CIMAT
Guanajuato, Gto.
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1. L. FEJÉR, "Potenzreihen mit mehrfach monotoner Koeffizientenfolge und ihre Legendre-Polynome," *Proc. Cambridge Philos. Soc.*, v. 31, 1935, pp. 307–316.

2. P. WYNN, "Accélération de la convergence de séries d'opérateurs en analyse numérique," *C. R. Acad. Sci. Paris Ser. A-B*, v. 276A, 1973, pp. 803–806.

3. W. GAUTSCHI, "Anomalous convergence of a continued fraction for ratios of Kummer functions," *Math. Comp.*, v. 31, 1977, pp. 994–999.

24[4.05.2, 4.10.3, 4.15.3].—E. P. DOOLAN, J. J. H. MILLER & W. H. A. SCHILDERS, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980, xvi + 324 pp., 24 cm. Price \$60.00.

This monograph systematically addresses a relatively new class of numerical methods for singularly perturbed initial and boundary value problems, typical examples of which are

$$\text{(IVP)} \qquad \varepsilon u_x(x) + a(x)u(x) = f(x) \quad \text{for } x > 0,\ u(0) = A,$$

and

$$\text{(BVP)} \qquad \varepsilon u_{xx}(x) + a(x)u_x(x) - b(x)u(x) = f(x) \quad \text{for } 0 < x < 1,$$
$$u(0) = A \text{ and } u(1) = B.$$

In these problems ε is a positive constant in $(0, 1]$ which may be very small, $a(x) > 0$, $b(x) \geqslant 0$, and A and B are given constants. When ε is small, near $x = 0$ the solution $u(x)$ of (IVP) and (BVP) displays a boundary layer, i.e., a large gradient.

The presentation is expository while centering around the authors' research on finite difference methods for problems of the type (IVP) and (BVP) whose convergence is *uniform* for ε in $(0, 1]$ in the sense described below. Many of the results are new and have appeared previously in at most an abbreviated form.

Denoting the approximate solution obtained using a given finite difference scheme on an equally spaced mesh of size h by u^h (having value u_i^h at the ith mesh point), the scheme is said to be *uniformly convergent with order p* if the difference between u^h and the exact solution u at all the grid points is bounded by Ch^p where C and p are independent of *h and* ε. Uniformly convergent methods can be expected to be reliable for all values of ε even on coarse meshes. Such methods may thus also provide a sound starting point for various mesh refinement algorithms.

When ε is small relative to the mesh size, use of classical "centered" difference methods is quickly seen to lead to instability; e.g., defining $\rho = h/\varepsilon$ and approximating the solution of (IVP) when $a \equiv 1$ and $f \equiv 0$ with

$$\text{(C1)} \qquad \varepsilon(u_{i+1} - u_i)/h + (u_{i+1} + u_i)/2 = 0, \quad u_0 = A,$$

leads to

$$\text{(C2)} \qquad u_{i+1} = (1 - \rho/2)u_i/(1 + \rho/2)$$

which oscillates when $\rho > 2$. This type of instability can be suppressed by the use of "upwinding", e.g.,

$$\text{(W1)} \qquad \varepsilon(u_{i+1} - u_i)/h + u_{i+1} = 0, \quad u_0 = A,$$

however this still does not achieve uniform (in ε) convergence, since when $\rho = 1$ the error at $x = h$ remains a fixed nonzero quantity as $h \to 0$.

In the text necessary conditions are given for a finite difference scheme to be uniformly convergent for (IVP) or (BVP) (and for related problems). The general idea is that the scheme should be exact for the constant coefficient homogeneous problem, or equivalently, that the fundamental (exponential) solution behavior should be built into the coefficients of the difference scheme. Such schemes are called *exponentially fitted*. A uniformly accurate scheme for (IVP) is

$$\varepsilon\sigma_i(\rho)(u_{i+1} - u_i)/h + a(x_i)u_i = f(x_i), \quad u_0 = A, \tag{U1}$$

where $\rho \equiv h/\varepsilon$ and the *exponential fitting factor* σ_i is defined by

$$\sigma_i(\rho) = \rho a(x_i)/[1 - \exp(-\rho a(x_i))]. \tag{U2}$$

For (BVP), the original uniform scheme, which was formulated by Allen and Southwell [1], is

