MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 160
OCTOBER 1982, PAGES 725-732

Class Number Formulae of Dirichlet Type

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Abstract. Applying a theorem of Johnson and Mitchell, some new class number formulae are derived.

Throughout this paper $h(D)$ denotes the class number of the quadratic field $Q(\sqrt{D})$ of discriminant D. In 1840 Dirichlet [4] proved (among others) the class number formulae

$$(1) \qquad h(-p) = \frac{1}{3} \sum_{0<n<p/2} \left(\frac{n}{p}\right), \qquad p \equiv 3 \pmod 8,$$

and

$$(2) \qquad h(-p) = \sum_{0<n<p/2} \left(\frac{n}{p}\right), \qquad p \equiv 7 \pmod 8,$$

where, here and throughout the paper, p denotes a prime greater than 3. Since then many other similar formulae have appeared in the literature; see for example [1], [3]–[16]. We just give two examples. In the work of Karpinski [14, p. 15] we find

$$(3) \qquad h(-5p) = -4 \sum_{4p/10<n<5p/10} \left(\frac{n}{p}\right), \qquad p \equiv 31, 39 \pmod{40},$$

while

$$(4) \qquad h(-3p) = 2 \sum_{0<n<p/3} \left(\frac{n}{p}\right), \qquad p \equiv 1 \pmod 4,$$

appears in the paper of Lerch [15, p. 402]. In some research of the second author the following formula arose:

$$(5) \qquad h(-7p) = -2 \sum_{5p/14<n<6p/14} \left(\frac{n}{p}\right), \qquad p \equiv 5 \pmod 8.$$

This result does not seem to have been stated explicitly before. The authors decided therefore to determine all such class number formulae of the type given in (1)–(5) within certain limits.

Received January 15, 1982.

1980 *Mathematics Subject Classification.* Primary 12A50, 12-04; Seconary 12A25.

Key words and phrases. Dirichlet type class number formulae, class numbers of imaginary quadratic number fields.

* Research supported by Natural Sciences and Engineering Research Council of Canada Grant No. A-7233.

0025-5718/82/0000-0249/$02.75

All of these formulae are of the form

$$(6) \qquad h(-dp) = c \sum_{(k-1)p/ld<n<kp/ld} \left(\frac{n}{p}\right), \qquad p \equiv r \left(\bmod 4 \prod_{q|ld} q\right),$$

where

(i) d is a positive integer such that $-dp$ is the discriminant of an imaginary quadratic field,

(ii) l is a positive integer,

(iii) k is a positive integer satisfying $1 \leqslant k \leqslant [\frac{1}{2}(ld+1)]$,

(iv) q runs through primes dividing ld,

(v) r is an integer satisfying $1 \leqslant r \leqslant 4\prod_{q|ld} q, (r, 2ld) = 1$,

(vi) c is a rational number.

We call a formula of the form (6) a class number formula of Dirichlet type. A computer search was designed to find all such formulae for which the parameters d and l satisfy $1 \leqslant d \leqslant 28$, $1 \leqslant ld \leqslant 30$. Within these limits our search revealed 127 formulae of Dirichlet type. (We note that it is not impossible for a data input error to have resulted in a missed formula.) Of these 127 formulae, which are summarized in the following table, 84 were found in the literature and 43 appear to be new. Following Table 1 we indicate how the new formulae are proved.

TABLE 1

Dirichlet type class number formulae

d	l	k	c	r	$4\prod_{q\vert ld} q$	References
1	2	1	1/3	3	8	Dirichlet [4]
1	2	1	1	7	8	“ ”
1	3	1	1/2	7	12	Lerch [15]
1	3	1	1	11	12	“ ”
1	4	1	1	7	8	Dirichlet [4]
1	4	2	1/3	3	8	“ ”
1	6	1	1	7	24	Holden [7]
1	6	1	1	11	24	“ ”
1	6	1	−1	19	24	“ ”
1	6	1	1	23	24	“ ”
1	6	2	1	7	24	“ ”
1	6	2	1/3	19	24	“ ”
1	6	3	−1	7	24	“ ”
1	6	3	1/2	11	24	“ ”
1	6	3	1	19	24	“ ”
1	10	1	1	11	40	Karpinski [14]
1	10	1	1	19	40	“ ”
1	12	3	−1	11	24	“ ”
1	12	3	1	19	24	“ ”
1	12	4	1	7	24	“ ”
1	12	4	1	11	24	“ ”
1	12	4	1/2	19	24	
1	14	3	−1	11	56	
1	14	3	−1	43	56	
1	14	3	−1	51	56	
1	14	6	1	11	56	
1	14	6	1	43	56	

