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Closed Expressions for $\int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt$

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Abstract. Closed expressions for the integral $\int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt$, whose general form is given elsewhere, are listed for $n = 1(1)9$, $p = 1(1)9$. A formula is derived which allows an easy evaluation of these expressions by formula manipulation on a computer.

1. Introduction. At the beginning of this century, Nielsen discussed, in a little-known monograph [9], properties of a family of functions

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1 - xt)\, dt \tag{1}$$

for positive integers n, p, and complex x. These functions include many special cases such as Euler's dilogarithm, Kummer's trilogarithm, the Spence functions and polylogarithms. As already proposed [4], it seems appropriate to call the family (1) Nielsen's generalized polylogarithms.

Although the monograph [9] contains quite a number of misprints and a few erroneous results, it does present a considerable amount of useful information, in particular transformation formulae relating $S_{n,p}(x)$ to $S_{n,p}(1/x)$ and $S_{n,p}(1 - x)$. It is remarkable that these formulae, and consequently also those for $S_{n,p}(1/(1 - x))$, $S_{n,p}((x - 1)/x)$, and $S_{n,p}(x/(x - 1))$ contain, apart from logarithms and constants, only functions $S_{\nu,\pi}(x)$. However, as far as the author knows, the important formulae of [9] have never found their way into any of the relevant handbooks.

Interest in these functions revived some time ago, at least for the case $p = 1$, in the context of multi-dimensional integration of rational functions in quantum electrodynamics (see, for example, [1], [8]). Their properties are also of interest in group theory and geometry [7]. The book of Lewin [6] gives many formulae and properties of $S_{n,1}(x)$. A general discussion of Nielsen's monograph is given in [4].

2. The Values $s_{n,p} = S_{n,p}(1)$. The purpose of this note is to give explicit expressions for the special values

$$s_{n,p} = S_{n,p}(1) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt, \tag{2}$$

at least for some n and p. It is easy to show that $s_{n,p} = s_{p,n}$, and hence we can restrict p to $n \geqslant p$. A closed expression for $s_{n,p}$ is given in [4] (in implicit form also in

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[6]), which reads

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \sum_{m_i} \frac{H_p(m_1,\ldots,m_k)}{m_1 \cdots m_k} \zeta(m_1) \cdots \zeta(m_k), \tag{3}$$

where

$$H_p(m_1,\ldots,m_k) = \sum_{p_i} \binom{m_1}{p_1} \cdots \binom{m_k}{p_k}. \tag{4}$$

The sum over m_i is to be taken over all sets of integers $\{m_i\}$ $(i = 1,\ldots,k)$ which satisfy

$$m_i \geq 2, \quad \sum_{i=1}^{k} m_i = n + p, \tag{5}$$

and the sum over p_i over all sets of integers $\{p_i\}$ $(i = 1,\ldots,k)$ which satisfy

$$1 \leq p_i \leq m_i - 1, \quad \sum_{i=1}^{k} p_i = p. \tag{6}$$

The function

$$\zeta(m) = \sum_{k=1}^{\infty} k^{-m} \tag{7}$$

is the Riemann zeta function for integer argument. Nielsen remarked that the functions $S_{n,p}(x)$ are probably the simplest analytic functions which coincide with $\zeta(m)$ for special values of its arguments. He added that he was not able to use his theory of $S_{n,p}(x)$ to find expressions for $\zeta(2\mu + 1)$ analogous to the known expressions for $\zeta(2\mu)$.

Nielsen [9] formulated a theorem about the structure of $s_{n,p}$ and gave the principle of the proof. He also calculated the cases $p \leq 3$. The case $p = 1$ is trivial, giving

$$\begin{aligned} &\int_0^1 t^{-1} \log^{n-1} t \log(1 - t)\, dt \\ &\qquad = (-1)^n (n-1)! s_{n,1} = (-1)^n (n-1)! \zeta(n+1). \end{aligned} \tag{8}$$

The case $p = 2$ can also be handled easily, but the $k = 3$ term in (3) for $p = 3$ is somewhat more involved, and Nielsen's final expression [9, Section 18 (19), (20)] is incorrect. However, the expression for s_{73} given as an example in [9] differs from the correct expression only by a difference in the coefficient of $\zeta(2)\zeta^2(4)$ ($\frac{1}{3}$ instead of $\frac{1}{2}$), and this could be due to a misprint.

Writing (3) as

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \alpha_k(n, p), \tag{9}$$

it is easy to find from (4) the following expressions for $\alpha_k(n, p)$ in the case of some special values of p and k:

$$\alpha_1(n, p) = \frac{(n + p - 1)!}{n! p!} \zeta(n + p), \tag{10}$$

(11) $$\alpha_2(n,2)=\sum_{\nu=2}^{n}\zeta(\nu)\zeta(n-\nu+2),$$

(12) $$\alpha_n(n,n)=\zeta^n(2).$$

For $k=2$, $p=3$, we have from (4), for $\nu=2,\dots,n+1$,

$$H_3(n-\nu+3,\nu)=\varepsilon_{\nu,2}\binom{n-\nu+3}{1}\binom{\nu}{2}+\varepsilon_{\nu,n+1}\binom{n-\nu+3}{2}\binom{\nu}{1}$$

and

(13) $$\frac{H_3(n-\nu+3,\nu)}{(n-\nu+3)\nu}=\begin{cases}\frac{1}{2}n & \text{if } \nu=2,\ \nu=n+1,\\ \frac{1}{2}(n+1) & \text{if } \nu=3,\dots,n,\end{cases}$$

where $\varepsilon_{\nu,\mu}=0$ for $\nu=\mu$ and $\varepsilon_{\nu,\mu}=1$ for $\nu\neq\mu$, so that

(14) $$\begin{aligned}\alpha_2(n,3)&=n\zeta(2)\zeta(n+1)+\frac{1}{2}(n+1)\sum_{\nu=3}^{n}\zeta(\nu)\zeta(n-\nu+3)\\ &=\sum_{\nu=2}^{n}(n-\nu+2)\zeta(\nu)\zeta(n-\nu+3).\end{aligned}$$

In the case $k=2$, $p=4$, one finds for $\nu=2,\dots,n+2$,

$$\begin{aligned}H_4(n-\nu+4,\nu)&=\varepsilon_{\nu,2}\varepsilon_{\nu,3}\binom{n-\nu+4}{1}\binom{\nu}{3}+\varepsilon_{\nu,2}\varepsilon_{\nu,n+2}\binom{n-\nu+4}{2}\binom{\nu}{2}\\ &\quad+\varepsilon_{\nu,n+1}\varepsilon_{\nu,n+2}\binom{n-\nu+4}{3}\binom{\nu}{1}.\end{aligned}$$

Thus

(15) $$\begin{aligned}&\frac{H_4(n-\nu+4,\nu)}{(n-\nu+4)\nu}\\ &\quad=\begin{cases}\frac{1}{6}n(n+1) & \text{if } \nu=2,\ \nu=n+2,\\ \frac{1}{6}n(n+2) & \text{if } \nu=3,\ \nu=n+1,\\ \frac{1}{12}\left[\nu^2-(n+4)\nu+2n^2+7n+7\right] & \text{if } \nu=4,\dots,n,\end{cases}\end{aligned}$$

and therefore

(16) $$\begin{aligned}\alpha_2(n,4)&=\frac{1}{3}n(n+1)\zeta(2)\zeta(n+2)+\frac{1}{3}n(n+2)\zeta(3)\zeta(n+1)\\ &\quad+\frac{1}{12}\sum_{\nu=4}^{n}\left[\nu^2-(n+4)\nu+2n^2+7n+7\right]\zeta(\nu)\zeta(n-\nu+4).\end{aligned}$$

For larger values of p, $\alpha_2(n,p)$ becomes more and more complicated.

For $k=p=3$, we see that $p_1=p_2=p_3=1$ and $H_3(m_1,m_2,m_3)=m_1m_2m_3$. The sum over m_i in (3) therefore equals the sum over the products $\zeta(m_1)\zeta(m_2)\zeta(m_3)$ for all partitions $\{m_1,m_2,m_3\}$ of $n+3$ satisfying $2\leqslant m_i\leqslant[(n+3)/3]$, with a weight for possible permutations, where $[\xi]$ denotes the integer part of ξ. This leads to

(17) $$\alpha_3(n,3)=\sum_{\mu=2}^{\mu^*}\zeta(\mu)\sum_{\nu=\mu}^{\nu^*}\omega(n;\mu,\nu)\zeta(\nu)\zeta(n+3-\nu-\mu),$$

where $\mu^* = [(n+3)/3]$, $\nu^* = [(n-\mu+3)/2]$, and

$$\omega(n;\mu,\nu) = \begin{cases} 1 & \text{if } \mu=\nu \text{ and } 3\mu = n+3, \\ 3 & \text{if } \mu=\nu \text{ and } 3\mu \neq n+3 \text{ or} \\ & \text{if } \mu\neq\nu \text{ and } 2\mu+\nu = n+3 \text{ or} \\ & \text{if } \mu\neq\nu \text{ and } \mu+2\nu = n+3 \\ 6 & \text{otherwise.} \end{cases} \tag{18}$$

From (1), (10), and (11) it follows that

$$\begin{aligned} \int_0^1 t^{-1}\log^{n-1}t\log^2(1-t)\,dt &= 2(-1)^{n-1}(n-1)!s_{n,2} \\ &= (-1)^{n-1}(n-1)!\left[(n+1)\zeta(n+2) - \sum_{\nu=2}^{n}\zeta(\nu)\zeta(n-\nu+2)\right], \end{aligned} \tag{19}$$

and from (10), (14), and (17),

$$\begin{aligned} \int_0^1 t^{-1}\log^{n-1}t\log^3(1-t)\,dt &= 6(-1)^n(n-1)!s_{n,3} \\ &= (-1)^n(n-1)!\Bigg[(n+1)(n+2)\zeta(n+3) \\ &\quad -3\sum_{\nu=2}^{n}(n-\nu+2)\zeta(\nu)\zeta(n-\nu+3) \\ &\quad +\sum_{\mu=2}^{\mu^*}\zeta(\mu)\sum_{\nu=\mu}^{\nu^*}\omega(n;\mu,\nu)\zeta(\nu)\zeta(n+3-\nu-\mu)\Bigg]. \end{aligned} \tag{20}$$

This last formula corrects formula [9, Section 18 (19)] of Nielsen.

