

On the Approximate Calculation of Double Integrals

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Abstract. Cubature formulas are obtained which are optimal or asymptotically optimal on given sets of functions. These formulas consist of line integrals which may be evaluated by optimal or asymptotically optimal quadrature formulas. The advantage of these formulas over the optimal and asymptotically optimal cubature formulas with rectangular-lattices of knots is shown.

1. Introduction. Notations and Definitions. The problem of constructing efficient cubature formulas is a difficult problem. The results in this field are reviewed in [1]-[4]. Below we consider the problem of obtaining optimal and near to optimal cubature formulas. The approach is based on the idea of blending [5], [6].

Let $1 < p \leq \infty$, $r, s \geq 1$, M, N, P be given, $p^{-1} + q^{-1} = 1$, $D = [0, 1] \times [0, 1]$. Let us introduce the notation of derivatives

$$f^{(i,j)}(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y), \quad i, j = 0, 1, \dots,$$

of norms

$$\|f(\cdot)\|_p = \begin{cases} \left(\int_0^1 |f(x)|^p dx \right)^{1/p}, & p < \infty, \\ \sup_{0 \leq x \leq 1} |f(x)|, & p = \infty, \end{cases}$$

$$\|f(\cdot, \cdot)\|_p = \begin{cases} \left(\iint_D |f(x, y)|^p dx dy \right)^{1/p}, & p < \infty, \\ \sup_{(x,y) \in D} |f(x, y)|, & p = \infty, \end{cases}$$

and the sets of functions

$$W_p^\mu = \{f(x) : f^{(\mu)}(x) \text{ piecewise continuous on } [0, 1], \|f^{(\mu)}(\cdot)\|_p \leq 1\},$$

$$V_p^{r,s} = \{f(x, y) : f^{(i,j)}(x, y) \text{ (} i \leq r, j \leq s \text{) piecewise continuous on } D, \\ \|f^{(r,s)}(\cdot, \cdot)\|_p \leq M\},$$

$$U_p^{r,s} = \{f(x, y) : f \in V_p^{r,s}, \|f^{(r,0)}(\cdot, y)\|_p \leq N, \forall y \in [0, 1], \\ \|f^{(0,s)}(x, \cdot)\|_p \leq P, \forall x \in [0, 1]\},$$

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$$\begin{aligned} \tilde{V}_p^{r,s} &= \left\{ f(x, y) : f \in V_p^{r,s}, f^{(i,0)}(0, y) \equiv f^{(i,0)}(1, y), i = 0, \dots, r-1, \right. \\ &\quad \left. f^{(0,j)}(x, 0) \equiv f^{(0,j)}(x, 1), j = 0, \dots, s-1 \right\}, \\ \tilde{U}_p^{r,s} &= \left\{ f(x, y) : f \in U_p^{r,s} \cap \tilde{V}_p^{r,s} \right\}, \\ W_p^{r,s} &= \left\{ f(x, y) : f \in V_p^{r,s}, \left\| \int_0^1 f^{(r,0)}(\cdot, y) dy \right\|_p \leq N, \left\| \int_0^1 f^{(0,s)}(x, \cdot) dx \right\|_p \leq P \right\}. \end{aligned}$$

Let Q be a set of linear functionals defined on a set H of functions f , and let L be a given linear functional defined on the set H . Consider approximate formulas of the form

$$(1.1) \quad Lf = \sum_{k=1}^l L_k f + r(f), \quad L_k \in Q, k = 1, \dots, l,$$

where $r(f)$ is the error of the formula.

Formula (1.1) is called an optimal formula on the set H , if the functionals $L_k \in Q$ are chosen so that the quantity

$$(1.2) \quad r[H] = \sup_{f \in H} |r(f)|$$

has the minimal value.

Denote the minimal value of $r[H]$ by $r^*[H]$.