TABLE 1
Dirichlet type class number formulae

d	l	k	c	r	$4\prod_{q\mid ld} q$	References
1	14	6	1	51	56	Karpinski [14]
1	18	8	−1	19	24	" "
1	20	6	1	11	40	
1	20	6	1	19	40	
1	30	10	1	19	120	
1	30	10	1/2	43	120	Johnson and Mitchell [13]
1	30	10	1/2	67	120	" "
1	30	10	1	83	120	
1	30	10	1	91	120	
1	30	10	1	107	120	
3	1	1	2	1	12	Lerch [15]
3	1	1	2	5	12	" "
3	1	2	−1	1	12	" "
3	1	2	−1	5	12	" "
3	2	1	1	1	24	" "
3	2	1	1	17	24	" "
3	2	2	−2	1	24	" "
3	2	2	−2	17	24	" "
3	2	2	2	5	24	" "
3	2	2	2	13	24	" "
3	2	3	−2	1	24	" "
3	2	3	−2	5	24	" "
3	2	3	−2	13	24	" "
3	2	3	−2	17	24	
3	4	1	2	17	24	Karpinski [14]
3	4	2	2	17	24	" "
3	4	5	−2	5	24	" "
3	4	5	−1	17	24	" "
3	4	6	2	17	24	" "
3	10	7	2	41	120	
3	10	7	2	89	120	
4	1	1	2	1	8	Dirichlet [4]
4	1	1	2	5	8	" "
4	1	2	−2	1	8	" "
4	1	2	−2	5	8	" "
4	3	1	2	13	24	Karpinski [14]
4	3	2	−2	13	24	" "
4	3	3	2	5	24	" "
4	3	3	2	13	24	" "
4	3	6	−2	13	24	" "
5	1	2	2	3	20	Lerch [15]
5	1	2	2	7	20	" "
5	1	2	2	11	20	" "
5	1	2	2	19	20	" "
5	2	2	−4	11	40	Karpinski [14]
5	2	2	−4	19	40	" "
5	2	2	4	31	40	" "
5	2	2	4	39	40	" "
5	2	3	4	11	40	" "

TABLE 1

Dirichlet type class number formulae

d	l	k	c	r	$4\prod_{q\|ld} q$	References
5	2	3	4	19	40	Karpinski [14]
5	2	3	4/3	31	40	" "
5	2	3	4/3	39	40	" "
5	2	4	4	11	40	" "
5	2	4	4	19	40	" "
5	2	4	−4	31	40	" "
5	2	4	−4	39	40	" "
5	2	5	−4	31	40	" "
5	2	5	−4	39	40	" "
5	3	6	4	11	60	
5	3	6	4	59	60	
5	4	5	4	31	40	
5	4	5	4	39	40	
5	4	6	2	7	40	
5	4	6	2	23	40	
5	4	6	2	31	40	
5	4	6	2	39	40	
5	6	6	−4	11	120	
5	6	6	−4	31	120	
5	6	6	−4	59	120	
5	6	6	−4	71	120	
5	6	6	−4	79	120	
5	6	6	−4	119	120	
5	6	10	−2	23	120	
5	6	10	−2	47	120	
5	6	10	−2	71	120	
5	6	10	−2	119	120	
7	2	6	−2	5	56	
7	2	6	−2	13	56	
7	2	6	−2	29	56	
7	2	6	−2	37	56	
7	2	6	−2	45	56	
7	2	6	−2	53	56	
8	1	2	4	7	8	Dirichlet [4]
8	1	3	4	7	8	" "
8	1	4	−4	7	8	" "
8	3	4	4	7	24	Karpinski [14]
8	3	4	4	23	24	" "
8	3	9	4	24	24	" "
12	1	2	4	7	24	" "
12	1	2	4	11	24	" "
12	1	2	4	23	24	" "
12	1	5	4	11	24	" "
12	1	5	4	19	24	" "
12	1	5	4	23	24	" "
12	1	6	−4	7	24	
12	1	6	−4	23	24	

TABLE 1
Dirichlet type class number formulae

d	l	k	c	r	$4\prod_{q\mid ld} q$	References
15	2	12	−4	13	120	Karpinski [14]
15	2	12	−4	37	120	" "
15	2	12	−4	61	120	
15	2	12	−4	109	120	

Henceforth we let $s(ld, k)$ denote the sum

$$\sum_{(k-1)p/ld<n<kp/ld} \left(\frac{n}{p}\right).$$

In order to prove the 43 formulae in Table 1 for which we are unable to find a reference, it was necessary to complete the tables in [13] and to derive a new one (see Table 2) for $ld = 30$ using the theorem of Johnson and Mitchell [13, pp. 117–118]. The 16 cases in Table 2 (listed as congruence classes modulo 120) arise from the 16 possible choices of sign for $(-1/p)$, $(2/p)$, $(3/p)$, $(5/p)$. Since, in each case, exactly eleven of the equations in the theorem of Johnson and Mitchell are linearly independent, the 15 variables $s(30,1),\ldots,s(30,15)$ are given in terms of four parameters A, B, C, D. When 0 appears in Table 2 it denotes the number "zero" and not the letter "oh".