For arbitrary n and p, it is obvious that (3) can, in practice, be evaluated only by means of a computer. Even then, the problem is complicated. The main task consists in constructing the sets $\{m_i\}$ and $\{p_i\}$. Because of the fact that all permutations have to be taken into account, the number of these sets grows rapidly with increasing values of $n+p$. We have constructed these sets up to $n = p = 9$ by means of a FORTRAN program. As an example, their number is shown for $n = p = 9$ in Table 1. Therefore $\Sigma\{m_i\}\{p_i\} = 85376$ sets would have to be analyzed in this case. Because of the condition $1 \leq p_i \leq m_i - 1$, only 12870 of these would contribute to the 88 different terms in the result (3) for $s_{9,9}$.

TABLE 1

k	1	2	3	4	5	6	7	8	9
$\{m_i\}$	1	15	91	286	495	462	210	36	1
$\{p_i\}$	1	8	28	56	70	56	28	8	1

The complicated calculations required for the evaluation of (3) may be avoided by using an alternative expression, well-adapted to evaluation by formula-manipulation systems such as REDUCE [8]. As in the derivation of (3), we start from the relation [4, 9]:

$$s_{n,p}=\frac{(-1)^{n+p-1}}{(n-1)!p!}\frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^{p}}\frac{1}{\beta}\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\bigg|_{\alpha=\beta=0}. \tag{21}$$

We now introduce the power series [2, No. 8.321]

$$\begin{aligned}\Gamma(1+x)&=\sum_{k=0}^{\infty}b_kx^k\qquad(|x|<1),\\ 1/\Gamma(1+x)&=\sum_{k=0}^{\infty}a_kx^k,\end{aligned}\tag{22}$$

where $a_0=b_0=1$, and

$$\begin{aligned}a_k&=\frac{1}{k}\sum_{m=1}^{k}(-1)^{m+1}\zeta(m)a_{k-m},\\ b_k&=-\frac{1}{k}\sum_{m=1}^{k}(-1)^{m+1}\zeta(m)b_{k-m}\qquad(k>0),\end{aligned}\tag{23}$$

with the definition $\zeta(1)=\gamma$ (Euler's constant). Then, performing the differentiations with respect to α in (21), and using the relation

$$\sum_{\rho=0}^{p}b_{p-\rho}a_\rho=0\qquad(p>0),\tag{24}$$

we obtain

$$\begin{aligned}&\frac{\partial^p}{\partial\alpha^p}\Gamma(1+\alpha)/\Gamma(1+\alpha+\beta)\bigg|_{\alpha=0}\\ &=\sum_{\rho=0}^{p}\binom{p}{\rho}\left(\sum_{k=0}^{\infty}a_k\sum_{\kappa=0}^{k}\binom{k}{\kappa}\alpha^\kappa\beta^{k-\kappa}\right)^{(\rho)}\left(\sum_{k=0}^{\infty}b_k\alpha^k\right)^{(p-\rho)}\bigg|_{\alpha=0}\\ &=p!\sum_{\rho=0}^{p}b_{p-\rho}\sum_{k=\rho+1}^{\infty}a_k\binom{k}{\rho}\beta^{k-\rho}=H(\beta).\end{aligned}\tag{25}$$

Similarly

$$\begin{aligned}&\frac{\partial^{n-1}}{\partial\beta^{n-1}}\frac{1}{\beta}H(\beta)\Gamma(1+\beta)\\ &=p!\sum_{\nu=0}^{n-1}\binom{n-1}{\nu}[H(\beta)/\beta]^{(\nu)}\left(\sum_{k=0}^{\infty}b_k\beta^k\right)^{(n-\nu-1)}\bigg|_{\beta=0},\end{aligned}\tag{26}$$

and therefore, finally,

$$s_{n,p}=(-1)^{n+p-1}\sum_{\nu=0}^{n-1}b_{n-\nu-1}\sum_{\rho=0}^{p}\binom{\nu+\rho+1}{\rho}b_{p-\rho}a_{\nu+\rho+1}.\tag{27}$$

This expression, although revealing less of the structure (already inferred by Nielsen [9]) of $s_{n,p}$ than formula (3), namely that $s_{n,p}$ can be expressed as a homogeneous polynomial of "degree" $n+p$ in the terms $\zeta(m)$, $(2\leqslant m\leqslant n+p)$, with rational coefficients, is much more suitable for actual computation. Using a formula-manipulation system, the evaluation of (27) is in fact straightforward once the expressions (23) for a_k $(0 \leqslant k \leqslant n+p)$ and b_k $[0 \leqslant k \leqslant \max(n-1, p)]$ in terms of $\zeta(m)$ have been initially established. It follows from (5) that, at least, all terms involving $\zeta(1) = \gamma$ will cancel in the final expression for (27). For example, the special cases $s_{n,1}$ and $s_{1,p}$ reduce to a single term:

$$s_{n,1} = (-1)^n \sum_{\nu=0}^{n-1} b_{n-\nu-1}[(\nu+2)a_{\nu+2} + b_1 a_{\nu+1}] = \zeta(n+1)$$

and

$$s_{1,p} = (-1)^p \sum_{\rho=0}^{p} (\rho+1) b_{p-\rho} a_{\rho+1} = \zeta(p+1). \tag{28}$$

The results obtained with REDUCE have been checked by evaluating the definition integral (2) by numerical integration, replacing the limits 0 and 1 by $\varepsilon = 10^{-8}$ and $1-\varepsilon$, respectively, and using Stieltjes' 32 decimal table [10] of $\zeta(m)$, $m = 2(1)70$, which is reproduced in [9], for the evaluation of $s_{n,p}$.

We add here that the substitution $t = \sin^2\theta$ in (2) leads to the integral [6],

$$s_{n,p} = -\frac{(-2)^{n+p}}{(n-1)!p!} \int_0^{\pi/2} \cot\theta \log^n \sin\theta \log^p \cos\theta \, d\theta. \tag{29}$$

A closed expression for a similar integral,

$$R_{n,p} = \int_0^{\pi/2} \log^n \sin t \log^p \cos t \, dt \qquad (n \geqslant 0, p \geqslant 0), \tag{30}$$

has been given in [5], with examples up to $n = p = 4$.

3. A Table of the Integral. We list the expressions for $s_{n,p}$, $n = 1(1)9$, $p = 1(1)n$. The values for the integral in (2) itself,

$$r_{n,p} = \int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\, dt = (-1)^{n+p-1}(n-1)!p! s_{n,p}, \tag{31}$$

would lead for higher n or p to rather large coefficients. The reference work [2, No. 4.2912] lists only the case $n = p = 1$, whereas Lewin [6] gives (31) for $n = 2, 3, 4$, and $p = 2$.

Using the well-known relation [2, No. 9.5421],

$$\zeta(2\mu) = \frac{2^{2\mu-1}\pi^{2\mu}|B_{2\mu}|}{(2\mu)!}, \tag{32}$$

where $B_{2\mu}$ are the Bernoulli numbers, the expressions for $r_{n,p}$ simplify to some extent. We also give these values for $n = 1(1)7$, $p = 1(1)n$.

$s_{11} = \zeta(2)$,

$s_{21} = \zeta(3)$,

$s_{22} = -\frac{1}{2}\zeta^2(2) + \frac{3}{2}\zeta(4)$,

$s_{31} = \zeta(4)$,

$s_{32} = -\zeta(2)\zeta(3) + 2\zeta(5)$,

$$s_{33} = \tfrac{1}{6}\zeta^3(2) - \tfrac{3}{2}\zeta(2)\zeta(4) - \zeta^2(3) + \tfrac{10}{3}\zeta(6),$$
$$s_{41} = \zeta(5),$$
$$s_{42} = -\zeta(2)\zeta(4) - \tfrac{1}{2}\zeta^2(3) + \tfrac{5}{2}\zeta(6),$$
$$s_{43} = \tfrac{1}{2}\zeta^2(2)\zeta(3) - 2\zeta(2)\zeta(5) - \tfrac{5}{2}\zeta(3)\zeta(4) + 5\zeta(7),$$
$$s_{44} = -\tfrac{1}{24}\zeta^4(2) + \tfrac{3}{4}\zeta^2(2)\zeta(4) + \zeta(2)\zeta^2(3) - \tfrac{10}{3}\zeta(2)\zeta(6) - 4\zeta(3)\zeta(5) - \tfrac{17}{8}\zeta^2(4) + \tfrac{35}{4}\zeta(8),$$
$$r_{11} = -\tfrac{1}{6}\pi^2,$$
$$r_{21} = \zeta(3),$$
$$r_{22} = -\tfrac{1}{180}\pi^4,$$
$$r_{31} = -\tfrac{1}{45}\pi^4,$$
$$r_{32} = -\tfrac{2}{3}\pi^2\zeta(3) + 8\zeta(5),$$
$$r_{33} = -\tfrac{23}{1260}\pi^6 + 12\zeta^2(3),$$
$$r_{41} = 6\zeta(5),$$
$$r_{42} = -\tfrac{1}{105}\pi^6 + 6\zeta^2(3),$$
$$r_{43} = -\tfrac{1}{2}\pi^4\zeta(3) - 12\pi^2\zeta(5) + 180\zeta(7),$$
$$r_{44} = -\tfrac{499}{12600}\pi^8 - 24\pi^2\zeta^2(3) + 576\zeta(3)\zeta(5).$$

The remaining expressions for $s_{n,p}$, $n = 5(1)9$, $p = 1(1)n$, and $r_{n,p}$, $n = 5(1)7$, $p = 1(1)n$, are given in the microfiche section at the end of this issue. Numerical values of $s_{n,p}$ with 21 digits are presented in Table 2.