Formula (1.1) is called asymptotically optimal on the set H , if the functionals $L_k \in Q$ are chosen so that

$$\lim_{l \rightarrow \infty} \frac{r^*[H]}{r[H]} = 1.$$

Designate

$$\begin{aligned} &A_1^{(l)}, \dots, A_l^{(l)}; x_1^{(l)}, \dots, x_l^{(l)}; r_1^{(l)}, \\ &B_1^{(l)}, \dots, B_l^{(l)}; y_1^{(l)}, \dots, y_l^{(l)}; r_2^{(l)}, \end{aligned}$$

the coefficients, knots and the value (1.2) of the optimal formulas

$$(1.3) \quad \int_0^1 f(x) dx = \sum_{k=1}^l A_k f(x_k) + r(f), \quad 0 \leq x_1 < \dots < x_l \leq 1,$$

on the sets W_p^r and W_p^s , respectively.

It follows by the results of [3, Theorem 5.2.1 and 3.6.1] that $r_1^{(l)} = O(l^{-r})$, $r_2^{(l)} = O(l^{-s})$. J. Girshovich [3] has obtained an asymptotically optimal formula (1.3) on the set W_p^μ with the knots $x_k = (k-1)/(l-1)$, $k = 1, \dots, l$, and given a simple scheme for the evaluation of the coefficients of this formula.

Designate by

$$(1.4) \quad \bar{A}_1^{(l)}, \dots, \bar{A}_l^{(l)}; \bar{r}_1^{(l)}, \quad \bar{B}_1^{(l)}, \dots, \bar{B}_l^{(l)}; \bar{r}_2^{(l)},$$

the coefficients and the value (1.2) of the asymptotically optimal formulas (1.3) on the sets W_p^r and W_p^s , respectively.

It is shown in [3], [7] that the optimal formula

$$(1.5) \quad \iint_D f(x, y) \, dx \, dy = \sum_{k=1}^l \sum_{j=1}^v C_{kj} f(x_k, y_j) + R(f),$$

$$0 \leq x_1 < \dots < x_l \leq 1, 0 \leq y_1 < \dots < y_v \leq 1,$$

on the set $W_p^{r,s}$ has coefficients $C_{kj} = A_k^{(l)} B_j^{(v)}$, knots $(x_k, y_j) = (x_k^{(l)}, y_j^{(v)})$ and exact bound for the remainder

$$R[W_p^{r,s}] = \sup_{f \in W_p^{r,s}} |R(f)| = O(l^{-r} + v^{-s}).$$

The asymptotically optimal formula (1.5) on the set $W_p^{r,s}$ has coefficients $C_{kj} = \bar{A}_k^{(l)} \bar{B}_j^{(v)}$, knots $(x_k, y_j) = ((k-1)/(l-1), (j-1)/(v-1))$ and exact bound for the remainder $O(l^{-r} + v^{-s})$ [3]. The optimal and asymptotically optimal formulas (1.5) on the set $V_p^{r,s}$ do not exist since by Lemma 1.5.1 of [3, pp. 18–19], these formulas must be exact for all functions $\varphi(x)y^j, \psi(y)x^i \in V_p^{r,s}$ ($i < r, j < s$). Also, we do not know the optimal and asymptotically optimal formulas (1.5) on the set $U_p^{r,s}$.

The first optimal quadrature formulas were found by A. Sard in the case of fixed knots [8], by S. Nikolsky [9] and by I. Schoenberg [10]. The first optimal cubature formula with arbitrary knots was presented in [11].