We just give the proofs of two of the class number formulae in Table 1 as the others are proved similarly using Table 2 and easily completed tables of Johnson and Mitchell. In particular, we prove

(7) $$h(-7p) = -2s(14,6), \qquad p \equiv 5 \pmod 8,$$

and

(8) $$h(-15p) = -4s(30,12), \qquad p \equiv 13 \pmod{24}.$$

To prove (7) and (8) we require the following result which is given in [2, Lemma 1]: if $p \equiv 1 \pmod 4$ is prime and $d \not\equiv 0 \pmod p$ is the discriminant of an imaginary quadratic field, then

(9) $$h(dp) = \sum_{k=1}^{|d|} \left(\sum_{j=k}^{|d|} \left(\frac{d}{j}\right) \right) s(|d|, k).$$

Taking $d = -7$ in (9), we obtain

$$h(-7p) = -s(7,2) - 2s(7,3) - s(7,4) - 2s(7,5) - s(7,6).$$

As $s(7, r) = s(7, 8-r)$ $(r = 1,2,\ldots,7)$ for $p \equiv 1 \pmod 4$, we have

$$h(-7p) = -2s(7,2) - 4s(7,3) - s(7,4).$$

Next, as

$$s(7, r) = s(14, 2r-1) + s(14, 2r) \qquad (r = 1,2,\ldots,7)$$

TABLE 2. $ld = 30$

$s(30, r)$ \ p (mod 120)	1,49	73,97	41,89	61,109	17,113	13,37	29,101	53,77
1	$-3A-6B-3C-2D$	A	A	A	A	A	A	A
2	B	B	B	B	B	B	B	B
3	C	C	$A+2B+3C+4D$	C	$-3A+C-2D$	C	C	C
4	B	B	$-B-2C-2D$	$-2A-B+2D$	$2A+B$	$4A+3B$	$-4A-3B-2C$	$-4A-3B-2C$
5	A	$A-4B-3C+2D$	C	$A-C-2D$	C	$-5A-4B-C$	$3A+2B+C$	$3A+2B+C$
6	D	$-A+2B+2C-2D$	D	D	$-A-B-C$	$3A+2B$	D	$-4A-3B-2C$
7	$B+C-D$	$-A+B+C$	$A+B+C+D$	$B+C-D$	0	$-3A-B+C$	$B+C+D$	$-4A-2B-C$
8	B	B	B	$2A+B$	B	$2A+E$	$2A+B$	$2A+B$
9	$A-B-C+2D$	$A-3B-2C$	$-A-3B-2C-2D$	$-A-B+2D$	$A-B$	$5A+3B$	$-A-B$	$-A-B$
10	$B+C-D$	D	$-A-D$	$2A+B-C-3D$	D	D	$3A+2B-D$	D
11	D	$B+C-D$	$-2A-3B-3C-3D$	$-2A+2C+3D$	$A-B-C+D$	$B+C-D$	$-5A-3B-C+D$	$-2A-B-C-D$
12	$B+C-D$	$-A+B+C$	$A+3B+3C+3D$	$-B-C-D$	$-2A$	$-3A-3B-C$	$2A+B+C-D$	$6A+4B+3C$
13	$A-C+D$	$-2B-2C$	$-A-2B-2C-3D$	$-A+D$	$2A-B-C+2D$	$2A+2B$	$-3A-2B-2C-D$	$A+B$
14	C	C	$A+C+2D$	$2A-C-2D$	$-A+C-2D$	$-4A-4B-C$	$4A+2B+C$	$4A+2B+C$
15	B	B	B	$-2A-B$	B	$-2A-B$	$-2A-B$	$-2A-B$