TABLE 2

n	p	$s_{n,p}$
1	1	1.64493 40668 48226 43647E+00
2	1	1.20205 69031 59594 28540E+00
2	2	2.70580 80842 77845 47879E-01
3	1	1.08232 32337 11138 19152E+00
3	2	9.65511 59989 44373 44656E-02
3	3	1.74898 53169 01140 44259E-02
4	1	1.03692 77551 43369 92633E+00
4	2	4.05368 97271 51973 78290E-02
4	3	4.12316 51524 32535 53202E-03
4	4	6.02891 53283 31913 91876E-04
5	1	1.01734 30619 84449 13971E+00
5	2	1.83559 28317 49446 58780E-02
5	3	1.10762 05206 81261 04542E-03
5	4	1.06090 22891 02175 20514E-04
5	5	1.29078 86926 10006 80019E-05
6	1	1.00834 92773 81922 82684E+00
6	2	8.65052 90995 61105 50088E-03
6	3	3.20419 48118 65540 68195E-04
6	4	2.08107 99998 53278 80665E-05
6	5	1.81177 17675 49254 62907E-06
6	6	1.88257 25261 51750 84100E-07

TABLE 2 (continued)

n	p	$s_{n,p}$
7	1	1.00407 73561 97944 33938E+00
7	2	4.17024 20454 82641 20903E-03
7	3	9.70014 34407 46026 74085E-05
7	4	4.37446 80142 37467 26660E-06
7	5	2.79046 61391 18230 10386E-07
7	6	2.19761 45278 08360 14044E-08
7	7	1.99035 60428 47009 48657E-09
8	1	1.00200 83928 26082 21442E+00
8	2	2.03771 21074 18497 21127E-03
8	3	3.02392 65882 11524 77408E-05
8	4	9.63193 58629 25147 64220E-07
8	5	4.58067 37635 27237 01146E-08
8	6	2.78188 36333 42565 39146E-09
8	7	1.98864 22337 79748 87998E-10
8	8	1.59526 66865 47416 62087E-11
9	1	1.00099 45751 27818 08534E+00
9	2	1.00397 69886 51568 46827E-03
9	3	9.61339 45728 45768 87138E-06
9	4	2.19049 55625 33962 98356E-07
9	5	7.86919 99763 14613 72568E-09
9	6	3.73432 32019 05082 31190E-10
9	7	2.13492 93645 52627 26763E-11
9	8	1.39313 33415 77287 65028E-12
9	9	1.00261 68238 56214 27731E-13

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$$s_{51} = \zeta(6)$$

$$s_{52} = -\zeta(2)\zeta(5) - \zeta(3)\zeta(4) + 3\zeta(7)$$

$$s_{53} = \tfrac{1}{2}\zeta^2(2)\zeta(4) + \tfrac{1}{2}\zeta(2)\zeta^2(3) - \tfrac{5}{2}\zeta(2)\zeta(6) - 3\zeta(3)\zeta(5) - \tfrac{3}{2}\zeta^2(4) + 7\zeta(8)$$

$$s_{54} = -\tfrac{1}{6}\zeta^3(2)\zeta(3) + \zeta^2(2)\zeta(5) + \tfrac{5}{2}\zeta(2)\zeta(3)\zeta(4) - 5\zeta(2)\zeta(7) + \tfrac{1}{2}\zeta^3(3) - \tfrac{35}{6}\zeta(3)\zeta(6) - 6\zeta(4)\zeta(5) + 14\zeta(9)$$

$$s_{55} = \tfrac{1}{120}\zeta^5(2) - \tfrac{1}{4}\zeta^3(2)\zeta(4) - \tfrac{1}{2}\zeta^2(2)\zeta^2(3) + \tfrac{5}{3}\zeta^2(2)\zeta(6) + 4\zeta(2)\zeta(3)\zeta(5) + \tfrac{17}{8}\zeta(2)\zeta^2(4) - \tfrac{35}{4}\zeta(2)\zeta(8) + \tfrac{5}{2}\zeta^2(3)\zeta(4) - 10\zeta(3)\zeta(7) - 10\zeta(4)\zeta(6) - 5\zeta^2(5) + \tfrac{126}{5}\zeta(10)$$

$$s_{61} = \zeta(7)$$

$$s_{62} = -\zeta(2)\zeta(6) - \zeta(3)\zeta(5) - \tfrac{1}{2}\zeta^2(4) + \tfrac{7}{2}\zeta(8)$$

$$s_{63} = \tfrac{1}{2}\zeta^2(2)\zeta(5) + \zeta(2)\zeta(3)\zeta(4) - 3\zeta(2)\zeta(7) + \tfrac{1}{6}\zeta^3(3) - \tfrac{7}{2}\zeta(3)\zeta(6) - \tfrac{7}{2}\zeta(4)\zeta(5) + \tfrac{28}{3}\zeta(9)$$

$$s_{64} = -\tfrac{1}{6}\zeta^3(2)\zeta(4) - \tfrac{1}{4}\zeta^2(2)\zeta^2(3) + \tfrac{5}{4}\zeta^2(2)\zeta(6) + 3\zeta(2)\zeta(3)\zeta(5) + \tfrac{3}{2}\zeta(2)\zeta^2(4) - 7\zeta(2)\zeta(8) + \tfrac{7}{4}\zeta^2(3)\zeta(4) - 8\zeta(3)\zeta(7) - \tfrac{57}{12}\zeta(4)\zeta(6) - 4\zeta^2(5) + 21\zeta(10)$$

$$s_{65} = \tfrac{1}{24}\zeta^4(2)\zeta(3) - \tfrac{1}{3}\zeta^3(2)\zeta(5) - \tfrac{5}{4}\zeta^2(2)\zeta(3)\zeta(4) + \tfrac{5}{2}\zeta^2(2)\zeta(7) - \tfrac{1}{2}\zeta(2)\zeta^3(3) + \tfrac{35}{6}\zeta(2)\zeta(3)\zeta(6) + 6\zeta(2)\zeta(4)\zeta(5) - 14\zeta(2)\zeta(9) + \tfrac{7}{2}\zeta^2(3)\zeta(5) + \tfrac{29}{8}\zeta(3)\zeta^2(4) - \tfrac{63}{4}\zeta(3)\zeta(8) - \tfrac{31}{2}\zeta(4)\zeta(7) - \tfrac{91}{6}\zeta(5)\zeta(6) + 42\zeta(11)$$

$$s_{66} = -\tfrac{1}{720}\zeta^6(2) + \tfrac{1}{16}\zeta^4(2)\zeta(4) + \tfrac{1}{6}\zeta^3(2)\zeta^2(3) - \tfrac{5}{9}\zeta^3(2)\zeta(6) - 2\zeta^2(2)\zeta(3)\zeta(5) - \tfrac{17}{16}\zeta^2(2)\zeta^2(4) + \tfrac{35}{8}\zeta^2(2)\zeta(8) - \tfrac{5}{2}\zeta(2)\zeta^2(3)\zeta(4) + 10\zeta(2)\zeta(3)\zeta(7) + 10\zeta(2)\zeta(4)\zeta(6) + 5\zeta(2)\zeta^2(5) - \tfrac{126}{5}\zeta(2)\zeta(10) - \tfrac{1}{4}\zeta^4(3) + \tfrac{35}{6}\zeta^2(3)\zeta(6) + 12\zeta(3)\zeta(4)\zeta(5) - 28\zeta(3)\zeta(9) + \tfrac{33}{16}\zeta^3(4) - \tfrac{217}{8}\zeta(4)\zeta(8) - 26\zeta(5)\zeta(7) - \tfrac{461}{36}\zeta^2(6) + 77\zeta(12)$$

$$s_{71} = \zeta(8)$$

$$s_{72} = -\zeta(2)\zeta(7) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) + 4\zeta(9)$$

$$\begin{aligned} s_{73} = {} & \tfrac{1}{2}\zeta^2(2)\zeta(6) + \zeta(2)\zeta(3)\zeta(5) + \tfrac{1}{2}\zeta(2)\zeta^2(4) - \tfrac{7}{2}\zeta(2)\zeta(8) \\ & + \tfrac{1}{2}\zeta^2(3)\zeta(4) - 4\zeta(3)\zeta(7) - 4\zeta(4)\zeta(6) - 2\zeta^2(5) + 12\zeta(10) \end{aligned}$$

$$\begin{aligned} s_{74} = {} & -\tfrac{1}{6}\zeta^3(2)\zeta(5) - \tfrac{1}{2}\zeta^2(2)\zeta(3)\zeta(4) + \tfrac{3}{2}\zeta^2(2)\zeta(7) - \tfrac{1}{6}\zeta(2)\zeta^3(3) \\ & + \tfrac{7}{2}\zeta(2)\zeta(3)\zeta(6) + \tfrac{7}{2}\zeta(2)\zeta(4)\zeta(5) - \tfrac{20}{3}\zeta(2)\zeta(9) + 2\zeta^2(3)\zeta(5) \\ & + 2\zeta(3)\zeta^2(4) - \tfrac{21}{2}\zeta(3)\zeta(8) - \tfrac{21}{2}\zeta(4)\zeta(7) - \tfrac{31}{3}\zeta(5)\zeta(6) + 30\zeta(11) \end{aligned}$$