2. Cubature Formulas Based on Optimal Quadrature Formulas. We shall find an optimal formula of the form

$$(2.1) \quad \iint_D f(x, y) \, dx \, dy = \sum_{k=1}^m \alpha_k \int_0^1 \alpha_k \int_0^1 f(x_k, y) \, dy + \sum_{j=1}^n \beta_j \int_0^1 f(x, y_j) \, dx$$

$$+ \sum_{k=1}^m \sum_{j=1}^n \gamma_{kj} f(x_k, y_j) + E(f),$$

$$0 \leq x_1 < \dots < x_m \leq 1, 0 \leq y_1 < \dots < y_n \leq 1,$$

on the set $V_p^{r,s}$. In other words, we shall find the formula (2.1) with the minimal value of

$$E[V_p^{r,s}] = \sup_{f \in V_p^{r,s}} |E(f)|.$$

THEOREM 2.1. *The coefficients and knots*

$$(2.2) \quad \alpha_k = A_k^{(m)}, \quad \beta_j = B_j^{(n)}, \quad \gamma_{kj} = -A_k^{(m)} B_j^{(n)},$$

$$x_k = x_k^{(m)}, \quad y_j = y_j^{(n)}, \quad k = 1, \dots, m; j = 1, \dots, n,$$

and the exact bound of the remainder

$$(2.3) \quad E[V_p^{r,s}] = M r_1^{(m)} r_2^{(n)}$$

are the coefficients, knots and exact bound of the remainder of the optimal formula (2.1) on the set $V_p^{r,s}$.

Proof. The optimal formula (2.1) must satisfy the condition

$$(2.4) \quad E(\varphi(x)y^v) = E(\varphi(y)x^\lambda) = 0 \quad \forall \varphi(x)y^v, \varphi(y)x^\lambda \in V_p^{r,s},$$

$$v = 0, \dots, s-1; \lambda = 0, \dots, r-1,$$

by Lemma 1.5.1 of [3]. In view of this we only consider formulas (2.1) with the property (2.4). Taking in (2.1) $f(x, y) = \varphi(x)y^\nu$ and $f(x, y) = \varphi(y)x^\lambda$, where $\varphi(x)$ is an arbitrary function, we obtain by (2.4)

$$(2.5) \quad \sum_{j=1}^n \beta_j y_j^\nu = \frac{1}{\nu + 1}, \quad \sum_{k=1}^m \alpha_k x_k^\lambda = \frac{1}{\lambda + 1},$$

$$\frac{\alpha_k}{\nu + 1} = - \sum_{j=1}^n \gamma_{kj} y_j^\nu, \quad \frac{\beta_j}{\lambda + 1} = - \sum_{k=1}^m \gamma_{kj} x_k^\lambda, \quad k = 1, \dots, m;$$

$$j = 1, \dots, n; \lambda = 0, \dots, r - 1; \nu = 0, \dots, s - 1.$$

Now let $f(x, y) \in V_p^{r,s}$,

$$U_+^k = \begin{cases} U^k, & U > 0, \\ 0, & U \leq 0. \end{cases}$$

By Taylor's formula

$$f(x, y) = \sum_{i=0}^{r-1} \psi_i(y)x^i + \sum_{j=0}^{s-1} \varphi_j(x)y^j$$

$$+ \frac{1}{(r-1)!(s-1)!} \iint_D f^{(r,s)}(t, u)(x-t)_+^{r-1}(y-u)_+^{s-1} dt du,$$

where

$$\psi_i(y) = \frac{1}{i!} f^{(i,0)}(0, y),$$

$$\varphi_j(x) = \frac{1}{j!(r-1)!} \int_0^1 f^{(r,j)}(t, 0)(x-t)_+^{r-1} dt.$$

Hence by (2.4)

$$(2.6) \quad E(f) = \frac{1}{(r-1)!(s-1)!} \iint_D f^{(r,s)}(t, u) E((x-t)_+^{r-1}(y-u)_+^{s-1}) dt du$$

$$= \iint_D f^{(r,s)}(t, u) K(t, u) dt du,$$

where $K(t, u)$ is monospline,

$$K(t, u) = \frac{1}{(r-1)!(s-1)!} \left[\frac{(1-t)^r(1-u)^s}{rs} - \frac{(1-u)^s}{s} \sum_{k=1}^m \alpha_k (x_k - t)_+^{r-1} \right.$$

$$\left. - \frac{(1-t)^r}{r} \sum_{j=1}^n \beta_j (y_j - u)_+^{s-1} - \sum_{k=1}^m \sum_{j=1}^n \gamma_{kj} (x_k - t)_+^{r-1} (y_j - u)_+^{s-1} \right].$$

The monospline $K(t, u)$ satisfies the condition

$$(2.7) \quad K^{(\lambda,0)}(0, y) \equiv K^{(0,\nu)}(x, 0) \equiv 0, \quad \lambda = 0, \dots, r - 1; \nu = 0, \dots, s - 1,$$

by the equalities (2.5).