$s(30, r)$ \ p (mod 120)	71,119	23,47	31,79	11,59	7,103	83,107	19,91	43,67
1	A	A	A	A	A	A	A	A
2	$-C-4D$	B	B	B	B	B	B	B
3	B	$-A-B+2D$	C	C	$-A-B+2D$	$-A-B$	C	$-A-B$
4	C	$B-2C$	$-B-2D$	$-B-2C-2D$	$-B-2D$	$2A+B-2D$	$-B-2C$	$-2A-B-2C$
5	$-C-2D$	C	B	$B+2C+2D$	C	C	$-2A-B$	C
6	D	D	D	D	D	D	D	D
7	$-B-C-5D$	$A+2B-3D$	$B-C+D$	$-B-3C-3D$	$A+2B-D$	$-A-2C+D$	$2A+B-C-D$	$3A+2B-D$
8	$2B+C+4D$	$-2A-3B+4D$	$-B+2C-4D$	$B+4C+4D$	$-2A-3B$	$-B+2C$	$B+4C$	$-B+2C$
9	$-B-2D$	$A+2B-C-2D$	$-C$	$-C-2D$	$A+C-2D$	$-A-C$	$-C$	$-A-C$
10	$2D$	$-B+C$	$A+B+C$	0	$2D$	$2A+B+C-2D$	$A+B+C$	$4A+2B+2C$
11	$B-C-2D$	$-A-B+C+2D$	$-A$	$B+C$	$-A$	$A+B+C-2D$	$-A$	$3A+2B+2C$
12	$-B+C+D$	$A+2B-2C-D$	$-B-C+D$	$-B-C-D$	$A-D$	$A-D$	$2A+B+C-D$	$A-D$
13	$B+C+D$	$-A-B-C+D$	$-B+C-D$	$B+3C+3D$	$-A-B-C+D$	$-A-B+C+D$	$-2A-B+C+D$	$-A-B+C+D$
14	$-B-2C-4D$	$A+B+2C-2D$	$2B-C+2D$	$-C-2D$	$A+3B$	$A+B-2D$	$-C$	$-3A-B-2C$
15	$C+4D$	$-B$	$-B$	$2A+B$	$-B$	$2A+B$	$2A+B$	$2A+B$

and $s(14, r) = s(14, 15 - r)$ $(r = 1, 2, \ldots, 14)$, we obtain

$$(10) \quad h(-7p) = -2s(14,3) - 2s(14,4) - 4s(14,5) - 4s(14,6) - 2s(14,7).$$

From the theorem of Johnson and Mitchell [13] (see the table on p. 122) we have, for $p \equiv 5, 13, 29, 37, 45, 53 \pmod{56}$, $s(14,1) = A$, $s(14,2) = B$, $s(14,3) = C$, $s(14,4) = 2A + B$, $s(14,5) = A + B$, $s(14,6) = -2A - 2B - C$, $s(14,7) = -2A - B$. Using this in (10), we obtain for $p \equiv 5 \pmod 8$

$$h(-7p) = 4A + 4B + 2C = -2s(14,6),$$

proving (7).

Next, taking $d = -15$ in (9), we obtain

$$\begin{aligned} h(-15p) = {} & -s(15,2) - 2s(15,3) - 2s(15,4) - 3s(15,5) \\ & -3s(15,6) - 3s(15,7) - 2s(15,8) \\ & -3s(15,9) - 3s(15,10) - 3s(15,11) \\ & -2s(15,12) - 2s(15,13) - s(15,14). \end{aligned}$$

As $s(15, r) = s(15, 16 - r)$ $(r = 1, 2, \ldots, 15)$ for $p \equiv 1 \pmod 4$, we have

$$\begin{aligned} h(-15p) = {} & -2s(15,2) - 4s(15,3) - 4s(15,14) - 6s(15,5) \\ & -6s(15,6) - 6s(15,7) - 2s(15,8). \end{aligned}$$

Next, as $s(15, r) = s(30, 2r - 1) + s(30, 2r)$ $(r = 1, 2, \ldots, 15)$ and $s(30, r) = s(30, 31 - r)$ $(r = 1, 2, \ldots, 30)$, we obtain

$$\begin{aligned} h(-15p) = {} & -2s(30,3) - 2s(30,4) - 4s(30,5) - 4s(30,6) \\ & -4s(30,7) - 4s(30,8) - 6s(30,9) - 6s(30,10) \\ & -6s(30,11) - 6s(30,12) - 6s(30,13) - 6s(30,14) - 4s(30,15). \end{aligned}$$

Appealing to Table 2 we have, after simplifying the above, for $p \equiv 13, 37 \pmod{120}$,

$$h(-15p) = 12A + 12B + 4C = -4s(30,12),$$

and, for $p \equiv 61, 109 \pmod{120}$,

$$h(-15p) \equiv 4B + 4C + 4D = -4s(30,12),$$

completing the proof of (8).

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