$$\begin{aligned} s_{75} = {} & \tfrac{1}{24}\zeta^4(2)\zeta(4) + \tfrac{1}{12}\zeta^3(2)\zeta^2(3) - \tfrac{5}{12}\zeta^3(2)\zeta(6) - \tfrac{3}{2}\zeta^2(2)\zeta(3)\zeta(5) \\ & - \tfrac{3}{4}\zeta^2(2)\zeta^2(4) + \tfrac{7}{2}\zeta^2(2)\zeta(8) - \tfrac{7}{4}\zeta(2)\zeta^2(3)\zeta(4) + 8\zeta(2)\zeta(3)\zeta(7) \\ & + \tfrac{97}{12}\zeta(2)\zeta(4)\zeta(6) + 4\zeta(2)\zeta^2(5) - 21\zeta(2)\zeta(10) - \tfrac{1}{6}\zeta^4(3) \\ & + \tfrac{14}{3}\zeta^2(3)\zeta(6) + \tfrac{19}{2}\zeta(3)\zeta(4)\zeta(5) - \tfrac{70}{3}\zeta(3)\zeta(9) + \tfrac{13}{8}\zeta^3(4) \\ & - \tfrac{91}{4}\zeta(4)\zeta(8) - 22\zeta(5)\zeta(7) - \tfrac{65}{6}\zeta^2(6) + 66\zeta(12) \end{aligned}$$

$$\begin{aligned} s_{76} = {} & -\tfrac{1}{120}\zeta^5(2)\zeta(3) + \tfrac{1}{12}\zeta^4(2)\zeta(5) + \tfrac{5}{12}\zeta^3(2)\zeta(3)\zeta(4) - \tfrac{5}{6}\zeta^3(2)\zeta(7) \\ & + \tfrac{1}{4}\zeta^2(2)\zeta^3(3) - \tfrac{35}{12}\zeta^2(2)\zeta(3)\zeta(6) - 3\zeta^2(2)\zeta(4)\zeta(5) + 7\zeta^2(2)\zeta(9) \\ & - \tfrac{7}{2}\zeta(2)\zeta^2(3)\zeta(5) - \tfrac{29}{8}\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{63}{4}\zeta(2)\zeta(3)\zeta(8) \\ & + \tfrac{31}{2}\zeta(2)\zeta(4)\zeta(7) + \tfrac{91}{6}\zeta(2)\zeta(5)\zeta(6) - 42\zeta(2)\zeta(11) - \tfrac{17}{12}\zeta^3(3)\zeta(4) \\ & + 9\zeta^2(3)\zeta(7) + \tfrac{217}{12}\zeta(3)\zeta(4)\zeta(6) + 9\zeta(3)\zeta^2(5) - \tfrac{231}{5}\zeta(3)\zeta(10) \\ & + \tfrac{37}{4}\zeta^2(4)\zeta(5) - \tfrac{133}{3}\zeta(4)\zeta(9) - 42\zeta(5)\zeta(8) - \tfrac{122}{3}\zeta(6)\zeta(7) \\ & + 132\zeta(13) \end{aligned}$$

$$\begin{aligned} s_{77} = {} & \tfrac{1}{5040}\zeta^7(2) - \tfrac{1}{80}\zeta^5(2)\zeta(4) - \tfrac{1}{24}\zeta^4(2)\zeta^2(3) + \tfrac{5}{36}\zeta^4(2)\zeta(6) \\ & + \tfrac{2}{3}\zeta^3(2)\zeta(3)\zeta(5) + \tfrac{17}{48}\zeta^3(2)\zeta^2(4) - \tfrac{35}{24}\zeta^3(2)\zeta(8) \\ & + \tfrac{5}{4}\zeta^2(2)\zeta^2(3)\zeta(4) - 5\zeta^2(2)\zeta(3)\zeta(7) - 5\zeta^2(2)\zeta(4)\zeta(6) - \tfrac{5}{2}\zeta^2(2)\zeta^2(5) \\ & + \tfrac{63}{5}\zeta^2(2)\zeta(10) + \tfrac{1}{4}\zeta(2)\zeta^4(3) - \tfrac{35}{6}\zeta(2)\zeta^2(3)\zeta(6) \\ & - 12\zeta(2)\zeta(3)\zeta(4)\zeta(5) + 28\zeta(2)\zeta(3)\zeta(9) - \tfrac{33}{16}\zeta(2)\zeta^3(4) \\ & + \tfrac{217}{8}\zeta(2)\zeta(4)\zeta(8) + 26\zeta(2)\zeta(5)\zeta(7) + \tfrac{461}{36}\zeta(2)\zeta^2(6) - 77\zeta(2)\zeta(12) \\ & - \tfrac{7}{3}\zeta^3(3)\zeta(5) - \tfrac{29}{8}\zeta^2(3)\zeta^2(4) + \tfrac{63}{4}\zeta^2(3)\zeta(8) + 31\zeta(3)\zeta(4)\zeta(7) \\ & + \tfrac{91}{3}\zeta(3)\zeta(5)\zeta(6) - 84\zeta(3)\zeta(11) + \tfrac{187}{12}\zeta^2(4)\zeta(6) + \tfrac{31}{2}\zeta(4)\zeta^2(5) \\ & - \tfrac{399}{5}\zeta(4)\zeta(10) - \tfrac{224}{3}\zeta(5)\zeta(9) - \tfrac{427}{6}\zeta(6)\zeta(8) - 35\zeta^2(7) \\ & + \tfrac{1716}{7}\zeta(14) \end{aligned}$$

$$s_{81} = \zeta(9)$$

$$s_{82} = -\zeta(2)\zeta(8) - \zeta(3)\zeta(7) - \zeta(4)\zeta(6) - \tfrac{1}{2}\,\zeta^2(5) + \tfrac{9}{2}\,\zeta(10)$$

$$\begin{aligned} s_{83} = {} & \tfrac{1}{2}\,\zeta^2(2)\zeta(7) + \zeta(2)\zeta(3)\zeta(6) + \zeta(2)\zeta(4)\zeta(5) - 4\zeta(2)\zeta(9) \\ & + \tfrac{1}{2}\,\zeta^2(3)\zeta(5) + \tfrac{1}{2}\,\zeta(3)\zeta^2(4) - \tfrac{9}{2}\,\zeta(3)\zeta(8) - \tfrac{9}{2}\,\zeta(4)\zeta(7) \\ & - \tfrac{9}{2}\,\zeta(5)\zeta(6) + 15\zeta(11) \end{aligned}$$

$$\begin{aligned} s_{84} = {} & -\tfrac{1}{6}\,\zeta^3(2)\zeta(6) - \tfrac{1}{2}\,\zeta^2(2)\zeta(3)\zeta(5) - \tfrac{1}{4}\,\zeta^2(2)\zeta^2(4) + \tfrac{7}{4}\,\zeta^2(2)\zeta(8) \\ & - \tfrac{1}{2}\,\zeta(2)\zeta^2(3)\zeta(4) + 4\zeta(2)\zeta(3)\zeta(7) + 4\zeta(2)\zeta(4)\zeta(6) + 2\zeta(2)\zeta^2(5) \\ & - 12\zeta(2)\zeta(10) - \tfrac{1}{24}\,\zeta^4(3) + \tfrac{9}{4}\,\zeta^2(3)\zeta(6) + \tfrac{9}{2}\,\zeta(3)\zeta(4)\zeta(5) \\ & - \tfrac{40}{3}\,\zeta(3)\zeta(9) + \tfrac{3}{4}\,\zeta^3(4) - \tfrac{53}{4}\,\zeta(4)\zeta(8) - 13\zeta(5)\zeta(7) - \tfrac{155}{24}\,\zeta^2(6) \\ & + \tfrac{165}{4}\,\zeta(12) \end{aligned}$$

$$\begin{aligned} s_{85} = {} & \tfrac{1}{24}\,\zeta^4(2)\zeta(5) + \tfrac{1}{6}\,\zeta^3(2)\zeta(3)\zeta(4) - \tfrac{1}{2}\,\zeta^3(2)\zeta(7) + \tfrac{1}{12}\,\zeta^2(2)\zeta^3(3) \\ & - \tfrac{7}{4}\,\zeta^2(2)\zeta(3)\zeta(6) - \tfrac{7}{4}\,\zeta^2(2)\zeta(4)\zeta(5) + \tfrac{14}{3}\,\zeta^2(2)\zeta(9) - 2\zeta(2)\zeta^2(3)\zeta(5) \\ & - 2\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{21}{2}\,\zeta(2)\zeta(3)\zeta(8) + \tfrac{21}{2}\,\zeta(2)\zeta(4)\zeta(7) \\ & + \tfrac{31}{3}\,\zeta(2)\zeta(5)\zeta(6) - 30\zeta(2)\zeta(11) - \tfrac{3}{4}\,\zeta^3(3)\zeta(4) + 6\zeta^2(3)\zeta(7) \\ & + \tfrac{145}{12}\,\zeta(3)\zeta(4)\zeta(6) + 6\zeta(3)\zeta^2(5) - 33\zeta(3)\zeta(10) + \tfrac{49}{8}\,\zeta^2(4)\zeta(5) \\ & - 32\zeta(4)\zeta(9) - \tfrac{123}{4}\,\zeta(5)\zeta(8) - 30\zeta(6)\zeta(7) + 99\zeta(13) \end{aligned}$$