$$\begin{aligned} \varepsilon\sigma_i(\rho)(u_{i-1} - 2u_i + u_{i+1})/h^2 + a(x_i)(u_{i+1} - u_{i-1})/(2h) - b(x_i)u_i \\ = f(x_i), \qquad i = 1,\ldots,N-1, \\ \rho \equiv h/\varepsilon, N \equiv 1/h, u_0 = A, u_N = B, \sigma_i(\rho) = \tfrac{1}{2}\rho a(x_i)\coth(\tfrac{1}{2}\rho a(x_i)). \end{aligned} \tag{U3}$$

Both these schemes are uniformly convergent with order 1.

The error analysis for these (and many other) finite difference methods is carried out through the use of, and in a manner designed to illustrate, three general approaches. All utilize a priori analysis of the behavior of the solution of the original problem, and the fact that in each case the differential equation and its difference approximation satisfy a maximum principle. The *two mesh method*, used first by Il'in [2] to prove uniform first order convergence for (U3), and posed as a systematic approach by Miller [4], states that a scheme is uniformly convergent with order p if and only if the scheme is convergent (for each fixed ε) and the difference in grid values for a successive mesh halving is uniformly of order p, i.e.,

$$|u_i^h - u_{2i}^{h/2}| \leqslant C_2 h^p$$

with C_2 and p independent of h, i, and ε.

The second approach, which the authors attribute to Emelyanov, Shishkin, and Titov, is to use a classical error bound based on the local truncation error for $\varepsilon \geqslant h^r$, for some appropriate choice of r, and then to use an asymptotic expansion of the solution to obtain an error bound for $\varepsilon \leqslant h^r$; the combination of the two estimates yielding the desired result.

The third approach hinges on the choice of comparison (barrier) functions derived specifically from the difference scheme being analyzed. This, together with certain a priori knowledge of the behavior of the solution, can be used to produce error estimates, as typified by the work of Kellogg and Tsan [3].

The text is divided into three parts, the first treating the initial value problem (cf. (IVP)). Basic properties and asymptotic expansions of the solution of the continuous problem are developed, and the behavior and limitations of classical difference schemes are described. Necessary conditions for a scheme to be uniformly convergent are given, and some specific exponentially fitted schemes are proven to be uniformly convergent. Other topics considered are extrapolation, uniformly accurate higher order schemes, systems, nonlinear problems, and open questions. In the second part of the text, boundary value problems (cf. (BVP)) are treated along an analogous program. In addition to (BVP), the selfadjoint problem

$$\text{(SA)} \quad -\varepsilon u_{xx}(x) + b(x)u(x) = f(x) \quad \text{for } 0 < x < 1, u(0) = A \text{ and } u(1) = B,$$

where $b(x) > 0$, is considered, as well as the conservation form equations corresponding to (BVP) and (SA). Mixed boundary conditions are also treated. The last section contains a wide range of numerical results illustrating the behavior of the finite difference methods discussed in the first two parts, along with a representative Fortran program listing. Very helpful lists of notation and terminology are included, as is an extensive bibliography.

Altogether, this monograph presents a very lucid account of the use and analysis of exponential fitting to obtain uniformly convergent finite difference schemes for singular perturbation problems. Many of the results are new and anyone working in this field will want to have ready access to this text. It also provides a concise and accessible introduction to this area of study. In particular, the first section dealing with initial value problems provides a superb introduction to the fundamental concepts while the algebra involved is quite tractable (in contrast to the convergence proofs for boundary value problems where the algebra is rather formidable, regardless of the approach taken to attain the result).