By Hölder’s inequality we obtain from (2.6) that

$$(2.8) \quad |E(f)| \leq M \|K(\cdot, \cdot)\|_q.$$

For the function

$$f_0(x, y) = \frac{M}{\|K(\cdot, \cdot)\|_q^{q/p}} \iint_D |K(t, u)|^{q-1} \operatorname{sgn} K(t, u) \frac{(x-t)_+^{r-1} (y-u)_+^{s-1}}{(r-1)!(s-1)!} dt du$$

belonging to $V_p^{r,s}$, it follows from (2.6) that

$$E(f_0) = M \|K(\cdot, \cdot)\|_q.$$

Then we have from (2.8) the equality

$$(2.9) \quad E[V_p^{r,s}] = M \|K(\cdot, \cdot)\|_q.$$

Now we have to minimize this value.

Consider only functions $K(x, y)$ satisfying condition (2.7) and consider the monosplines

$$(2.10) \quad K_1(t) = \frac{(1-t)^r}{r!} - \frac{1}{(r-1)!} \sum_{k=1}^m \alpha_k (x_k - t)_+^{r-1},$$

$$K_1^{(i)}(0) = 0, \quad i = 0, \dots, r-1,$$

$$(2.11) \quad K_2(u) = \frac{(1-u)^s}{s!} - \frac{1}{(s-1)!} \sum_{j=1}^n \beta_j (y_j - u)_+^{s-1},$$

$$K_2^{(l)}(0) = 0, \quad l = 0, \dots, s-1.$$

Let

$$\inf_{\{\alpha_k, \beta_j, \gamma_{kj}, x_k, y_j\}} \|K(\cdot, \cdot)\|_q = \|K^*(\cdot, \cdot)\|_q,$$

$$\inf_{\{\alpha_k, x_k\}} \|K_1(\cdot)\|_q = \|K_1^*(\cdot)\|_q, \quad \inf_{\{\beta_j, y_j\}} \|K_2(\cdot)\|_q = \|K_2^*(\cdot)\|_q.$$

It is shown in [3] that

$$K_1^*(t) = \frac{(1-t)^r}{r!} - \frac{1}{(r-1)!} \sum_{k=1}^m A_k^{(m)} (x_k^{(m)} - t)_+^{r-1},$$

$$K_2^*(u) = \frac{(1-u)^s}{s!} - \frac{1}{(s-1)!} \sum_{j=1}^n B_j^{(n)} (y_j^{(n)} - u)_+^{s-1},$$

$$(2.12) \quad \|K_1^*(\cdot)\|_q = r_1^{(m)}, \quad \|K_2^*(\cdot)\|_q = r_2^{(n)}.$$

Also we have the equality

$$(2.13) \quad K^*(t, u) = K_1^*(t) K_2^*(u)$$

by the result on polynomials of least derivation from zero [7]. This, together with (2.12), give us the values (2.2) and (2.3). The formula (2.1) with coefficients and knots (2.2) satisfies the condition (2.4).

The theorem is proved.