$$\begin{aligned} s_{86} = {} & -\tfrac{1}{120}\,\zeta^5(2)\zeta(4) - \tfrac{1}{48}\,\zeta^4(2)\zeta^2(3) + \tfrac{5}{48}\,\zeta^4(2)\zeta(6) + \tfrac{1}{2}\,\zeta^3(2)\zeta(3)\zeta(5) \\ & + \tfrac{1}{4}\,\zeta^3(2)\zeta^2(4) - \tfrac{7}{6}\,\zeta^3(2)\zeta(8) + \tfrac{7}{8}\,\zeta^2(2)\zeta^2(3)\zeta(4) - 4\zeta^2(2)\zeta(3)\zeta(7) \\ & - \tfrac{97}{24}\,\zeta^2(2)\zeta(4)\zeta(6) - 2\zeta^2(2)\zeta^2(5) + \tfrac{21}{2}\,\zeta^2(2)\zeta(10) + \tfrac{5}{6}\,\zeta(2)\zeta^4(3) \\ & - \tfrac{14}{3}\,\zeta(2)\zeta^2(3)\zeta(6) - \tfrac{19}{2}\,\zeta(2)\zeta(3)\zeta(4)\zeta(5) + \tfrac{70}{3}\,\zeta(2)\zeta(3)\zeta(9) \\ & - \tfrac{13}{8}\,\zeta(2)\zeta^3(4) + \tfrac{91}{4}\,\zeta(2)\zeta(4)\zeta(8) + 22\zeta(2)\zeta(5)\zeta(7) + \tfrac{65}{6}\,\zeta(2)\zeta^2(6) \\ & - 66\zeta(2)\zeta(12) - \tfrac{11}{6}\,\zeta^3(3)\zeta(5) - \tfrac{45}{16}\,\zeta^2(3)\zeta^2(4) + \tfrac{105}{8}\,\zeta^2(3)\zeta(8) \\ & + 26\zeta(3)\zeta(4)\zeta(7) + \tfrac{51}{2}\,\zeta(3)\zeta(5)\zeta(6) - 72\zeta(3)\zeta(11) + \tfrac{289}{16}\,\zeta^2(4)\zeta(6) \\ & + 13\zeta(4)\zeta^2(5) - \tfrac{687}{10}\,\zeta(4)\zeta(10) - \tfrac{199}{3}\,\zeta(5)\zeta(9) - \tfrac{1487}{24}\,\zeta(6)\zeta(8) \\ & - \tfrac{61}{2}\,\zeta^2(7) + \tfrac{429}{2}\,\zeta(14) \end{aligned}$$

$$
\begin{aligned}
s_{87} = {} & \tfrac{1}{720}\,\zeta^6(2)\zeta(3) - \tfrac{1}{60}\,\zeta^5(2)\zeta(5) - \tfrac{5}{48}\,\zeta^4(2)\zeta(3)\zeta(4) + \tfrac{5}{24}\,\zeta^4(2)\zeta(7) \\
& - \tfrac{1}{12}\,\zeta^3(2)\zeta^3(3) + \tfrac{35}{36}\,\zeta^3(2)\zeta(3)\zeta(6) + \zeta^3(2)\zeta(4)\zeta(5) - \tfrac{7}{3}\,\zeta^3(2)\zeta(9) \\
& + \tfrac{7}{4}\,\zeta^2(2)\zeta^2(3)\zeta(5) + \tfrac{29}{16}\,\zeta^2(2)\zeta(3)\zeta^2(4) - \tfrac{63}{8}\,\zeta^2(2)\zeta(3)\zeta(8) \\
& - \tfrac{31}{4}\,\zeta^2(2)\zeta(4)\zeta(7) - \tfrac{91}{12}\,\zeta^2(2)\zeta(5)\zeta(6) + 21\zeta^2(2)\zeta(11) \\
& + \tfrac{17}{12}\,\zeta(2)\zeta^3(3)\zeta(4) - 9\zeta(2)\zeta^2(3)\zeta(7) - \tfrac{217}{12}\,\zeta(2)\zeta(3)\zeta(4)\zeta(6) \\
& - 9\zeta(2)\zeta(3)\zeta^2(5) + \tfrac{231}{5}\,\zeta(2)\zeta(3)\zeta(10) - \tfrac{37}{4}\,\zeta(2)\zeta^2(4)\zeta(5) \\
& + \tfrac{133}{3}\,\zeta(2)\zeta(4)\zeta(9) + 42\zeta(2)\zeta(5)\zeta(8) + \tfrac{122}{3}\,\zeta(2)\zeta(6)\zeta(7) \\
& - 132\zeta(2)\zeta(13) + \tfrac{1}{12}\,\zeta^5(3) - \tfrac{7}{2}\,\zeta^3(3)\zeta(6) - \tfrac{43}{4}\,\zeta^2(3)\zeta(4)\zeta(5) \\
& + \tfrac{77}{3}\,\zeta^2(3)\zeta(9) - \tfrac{59}{16}\,\zeta(3)\zeta^3(4) + \tfrac{399}{8}\,\zeta(3)\zeta(4)\zeta(8) + 48\zeta(3)\zeta(5)\zeta(7) \\
& + \tfrac{851}{36}\,\zeta(3)\zeta^2(6) - 143\zeta(3)\zeta(12) + \tfrac{197}{8}\,\zeta^2(4)\zeta(7) + \tfrac{193}{4}\,\zeta(4)\zeta(5)\zeta(6) \\
& - 135\zeta(4)\zeta(11) + 8\zeta^3(5) - \tfrac{627}{5}\,\zeta(5)\zeta(10) - \tfrac{355}{3}\,\zeta(6)\zeta(9) \\
& - \tfrac{459}{4}\,\zeta(7)\zeta(8) + 429\zeta(15)
\end{aligned}
$$

$$
\begin{aligned}
s_{88} = {} & -\tfrac{1}{40320}\,\zeta^8(2) + \tfrac{1}{480}\,\zeta^6(2)\zeta(4) + \tfrac{1}{120}\,\zeta^5(2)\zeta^2(3) - \tfrac{1}{36}\,\zeta^5(2)\zeta(6) \\
& - \tfrac{1}{6}\,\zeta^4(2)\zeta(3)\zeta(5) - \tfrac{17}{192}\,\zeta^4(2)\zeta^2(4) + \tfrac{35}{96}\,\zeta^4(2)\zeta(8) \\
& - \tfrac{5}{12}\,\zeta^3(2)\zeta^2(3)\zeta(4) + \tfrac{5}{3}\,\zeta^3(2)\zeta(3)\zeta(7) + \tfrac{5}{3}\,\zeta^3(2)\zeta(4)\zeta(6) \\
& + \tfrac{5}{6}\,\zeta^3(2)\zeta^2(5) - \tfrac{21}{5}\,\zeta^3(2)\zeta(10) - \tfrac{1}{8}\,\zeta^2(2)\zeta^4(3) + \tfrac{35}{12}\,\zeta^2(2)\zeta^2(3)\zeta(6) \\
& + 6\zeta^2(2)\zeta(3)\zeta(4)\zeta(5) - 14\zeta^2(2)\zeta(3)\zeta(9) + \tfrac{33}{32}\,\zeta^2(2)\zeta^3(4) \\
& - \tfrac{217}{16}\,\zeta^2(2)\zeta(4)\zeta(8) - 13\zeta^2(2)\zeta(5)\zeta(9) - \tfrac{461}{72}\,\zeta^2(2)\zeta^2(6) \\
& + \tfrac{77}{2}\,\zeta^2(2)\zeta(12) + \tfrac{7}{3}\,\zeta(2)\zeta^3(3)\zeta(5) + \tfrac{29}{8}\,\zeta(2)\zeta^2(3)\zeta^2(4) \\
& - \tfrac{63}{4}\,\zeta(2)\zeta^2(3)\zeta(8) - 31\zeta(2)\zeta(3)\zeta(4)\zeta(7) - \tfrac{91}{3}\,\zeta(2)\zeta(3)\zeta(5)\zeta(6) \\
& + 84\zeta(2)\zeta(3)\zeta(11) - \tfrac{187}{12}\,\zeta(2)\zeta^2(4)\zeta(6) - \tfrac{31}{2}\,\zeta(2)\zeta(4)\zeta^2(5) \\
& + \tfrac{399}{5}\,\zeta(2)\zeta(4)\zeta(10) + \tfrac{224}{3}\,\zeta(2)\zeta(5)\zeta(9) + \tfrac{427}{6}\,\zeta(2)\zeta(6)\zeta(8) \\
& + 35\zeta(2)\zeta^2(7) - \tfrac{1716}{7}\,\zeta(2)\zeta(14) + \tfrac{17}{24}\,\zeta^4(3)\zeta(4) - 6\zeta^3(3)\zeta(7) \\
& - \tfrac{217}{12}\,\zeta^2(3)\zeta(4)\zeta(6) - 9\zeta^2(3)\zeta^2(5) + \tfrac{231}{5}\,\zeta^2(3)\tau(10) \\
& - \tfrac{37}{2}\,\zeta(3)\zeta^2(4)\zeta(5) + \tfrac{266}{3}\,\zeta(3)\zeta(4)\zeta(9) + 84\zeta(3)\zeta(5)\zeta(8) \\
& + \tfrac{244}{3}\,\zeta(3)\zeta(6)\zeta(7) - 264\zeta(3)\zeta(13) - \tfrac{203}{128}\,\zeta^4(4) + \tfrac{1379}{32}\,\zeta^2(4)\zeta(8) \\
& + 83\zeta(4)\zeta(5)\zeta(7) + \tfrac{327}{8}\,\zeta(4)\zeta^2(6) - \tfrac{495}{2}\,\zeta(4)\zeta(12) + \tfrac{122}{3}\,\zeta^2(5)\zeta(6) \\
& - 228\zeta(5)\zeta(11) - 213\zeta(6)\zeta(10) - 204\zeta(7)\zeta(9) - \tfrac{3217}{32}\,\zeta^2(8) \\
& + \tfrac{6435}{8}\,\zeta(16)
\end{aligned}
$$

$$s_{91} = \zeta(10)$$

$$s_{92} = -\zeta(2)\zeta(9) - \zeta(3)\zeta(8) - \zeta(4)\zeta(7) - \zeta(5)\zeta(6) + 5\zeta(11)$$

