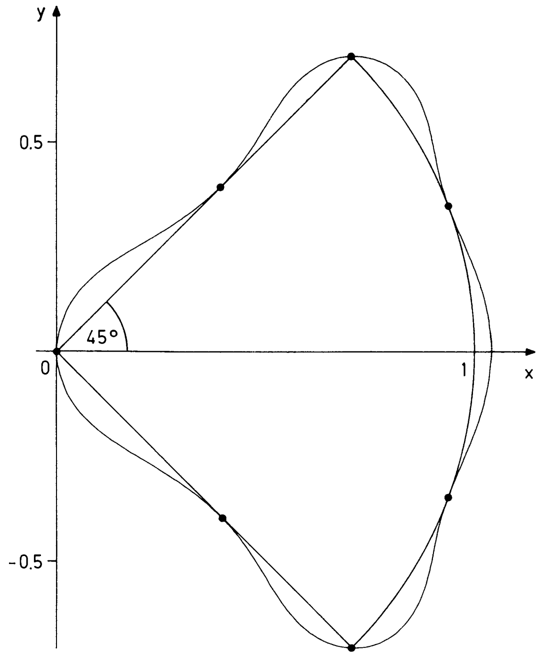

FIGURE 9
Lemniscate $\left|T_4^{\pi/4}(z)\right| = \left\|T_4^{\pi/4}\right\|_\infty$

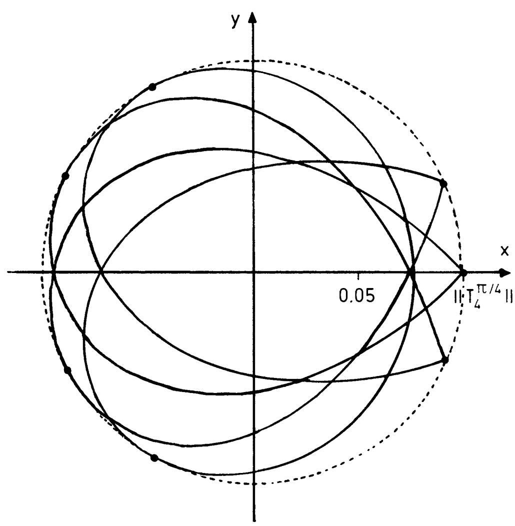

FIGURE 10
Error curve $w = T_4^{\pi/4}(z),\ z \in \partial S^{\pi/4}$

Institute for Applied Mathematics
Bundesstrasse 55
D-2000 Hamburg 13, West Germany

1. M. ABRAMOWITZ & I. A. STEGUN (Editors), *Handbook of Mathematical Functions*, fifth printing, Dover, New York, 1968.

2. J. P. COLEMAN & A. J. MONAGHAN, *Chebyshev Expansions for the Bessel Function $J_n(z)$ in the Complex Plane*, University of Durham, 1980. (Preprint.)

3. G. H. ELLIOTT, *Polynomial Approximation in the Complex Plane Using Generalised Humbert Polynomials*, Lecture at Dundee Biennial Conference on Numerical Analysis, 1981.

4. C. GEIGER & G. OPFER, "Complex Chebyshev polynomials on circular sectors," *J. Approx. Theory*, v. 24, 1978, pp. 93–118.

5. K. GLASHOFF & S. Å. GUSTAFSON, *Einführung in die lineare Optimierung*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1978.

6. K. GLASHOFF & K. ROLEFF, "A new method for Chebyshev approximation of complex-valued functions," *Math. Comp.*, v. 36, 1981, pp. 233–239.

7. R. HETTICH, "Numerical methods for nonlinear Chebyshev approximation," in *Approximation in Theorie und Praxis* (G. Meinardus, Ed.), Bibliographisches Institut Mannheim, Wien, Zürich, 1979, pp. 139–156.

8. G. MEINARDUS, *Approximation of Functions: Theory and Numerical Methods*, Springer-Verlag, Berlin, Heidelberg and New York, 1967.

9. G. OPFER, "An algorithm for the construction of best approximations based on Kolmogorov's criterion," *J. Approx. Theory*, v. 23, 1978, pp. 299–317.

10. R. L. STREIT & A. H. NUTTALL, *Linear Chebyshev Complex Function Approximation*, Technical Report 6403, Naval Underwater Systems Center, Newport, R. I., New London, Conn., 1981.