COROLLARY. Consider the formula

$$(2.14) \quad \iint_D f(x, y) dx dy = \sum_{k=1}^m \sum_{j=1}^{n^2} A_k^{(m)} B_j^{(n^2)} f(x_k^{(m)}, y_j^{(n^2)}) + \sum_{k=1}^{m^2} \sum_{j=1}^n A_k^{(m^2)} B_j^{(n)} f(x_k^{(m^2)}, y_j^{(n)}) - \sum_{k=1}^m \sum_{j=1}^n A_k^{(m)} B_j^{(n)} f(x_k^{(m)}, y_j^{(n)}) + E_1(f).$$

Let $f(x, y) \in U_p^{r,s}$. Then

$$(2.15) \quad |E_1(f)| \leq N r_1^{(m^2)} + P r_2^{(n^2)} + M r_1^{(m)} r_2^{(n)}.$$

Proof. We can consider the formula (2.1) with coefficients and knots (2.2) on the set $U_p^{r,s}$ as $U_p^{r,s} \subset V_p^{r,s}$.

Applying optimal quadrature formulas [3]

$$\int_0^1 f(x, y_j^{(n)}) dx = \sum_{k=1}^{m^2} A_k^{(m^2)} f(x_k^{(m^2)}, y_j^{(n)}) + r_1(f),$$

$$|r_1(f)| \leq N r_1^{(m^2)}, N^{-1} f(x, y) \in W_p^r \quad \forall y \in [0, 1],$$

$$\int_0^1 f(x_k^{(m)}, y) dy = \sum_{j=1}^{n^2} B_j^{(n^2)} f(x_k^{(m)}, y_j^{(n^2)}) + r_2(f),$$

$$|r_2(f)| \leq P r_2^{(n^2)}, P^{-1} f(x, y) \in W_p^s \quad \forall x \in [0, 1],$$

to the integrals in formula (2.1) with coefficients and knots (2.2), we obtain formula (2.14) and the estimate

$$(2.16) \quad |E_1(f)| \leq P r_2^{(n^2)} \sum_{k=1}^m |A_k^{(m)}| + N r_1^{(m^2)} \sum_{j=1}^n |B_j^{(n)}| + M r_1^{(m)} r_2^{(n)}.$$

Since by [12] $A_k^{(m)} > 0, B_j^{(n)} > 0, k = 1, \dots, m; j = 1, \dots, n; A_1^{(m)} + \dots + A_m^{(m)} = B_1^{(n)} + \dots + B_n^{(n)} = 1$, (2.15) follows.

The corollary is proved.

THEOREM 2.2. *The formula*

$$\iint_D f(x, y) dx dy = \frac{1}{m} \sum_{k=1}^m \int_0^1 f\left(\frac{k}{m}, y\right) dy + \frac{1}{n} \sum_{j=1}^n \int_0^1 f\left(x, \frac{j}{n}\right) dx - \frac{1}{mn} \sum_{k=1}^m \sum_{j=1}^n f\left(\frac{k}{m}, \frac{j}{n}\right) + E(f)$$

is the optimal formula (2.1) on the set $\tilde{V}_p^{r,s}$. The exact bound for the remainder of this formula is

$$E[\tilde{V}_p^{r,s}] = \frac{M}{m^r n^s} B_{r,q} B_{s,q},$$

where $B_{\mu,q} = \min_c \|B_\mu(\cdot) - c\|_q$, and $B_\mu(x)$ is the Bernoulli polynomial of degree μ .

The proof is similar to the proof of Theorem 2.1 and uses the optimal quadrature formulas for periodic functions [3] and the formula

$$f(x, y) = \varphi(x) + \psi(y) + \frac{1}{r!s!} \iint_D f^{(r,s)}(t, u) B_r^*(x-t) B_s^*(y-u) dt du,$$

where

$$\varphi(x) = -\frac{1}{r!} \iint_D f^{(r,0)}(t, u) B_r^*(x-t) dt du, \quad \psi(y) = \int_0^1 f(t, y) dt,$$

$$B_\mu^*(x) \equiv B_\mu(x) \quad \text{if } x \in [0, 1), \quad B_\mu^*(x+1) \equiv B_\mu^*(x).$$

The latter can be obtained by the analogous one-dimensional formula [13].