$$\begin{aligned} s_{93} = {} & \tfrac{1}{2}\,\zeta^2(2)\zeta(8) + \zeta(2)\zeta(3)\zeta(7) + \zeta(2)\zeta(4)\zeta(6) + \tfrac{1}{2}\,\zeta(2)\zeta^2(5) \\ & - \tfrac{9}{2}\,\zeta(2)\zeta(10) + \tfrac{1}{2}\,\zeta^2(3)\zeta(6) + \zeta(3)\zeta(4)\zeta(5) - 5\zeta(3)\zeta(9) + \tfrac{1}{6}\,\zeta^3(4) \\ & - 5\zeta(4)\zeta(8) - 5\zeta(5)\zeta(7) - \tfrac{5}{2}\,\zeta^2(6) + \tfrac{55}{3}\,\zeta(12) \end{aligned}$$

$$\begin{aligned} s_{94} = {} & -\tfrac{1}{6}\,\zeta^3(2)\zeta(7) - \tfrac{1}{2}\,\zeta^2(2)\zeta(3)\zeta(6) - \tfrac{1}{2}\,\zeta^2(2)\zeta(4)\zeta(5) + 2\zeta^2(2)\zeta(9) \\ & - \tfrac{1}{2}\,\zeta(2)\zeta^2(3)\zeta(5) - \tfrac{1}{2}\,\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{9}{2}\,\zeta(2)\zeta(3)\zeta(8) \\ & + \tfrac{9}{2}\,\zeta(2)\zeta(4)\zeta(7) + \tfrac{9}{2}\,\zeta(2)\zeta(5)\zeta(6) - 15\zeta(2)\zeta(11) - \tfrac{1}{6}\,\zeta^3(3)\zeta(4) \\ & + \tfrac{5}{2}\,\zeta^2(3)\zeta(7) + 5\zeta(3)\zeta(4)\zeta(6) + \tfrac{5}{2}\,\zeta(3)\zeta^2(5) - \tfrac{33}{2}\,\zeta(3)\zeta(10) \\ & + \tfrac{5}{2}\,\zeta^2(4)\zeta(5) - \tfrac{49}{3}\,\zeta(4)\zeta(9) - 16\zeta(5)\zeta(8) - \tfrac{95}{6}\,\zeta(6)\zeta(7) + 55\zeta(13) \end{aligned}$$

$$\begin{aligned} s_{95} = {} & \tfrac{1}{24}\,\zeta^4(2)\zeta(6) + \tfrac{1}{6}\,\zeta^3(2)\zeta(3)\zeta(5) + \tfrac{1}{12}\,\zeta^3(2)\zeta^2(4) - \tfrac{7}{12}\,\zeta^3(2)\zeta(8) \\ & + \tfrac{1}{4}\,\zeta^2(2)\zeta^2(3)\zeta(4) - 2\zeta^2(2)\zeta(3)\zeta(7) - 2\zeta^2(2)\zeta(4)\zeta(6) - \zeta^2(2)\zeta^2(5) \\ & + 6\zeta^2(2)\zeta(10) + \tfrac{1}{24}\,\zeta(2)\zeta^4(3) - \tfrac{9}{4}\,\zeta(2)\zeta^2(3)\zeta(6) - \tfrac{9}{2}\,\zeta(2)\zeta(3)\zeta(4)\zeta(5) \\ & + \tfrac{40}{3}\,\zeta(2)\zeta(3)\zeta(9) - \tfrac{3}{4}\,\zeta(2)\zeta^3(4) + \tfrac{53}{4}\,\zeta(2)\zeta(4)\zeta(8) \\ & + 13\zeta(2)\zeta(5)\zeta(7) + \tfrac{155}{24}\,\zeta(2)\zeta^2(6) - \tfrac{165}{4}\,\zeta(2)\zeta(12) - \tfrac{5}{6}\,\zeta^3(3)\zeta(5) \\ & - \tfrac{5}{4}\,\zeta^2(3)\zeta^2(4) + \tfrac{15}{2}\,\zeta^2(3)\zeta(8) + 15\zeta(3)\zeta(4)\zeta(7) + \tfrac{89}{6}\,\zeta(3)\zeta(5)\zeta(6) \\ & - 45\zeta(3)\zeta(11) + \tfrac{181}{24}\,\zeta^2(4)\zeta(6) + \tfrac{15}{2}\,\zeta(4)\zeta^2(5) - \tfrac{87}{2}\,\zeta(4)\zeta(10) \\ & - \tfrac{125}{3}\,\zeta(5)\zeta(9) - \tfrac{485}{12}\,\zeta(6)\zeta(8) - 20\zeta^2(7) + 143\zeta(14) \end{aligned}$$

$$\begin{aligned} s_{96} = {} & -\tfrac{1}{120}\,\zeta^5(2)\zeta(5) - \tfrac{1}{24}\,\zeta^4(2)\zeta(3)\zeta(4) + \tfrac{1}{8}\,\zeta^4(2)\zeta(7) - \\ & - \tfrac{1}{36}\,\zeta^3(2)\zeta^3(3) + \tfrac{7}{12}\,\zeta^3(2)\zeta(3)\zeta(6) + \tfrac{7}{12}\,\zeta^3(2)\zeta(4)\zeta(5) \\ & - \tfrac{14}{9}\,\zeta^3(2)\zeta(9) + \zeta^2(2)\zeta^2(3)\zeta(5) + \zeta^2(2)\zeta(3)\zeta^2(4) - \tfrac{21}{4}\,\zeta^2(2)\zeta(3)\zeta(8) \\ & - \tfrac{21}{4}\,\zeta^2(2)\zeta(4)\zeta(7) - \tfrac{31}{6}\,\zeta^2(2)\zeta(5)\zeta(6) + 15\zeta^2(2)\zeta(11) \\ & + \tfrac{3}{4}\,\zeta(2)\zeta^3(3)\zeta(4) - 6\zeta(2)\zeta^2(3)\zeta(7) - \tfrac{145}{12}\,\zeta(2)\zeta(3)\zeta(4)\zeta(6) \\ & - 6\zeta(2)\zeta(3)\zeta^2(5) + 33\zeta(2)\zeta(3)\zeta(10) - \tfrac{49}{8}\,\zeta(2)\zeta^2(4)\zeta(5) \\ & + 32\zeta(2)\zeta(4)\zeta(9) + \tfrac{123}{4}\,\zeta(2)\zeta(5)\zeta(8) + 30\zeta(2)\zeta(6)\zeta(7) - 99\zeta(2)\zeta(13) \\ & + \tfrac{1}{24}\,\zeta^5(3) - \tfrac{83}{36}\,\zeta^3(3)\zeta(6) - 7\zeta^2(3)\zeta(4)\zeta(5) + \tfrac{55}{3}\,\zeta^2(3)\zeta(9) \\ & - \tfrac{19}{8}\,\zeta(3)\zeta^3(4) + 36\zeta(3)\zeta(4)\zeta(8) + 35\zeta(3)\zeta(5)\zeta(7) + \tfrac{415}{24}\,\zeta(3)\zeta^2(6) \\ & - \tfrac{429}{8}\,\zeta(3)\zeta(12) + \tfrac{143}{8}\,\zeta^2(4)\zeta(8) + \tfrac{211}{6}\,\zeta(4)\zeta(5)\zeta(6) - 102\zeta(4)\zeta(11) \\ & + \tfrac{35}{8}\,\zeta^3(5) - \tfrac{957}{10}\,\zeta(5)\zeta(10) - \tfrac{820}{9}\,\zeta(6)\zeta(9) - \tfrac{355}{4}\,\zeta(7)\zeta(8) \\ & + \tfrac{1001}{3}\,\zeta(15) \end{aligned}$$

$$
\begin{aligned}
s_{97} = {} & \tfrac{1}{720}\,\zeta^6(2)\zeta(4) + \tfrac{1}{240}\,\zeta^5(2)\zeta^2(3) - \tfrac{1}{48}\,\zeta^5(2)\zeta(6) - \tfrac{1}{8}\,\zeta^4(2)\zeta(3)\zeta(5) \\
& - \tfrac{1}{16}\,\zeta^4(2)\zeta^2(4) + \tfrac{7}{24}\,\zeta^4(2)\zeta(8) - \tfrac{7}{24}\,\zeta^3(2)\zeta^2(3)\zeta(4) \\
& + \tfrac{4}{3}\,\zeta^3(2)\zeta(3)\zeta(7) + \tfrac{97}{72}\,\zeta^3(2)\zeta(4)\zeta(6) + \tfrac{2}{3}\,\zeta^3(2)\zeta^2(5) \\
& - \tfrac{7}{2}\,\zeta^3(2)\zeta(10) - \tfrac{1}{12}\,\zeta^2(2)\zeta^4(3) + \tfrac{7}{3}\,\zeta^2(2)\zeta^2(3)\zeta(6) \\
& + \tfrac{19}{4}\,\zeta^2(2)\zeta(3)\zeta(4)\zeta(5) - \tfrac{35}{3}\,\zeta^2(2)\zeta(3)\zeta(9) + \tfrac{13}{16}\,\zeta^2(2)\zeta^3(4) \\
& - \tfrac{91}{8}\,\zeta^2(2)\zeta(4)\zeta(8) - 11\zeta^2(2)\zeta(5)\zeta(7) - \tfrac{65}{12}\,\zeta^2(2)\zeta^2(6) \\
& + 33\zeta^2(2)\zeta(12) + \tfrac{11}{6}\,\zeta(2)\zeta^3(3)\zeta(5) + \tfrac{45}{16}\,\zeta(2)\zeta^2(3)\zeta^2(4) \\
& - \tfrac{105}{8}\,\zeta(2)\zeta^2(3)\zeta(8) - 26\zeta(2)\zeta(3)\zeta(4)\zeta(7) - \tfrac{51}{2}\,\zeta(2)\zeta(3)\zeta(5)\zeta(6) \\
& + 72\zeta(2)\zeta(3)\zeta(11) - \tfrac{209}{16}\,\zeta(2)\zeta^2(4)\zeta(6) - 13\zeta(2)\zeta(4)\zeta^2(5) \\
& + \tfrac{687}{10}\,\zeta(2)\zeta(4)\zeta(10) + \tfrac{194}{3}\,\zeta(2)\zeta(5)\zeta(9) + \tfrac{1487}{24}\,\zeta(2)\zeta(6)\zeta(8) \\
& + \tfrac{61}{2}\,\zeta(2)\zeta^2(7) - \tfrac{429}{2}\,\zeta(2)\zeta(14) + \tfrac{13}{24}\,\zeta^4(3)\zeta(4) - 5\zeta^3(3)\zeta(7) \\
& - \tfrac{181}{12}\,\zeta^2(3)\zeta(4)\zeta(6) - \tfrac{15}{2}\,\zeta^2(3)\zeta^2(5) + \tfrac{198}{5}\,\zeta^2(3)\zeta(10) \\
& - \tfrac{123}{8}\,\zeta(3)\zeta^2(4)\zeta(5) + \tfrac{229}{3}\,\zeta(3)\zeta(4)\zeta(9) + \tfrac{291}{4}\,\zeta(3)\zeta(5)\zeta(8) \\
& + \tfrac{212}{3}\,\zeta(3)\zeta(6)\zeta(7) - 231\zeta(3)\zeta(13) - \tfrac{21}{16}\,\zeta^4(4) - \tfrac{149}{4}\,\zeta^2(4)\zeta(8) \\
& + 72\zeta(4)\zeta(5)\zeta(7) + \tfrac{2557}{72}\,\zeta(4)\zeta^2(6) - \tfrac{869}{4}\,\zeta(4)\zeta(12) \\
& + \tfrac{106}{3}\,\zeta^2(5)\zeta(6) - 201\zeta(5)\zeta(11) - \tfrac{377}{2}\,\zeta(6)\zeta(10) - 181\zeta(7)\zeta(9) \\
& - \tfrac{357}{4}\,\zeta^2(8) + 715\zeta(16)
\end{aligned}
$$