While no errors affecting the validity of the results were noted, the following comments might perhaps save the reader some effort in following a few parts of the exposition. On page 24, u_1^h should be $e^{-\rho}$ etc. The equality on page 28 for Q_t can be verified by comparing terms involving $\sigma(\rho)$ and by using the identity $\coth(z) = (e^z + e^{-z})/(e^z - e^{-z})$. On page 42 the second term inside the braces expressing $V_{1,i}$ should read $-\exp(-\rho a(x_i))$. Also Theorem 1 in Appendix B is not correct as stated (e.g. take $\rho = 1$, $a \equiv 1$, $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = -1$, $\beta_1 = 3$, $\beta_2 = 1$, $\beta_3 = 5$; then (b) fails); however, wherever it is invoked the approach of writing $e^x - e^y = (x - y)e^\xi = (x - y)e^y + .5(x - y)^2 e^\eta$ (for some ξ and η between x and y) and recalling the fact that $x^r \exp(-cx)$ is bounded for $x \geqslant 0$ (for r and c fixed positive constants) can be used to obtain the desired bound. The equality used in the proof of Lemma 10.1 on page 60 did not seem to be obvious; it can be verified by multiplying through by Qe^s, comparing coefficients of $s^p Q^q$ for $p, q \geqslant 0$, and then using induction on p to establish the necessary combinatorial identity. The inequality on the top of page 107 in the brief sketch of the proof that the scheme (U3) is uniformly accurate is not right. The direction of the inequality should be reversed, and then the result is still only valid for $h \geqslant \varepsilon$ (e.g., it is clearly not correct for $\varepsilon = 1$). The (lengthy) complete proof can be found in Miller [4] (two mesh method) and Kellogg and Tsan [3] (comparison functions). Also the second term on the right side of (7.5) is bounded by a constant times the first and so can be omitted; the error estimate for (U3) is thus

$$|u(x_i) - u_i^h| \leqslant Ch^2/(h + \varepsilon) \quad \text{for each } i.$$

In the discussion below (7.6) on page 109 there is no contradiction since the result quoted also requires that the Q weights be nonnegative and evaluations of f occur only at x_{i-1}, x_i and x_{i+1} (this discussion is later correctly continued on pages 181–182).

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1. D. N. DE G. ALLEN & R. V. SOUTHWELL, "Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder," *Quart J. Mech. Appl. Math.*, v. 8, 1955, pp. 129–145.

2. A. M. IL'IN, "Differencing scheme for a differential equation with a small parameter affecting the highest derivative," *Mat. Zametki*, v. 6, 1969, pp. 237–248 = *Math. Notes*, v. 6, 1969, pp. 596–602.

3. R. B. KELLOGG & A. TSAN, "Analysis of some difference approximations for a singular perturbation problem without turning points," *Math. Comp.*, v. 32, 1978, pp. 1025–1039.

4. J. J. H. MILLER, "Sufficient conditions for the convergence, uniformly in epsilon, of a three point difference scheme for a singular perturbation problem," *Numerical Treatment of Differential Equations in Applications* (R. Ansorge and W. Tornig, Eds.), Lecture Notes in Math., vol. 679, Springer-Verlag, Berlin and New York, 1978, pp. 85–91.

25[5.00, 6.30].—R. GLOWINSKI, J. L. LIONS & R. TREMOLIERS, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981, xxx + 776 pp., 23 cm. Price $109.75, Dfl. 225.–.

This book is really a compilation of three volumes. Chapters 1–3 and Chapters 4–6 are the respective English translations of volumes I and II of the French edition which appeared in 1976. Following these chapters there are six appendices covering material on variational inequalities developed since the publication of the French edition.

Since a review of the French edition appeared in *Math. Comp.*, v. 32, 1978, pp. 313–314, we give only a brief synopsis of the first six chapters and concentrate on the additional material contained in the appendices.

Chapter 1 deals with the general theory of stationary variational inequalities, Chapter 2 with solving the finite dimensional optimization problems which result from the approximation schemes, and Chapter 3 with the specific model problem of elasto-plastic torsion of a cylindrical bar. The problem of a nondifferentiable cost functional is considered in Chapters 4 and 5, with examples such as the steady flow of a Bingham fluid in a cylindrical duct. Chapter 6 contains a discussion of some general approximation schemes for time dependent variational inequalities.

It is the goal of the appendices to treat what the authors consider to be the most important contributions to the subject since the publication of the original French edition. That substantial progress has been made is evidenced by the fact that the appendices comprise about one third of this book.

For example, one important development has been the estimation of approximation errors in connection with the use of finite element approximation schemes. This material is now heavily represented with results for the obstacle problem in Appendix 1, the elasto-plastic torsion problem in Appendix 2, and the steady flow of a Bingham fluid in Appendix 4.