COROLLARY. Consider the formula

$$(2.17) \quad \iint_D f(x, y) dx dy = \frac{1}{mn^2} \sum_{k=1}^m \sum_{j=1}^{n^2} f\left(\frac{k}{m}, \frac{j}{n^2}\right) + \frac{1}{m^2n} \sum_{k=1}^{m^2} \sum_{j=1}^n f\left(\frac{k}{m^2}, \frac{j}{n}\right) - \frac{1}{mn} \sum_{k=1}^m \sum_{j=1}^n f\left(\frac{k}{m}, \frac{j}{n}\right) + E_1(f).$$

Let $f(x, y) \in \tilde{U}_p^{r,s}$. Then

$$|E_1(f)| \leq \frac{N}{r!m^{2r}} B_{r,q} + \frac{P}{s!n^{2s}} B_{s,q} + \frac{M}{r!s!m^r n^s} B_{r,q} B_{s,q}.$$

The proof is similar to the proof of the corollary of Theorem 2.1.

3. Cubature Formulas Based on Asymptotically Optimal Quadrature Formulas. The optimal formula (2.1) and formula (2.14) are based on the optimal quadrature formulas (1.3) on W_p^μ , which are known only for few values of μ [3], [4]. Therefore, it is useful to obtain formulas (2.1) and formulas of type (2.14) based on asymptotically optimal quadratures on W_p^μ , which are known for all integers μ [3, p. 108].

THEOREM 3.1. The coefficients and knots

$$(3.1) \quad \alpha_k = \bar{A}_k^{(m)}, \quad \beta_j = \bar{B}_j^{(n)}, \quad \gamma_{kj} = -\bar{A}_k^{(m)} \bar{B}_j^{(n)}, \\ x_k = \frac{k-1}{m-1}, \quad y_j = \frac{j-1}{n-1}, \quad k = 1, \dots, m; j = 1, \dots, n,$$

and the exact bound for the remainder

$$(3.2) \quad E[V_p^{r,s}] = \frac{MB_{r,q} B_{s,q}}{r!s!m^r n^s} \left(1 + \frac{C_1}{m}\right)^{1/q} \left(1 + \frac{C_2}{n}\right)^{1/q}$$

are the coefficients, knots and exact bound for the remainder of the asymptotically optimal formula (2.1) on the set $V_p^{r,s}$.

The $\bar{A}_k^{(m)}, \bar{B}_j^{(n)}$ are those alluded to in Section 1, the numbers $B_{\mu,q}$ are determined in Theorem 2.1, and C_1 and C_2 are constants independent of m and n .

Proof. Consider formula (2.1) on the set $V_p^{r,s}$. By (2.9) and (2.13) we have the exact bound for the remainder for the optimal formula (2.1) on the set $V_p^{r,s}$.

$$(3.3) \quad E^*[V_p^{r,s}] = M \|K_1^*(\cdot)\|_q \|K_2^*(\cdot)\|_q.$$

Consider formula (2.1) with the corresponding monospline

$$K(t, u) = \bar{K}(t, u) = \bar{K}_1(t)\bar{K}_2(u),$$

where $\bar{K}_1(t)$ and $\bar{K}_2(u)$ are the monosplines (2.10) and (2.11) with coefficients and knots (3.1); that is, formula (2.1) with coefficients and knots (3.1). The exact bound for the remainder of this formula is

$$(3.4) \quad \bar{E}[V_p^{r,s}] = M \|\bar{K}_1(\cdot)\|_q \|\bar{K}_2(\cdot)\|_q.$$