$$
\begin{aligned}
s_{98} = & -\tfrac{1}{5040}\,\zeta^7(2)\zeta(3) + \tfrac{1}{360}\,\zeta^6(2)\zeta(5) + \tfrac{1}{48}\,\zeta^5(2)\zeta(3)\zeta(4) \\
& - \tfrac{1}{24}\,\zeta^5(2)\zeta(7) + \tfrac{1}{48}\,\zeta^4(2)\zeta^3(3) - \tfrac{35}{144}\,\zeta^4(2)\zeta(3)\zeta(6) \\
& - \tfrac{1}{4}\,\zeta^4(2)\zeta(4)\zeta(5) + \tfrac{7}{12}\,\zeta^4(2)\zeta(9) - \tfrac{7}{12}\,\zeta^3(2)\zeta^2(3)\zeta(5) \\
& - \tfrac{29}{48}\,\zeta^3(2)\zeta(3)\zeta^2(4) + \tfrac{21}{8}\,\zeta^3(2)\zeta(3)\zeta(8) + \tfrac{31}{12}\,\zeta^3(2)\zeta(4)\zeta(7) \\
& + \tfrac{91}{36}\,\zeta^3(2)\zeta(5)\zeta(6) - 7\zeta^3(2)\zeta(11) - \tfrac{17}{24}\,\zeta^2(2)\zeta^3(3)\zeta(4) \\
& + \tfrac{9}{2}\,\zeta^2(2)\zeta^2(3)\zeta(7) + \tfrac{217}{24}\,\zeta^2(2)\zeta(3)\zeta(4)\zeta(6) + \tfrac{9}{2}\,\zeta^2(2)\zeta(3)\zeta^2(5) \\
& - \tfrac{231}{10}\,\zeta^2(2)\zeta(3)\zeta(10) + \tfrac{37}{8}\,\zeta^2(2)\zeta^2(4)\zeta(5) + \tfrac{133}{6}\,\zeta^2(2)\zeta(4)\zeta(9) \\
& - 21\zeta^2(2)\zeta(5)\zeta(8) - \tfrac{61}{3}\,\zeta^2(2)\zeta(6)\zeta(7) + 66\zeta^2(2)\zeta(13) - \tfrac{1}{12}\,\zeta(2)\zeta^5(3) \\
& + \tfrac{7}{2}\,\zeta(2)\zeta^3(3)\zeta(6) + \tfrac{43}{4}\,\zeta(2)\zeta^2(3)\zeta(4)\zeta(5) - \tfrac{77}{3}\,\zeta(2)\zeta^2(3)\zeta(9) \\
& + \tfrac{59}{16}\,\zeta(2)\zeta(3)\zeta^3(4) - \tfrac{399}{8}\,\zeta(2)\zeta(3)\zeta(4)\zeta(8) - 48\zeta(2)\zeta(3)\zeta(5)\zeta(7) \\
& - \tfrac{851}{36}\,\zeta(2)\zeta(3)\zeta^2(6) + 143\zeta(2)\zeta(3)\zeta(12) - \tfrac{197}{8}\,\zeta(2)\zeta^2(4)\zeta(7) \\
& - \tfrac{193}{4}\,\zeta(2)\zeta(4)\zeta(5)\zeta(6) + 135\zeta(2)\zeta(4)\zeta(11) - 8\zeta(2)\zeta^3(5) \\
& + \tfrac{627}{5}\,\zeta(2)\zeta(5)\zeta(10) + \tfrac{355}{3}\,\zeta(2)\zeta(6)\zeta(9) + \tfrac{459}{4}\,\zeta(2)\zeta(7)\zeta(8) \\
& - 429\zeta(2)\zeta(15) + \tfrac{25}{24}\,\zeta^4(3)\zeta(5) + \tfrac{103}{48}\,\zeta^3(3)\zeta^2(4) - \tfrac{77}{8}\,\zeta^3(3)\zeta(8) \\
& - \tfrac{57}{2}\,\zeta^2(3)\zeta(4)\zeta(7) - \tfrac{335}{12}\,\zeta^2(3)\zeta(5)\zeta(6) + 78\zeta^2(3)\zeta(11) \\
& - \tfrac{1375}{48}\,\zeta(3)\zeta^2(4)\zeta(6) - \tfrac{57}{2}\,\zeta(3)\zeta(4)\zeta^2(5) + \tfrac{297}{2}\,\zeta(3)\zeta(4)\zeta(10) \\
& + \tfrac{418}{3}\,\zeta(3)\zeta(5)\zeta(9) + \tfrac{1065}{8}\,\zeta(3)\zeta(6)\zeta(8) + \tfrac{131}{2}\,\zeta(3)\zeta^2(7) \\
& - \tfrac{6435}{14}\,\zeta(3)\zeta(14) - \tfrac{39}{4}\,\zeta^3(4)\zeta(5) + \tfrac{857}{12}\,\zeta^2(4)\zeta(9) \\
& + \tfrac{543}{4}\,\zeta(4)\zeta(5)\zeta(8) + \tfrac{395}{3}\,\zeta(4)\zeta(6)\zeta(7) - 429\zeta(4)\zeta(13) \\
& + \tfrac{131}{2}\,\zeta^2(5)\zeta(7) + \tfrac{4649}{72}\,\zeta(5)\zeta^2(6) - \tfrac{1573}{4}\,\zeta(5)\zeta(12) \\
& - 365\zeta(6)\zeta(11) - \tfrac{693}{2}\,\zeta(7)\zeta(10) - \tfrac{675}{2}\,\zeta(8)\zeta(9) + 1430\zeta(17)
\end{aligned}
$$

$$s_{99} = \tfrac{1}{362880}\zeta^9(2) - \tfrac{1}{3360}\zeta^7(2)\zeta(4) - \tfrac{1}{720}\zeta^6(2)\zeta^2(3) + \tfrac{1}{216}\zeta^6(2)\zeta(6)$$
$$+ \tfrac{1}{30}\zeta^5(2)\zeta(3)\zeta(5) + \tfrac{17}{960}\zeta^5(2)\zeta^2(4) - \tfrac{7}{96}\zeta^5(2)\zeta(8)$$
$$+ \tfrac{5}{48}\zeta^4(2)\zeta^2(3)\zeta(4) - \tfrac{5}{12}\zeta^4(2)\zeta(3)\zeta(7) - \tfrac{5}{12}\zeta^4(2)\zeta(4)\zeta(6)$$
$$- \tfrac{5}{24}\zeta^4(2)\zeta^2(5) + \tfrac{21}{20}\zeta^4(2)\zeta(10) + \tfrac{1}{24}\zeta^3(2)\zeta^4(3)$$
$$- \tfrac{35}{36}\zeta^3(2)\zeta^2(3)\zeta(6) - 2\zeta^3(2)\zeta(3)\zeta(4)\zeta(5) + \tfrac{14}{3}\zeta^3(2)\zeta(3)\zeta(9)$$
$$- \tfrac{11}{32}\zeta^3(2)\zeta^3(4) + \tfrac{217}{48}\zeta^3(2)\zeta(4)\zeta(8) + \tfrac{13}{3}\zeta^3(2)\zeta(5)\zeta(7)$$
$$+ \tfrac{461}{216}\zeta^3(2)\zeta^2(6) - \tfrac{77}{6}\zeta^3(2)\zeta(12) - \tfrac{7}{6}\zeta^2(2)\zeta^3(3)\zeta(5)$$
$$- \tfrac{29}{16}\zeta^2(2)\zeta^2(3)\zeta^2(4) + \tfrac{63}{8}\zeta^2(2)\zeta^2(3)\zeta(8) + \tfrac{31}{2}\zeta^2(2)\zeta(3)\zeta(4)\zeta(7)$$
$$+ \tfrac{91}{6}\zeta^2(2)\zeta(3)\zeta(5)\zeta(6) - 42\zeta^2(2)\zeta(3)\zeta(11) + \tfrac{187}{24}\zeta^2(2)\zeta^2(4)\zeta(6)$$
$$- \tfrac{31}{4}\zeta^2(2)\zeta(4)\zeta^2(5) - \tfrac{399}{10}\zeta^2(2)\zeta(4)\zeta(10) - \tfrac{112}{3}\zeta^2(2)\zeta(5)\zeta(9)$$
$$- \tfrac{427}{12}\zeta^2(2)\zeta(6)\zeta(8) - \tfrac{35}{2}\zeta^2(2)\zeta^2(7) + \tfrac{859}{7}\zeta^2(2)\zeta(14)$$
$$- \tfrac{17}{24}\zeta(2)\zeta^4(3)\zeta(4) + 6\zeta(2)\zeta^3(3)\zeta(7) + \tfrac{217}{12}\zeta(2)\zeta^2(3)\zeta(4)\zeta(6)$$
$$+ 9\zeta(2)\zeta^2(3)\zeta^2(5) - \tfrac{231}{5}\zeta(2)\zeta^2(3)\zeta(10) + \tfrac{37}{2}\zeta(2)\zeta(3)\zeta^2(4)\zeta(5)$$
$$- \tfrac{266}{3}\zeta(2)\zeta(3)\zeta(4)\zeta(9) - 84\zeta(2)\zeta(3)\zeta(5)\zeta(8) - \tfrac{244}{3}\zeta(2)\zeta(3)\zeta(6)\zeta(7)$$
$$+ 264\zeta(2)\zeta(3)\zeta(13) + \tfrac{203}{128}\zeta(2)\zeta^4(4) - \tfrac{1379}{32}\zeta(2)\zeta^2(4)\zeta(8)$$
$$- 83\zeta(2)\zeta(4)\zeta(5)\zeta(7) - \tfrac{327}{8}\zeta(2)\zeta(4)\zeta^2(6) + \tfrac{495}{2}\zeta(2)\zeta(4)\zeta(12)$$
$$- \tfrac{122}{3}\zeta(2)\zeta^2(5)\zeta(6) + 228\zeta(2)\zeta(5)\zeta(11) + 213\zeta(2)\zeta(6)\zeta(10)$$
$$+ 204\zeta(2)\zeta(7)\zeta(9) + \tfrac{3217}{32}\zeta(2)\zeta^2(8) - \tfrac{6435}{8}\zeta(2)\zeta(16) - \tfrac{1}{36}\zeta^6(3)$$
$$- \tfrac{7}{4}\zeta^4(3)\zeta(6) + \tfrac{43}{6}\zeta^3(3)\zeta(4)\zeta(5) - \tfrac{154}{9}\zeta^3(3)\zeta(9)$$
$$+ \tfrac{59}{16}\zeta^2(3)\zeta^3(4) - \tfrac{399}{8}\zeta^2(3)\zeta(4)\zeta(8) - 48\zeta^2(3)\zeta(5)\zeta(7)$$
$$- \tfrac{851}{36}\zeta^2(3)\zeta^2(6) + 143\zeta^2(3)\zeta(12) - \tfrac{197}{4}\zeta(3)\zeta^2(4)\zeta(7)$$
$$- \tfrac{193}{2}\zeta(3)\zeta(4)\zeta(5)\zeta(6) + 270\zeta(3)\zeta(4)\zeta(11) - 16\zeta(3)\zeta^3(5)$$
$$+ \tfrac{1254}{5}\zeta(3)\zeta(5)\zeta(10) + \tfrac{710}{3}\zeta(3)\zeta(6)\zeta(9) + \tfrac{459}{2}\zeta(3)\zeta(7)\zeta(8)$$
$$- 858\zeta(3)\zeta(15) - \tfrac{33}{2}\zeta^3(4)\zeta(6) - \tfrac{197}{8}\zeta^2(4)\zeta^2(5) + \tfrac{2571}{20}\zeta^2(4)\zeta(10)$$
$$+ \tfrac{724}{3}\zeta(4)\zeta(5)\zeta(9) + \tfrac{692}{3}\zeta(4)\zeta(6)\zeta(8) + \tfrac{227}{2}\zeta(4)\zeta^2(7)$$
$$- \tfrac{5577}{7}\zeta(4)\zeta(14) + \tfrac{459}{4}\zeta^2(5)\zeta(8) + \tfrac{668}{3}\zeta(5)\zeta(6)\zeta(7)$$
$$- 726\zeta(5)\zeta(13) + \tfrac{11055}{324}\zeta^3(6) - \tfrac{4015}{6}\zeta(6)\zeta(12) - 630\zeta(7)\zeta(11)$$
$$- \tfrac{1215}{2}\zeta(8)\zeta(10) - \tfrac{2701}{9}\zeta^2(9) + \tfrac{24310}{9}\zeta(18)$$

$$r_{51} = -\tfrac{8}{315}\,\pi^6$$

$$r_{52} = -\tfrac{8}{15}\,\pi^4\zeta(3) - 8\pi^2\zeta(5) + 144\zeta(7)$$

$$r_{53} = -\tfrac{61}{1575}\,\pi^8 - 12\pi^2\zeta^2(3) + 432\zeta(3)\zeta(5)$$

$$r_{54} = -\tfrac{4}{3}\,\pi^6\zeta(3) - \tfrac{112}{5}\,\pi^4\zeta(5) - 480\,\pi^2\zeta(7) + 288\zeta^3(3) + 8064\zeta(9)$$

$$r_{55} = -\tfrac{149}{660}\,\pi^{10} - 40\pi^4\zeta^2(3) - 1920\pi^2\zeta(3)\zeta(5) + 28800\zeta(3)\zeta(7) + 14400\zeta^2(5)$$

$$r_{61} = 120\zeta(7)$$

$$r_{62} = -\tfrac{2}{63}\,\pi^8 + 240\zeta(3)\zeta(5)$$

$$r_{63} = -\tfrac{4}{3}\,\pi^6\zeta(3) - 18\pi^4\zeta(5) - 360\pi^2\zeta(7) + 120\zeta^3(3) + 6720\zeta(9)$$

$$r_{64} = -\tfrac{278}{1485}\,\pi^{10} - 36\pi^4\zeta^2(3) - 1440\pi^2\zeta(3)\zeta(5) + 23040\zeta(3)\zeta(7) + 11520\zeta^2(5)$$

$$r_{65} = -\tfrac{47}{6}\,\pi^8\zeta(3) - \tfrac{208}{3}\,\pi^6\zeta(5) - 1480\pi^4\zeta(7) - 1200\pi^2\zeta^3(3) - 33600\pi^2\zeta(9)$$
$$+ 50400\zeta^2(3)\zeta(5) + 604800\zeta(11)$$

$$r_{66} = -\tfrac{4714153}{2522520}\,\pi^{12} - 200\pi^6\zeta^2(3) - 6720\pi^4\zeta(3)\zeta(5) - 144000\pi^2\zeta(3)\zeta(7)$$
$$- 72000\pi^2\zeta^2(5) + 21600\zeta^4(3) + 2419200\zeta(3)\zeta(9) + 2246400\zeta(5)\zeta(7)$$

$$r_{71} = -\tfrac{8}{105}\,\pi^8$$

$$r_{72} = -\tfrac{32}{21}\,\pi^6\zeta(3) - 16\pi^4\zeta(5) - 240\pi^2\zeta(7) + 5760\zeta(9)$$

$$r_{73} = -\tfrac{74}{385}\,\pi^{10} - 24\pi^4\zeta^2(3) - 720\pi^2\zeta(3)\zeta(5) + 17280\zeta(3)\zeta(7) + 8640\zeta^2(5)$$

$$r_{74} = -\tfrac{104}{15}\,\pi^8\zeta(3) - \tfrac{632}{7}\,\pi^6\zeta(5) - 1296\pi^4\zeta(7) - 480\pi^2\zeta^3(3) - 26880\pi^2\zeta(9)$$
$$+ 34560\zeta^2(3)\zeta(5) + 518400\zeta(11)$$

$$r_{75} = -\tfrac{1048079}{630630}\,\pi^{12} - 180\pi^6\zeta^2(3) - 5520\pi^4\zeta(3)\zeta(5) - 115200\pi^2\zeta(3)\zeta(7)$$
$$- 57600\pi^2\zeta^2(5) + 14400\zeta^4(3) + 2016000\zeta(3)\zeta(9) + 1900800\zeta(5)\zeta(7)$$

$$r_{76} = -\tfrac{224}{3}\,\pi^{10}\zeta(3) - 772\pi^8\zeta(5) - \tfrac{66008}{7}\,\pi^6\zeta(7) - 4560\pi^4\zeta^3(3) - 154560\pi^4\zeta(9)$$
$$- 302400\pi^2\zeta^2(3)\zeta(5) - 3628800\pi^2\zeta(11) + 4665600\zeta^2(3)\zeta(7)$$
$$+ 4665600\zeta(3)\zeta^2(5) + 68428800\zeta(13)$$

$$r_{77} = -\tfrac{5181555}{216216}\,\pi^{14} - 1974\pi^8\zeta^2(3) - 47040\pi^6\zeta(3)\zeta(5) - 745920\pi^4\zeta(3)\zeta(7)$$
$$- 372960\pi^4\zeta^2(5) - 151200\pi^2\zeta^4(3) - 16934400\pi^2\zeta(3)\zeta(9)$$
$$- 15724800\pi^2\zeta(5)\zeta(7) + 8467200\zeta^3(3)\zeta(5) + 304819200\zeta(3)\zeta(11)$$
$$+ 270950400\zeta(5)\zeta(9) + 127008000\zeta^2(7)$$