As the monosplines $\bar{K}_1(t)$ and $\bar{K}_2(u)$ define the asymptotically optimal quadrature formulas (3.1) with $l = m$ and $l = n$ on the sets W_p^r and W_p^s , respectively, [3, pp. 22–26], we have by [3, pp. 108–109] the equalities

$$\begin{aligned} \|\bar{K}_1(\cdot)\|_q &= \frac{B_{r,q}}{r!m^r} \left(1 + \frac{C_1}{m}\right)^{1/q}, & \|\bar{K}_2(\cdot)\|_q &= \frac{B_{s,q}}{s!n^s} \left(1 + \frac{C_2}{n}\right)^{1/q}, \\ \lim_{m \rightarrow \infty} \frac{\|K_1^*(\cdot)\|_q}{\|\bar{K}_1(\cdot)\|_q} &= \lim_{n \rightarrow \infty} \frac{\|K_2^*(\cdot)\|_q}{\|\bar{K}_2(\cdot)\|_q} = 1. \end{aligned}$$

Then by (3.3) and (3.4) the equality

$$\lim_{m,n \rightarrow \infty} E^*[V_p^{r,s}] / \bar{E}[V_p^{r,s}] = 1$$

and the equality (3.2) hold.

The theorem is proved.

COROLLARY. Consider the formula

$$(3.5) \quad \begin{aligned} \iint_D f(x, y) \, dx \, dy &= \sum_{k=1}^m \sum_{j=1}^{n^2} \bar{A}_k^{(m)} \bar{B}_j^{(n^2)} f\left(\frac{k-1}{m-1}, \frac{j-1}{n^2-1}\right) \\ &+ \sum_{k=1}^{m^2} \sum_{j=1}^n \bar{A}_k^{(m^2)} \bar{B}_j^{(n)} f\left(\frac{k-1}{m^2-1}, \frac{j-1}{n-1}\right) \\ &- \sum_{k=1}^m \sum_{j=1}^n \bar{A}_k^{(m)} \bar{B}_j^{(n)} f\left(\frac{k-1}{m-1}, \frac{j-1}{n-1}\right) + E_1(f). \end{aligned}$$

Let $f(x, y) \in U_p^{r,s}$. Then

$$\begin{aligned} |E_1(f)| &\leq \frac{ANB_{r,q}}{r!m^{2r}} \left(1 + \frac{C_1}{m}\right)^{1/q} + \frac{BPB_{s,q}}{s!n^{2s}} \left(1 + \frac{C_2}{n}\right)^{1/q} \\ &+ \frac{MB_{r,q}B_{s,q}}{r!s!m^r n^s} \left(1 + \frac{C_1}{m}\right)^{1/q} \left(1 + \frac{C_2}{n}\right)^{1/q}, \end{aligned}$$

where A and B do not depend on m and n ,

$$|\bar{A}_1^{(m)}| + \dots + |\bar{A}_m^{(m)}| \leq A, \quad |\bar{B}_1^{(n)}| + \dots + |\bar{B}_n^{(n)}| \leq B.$$

The proof is the same as the proof of the Corollary following Theorem 2.1.

4. Comparison of the Formulas. Example. Consider the formula (1.5) with $l = m^2$, $\nu = n^2$ and the formulas obtained in the Corollaries of Sections 2 and 3. For simplicity we take $m = n$, $r = s$. As follows from [3] and Sections 1–3 the optimal

formula (1.5) and the corresponding formulas (2.14), (2.17) and (3.5) have the same estimate $O(m^{-2r})$. However, formulas (2.14), (2.17) and (3.5) have a remarkable advantage over the optimal formula (1.5). The latter uses m^4 values of $f(x, y)$, while formulas (2.14), (2.17) and (3.5) use only $m^2(2m + 1)$ values of function $f(x, y)$.

Example. We compare the evaluation of the integral

$$I = \iint_D \frac{1 + (1 - x - y + xy)^{2.5}}{0.5 + \sin(xy)} dx dy = 1.595958 \dots$$

by the optimal formula (1.5) with $l = \nu = m^2$ and by formula (2.14) with $m = n$. Both formulas are taken for the set $W_\infty^{2,2}$. By [3, pp. 50–51] we have

$$\begin{aligned} x_k^{(m)} &= y_k^{(m)} = (2k - 2 + 0.5\sqrt{3})h_m, & k = 1, \dots, m, \\ A_1^{(m)} &= A_m^{(m)} = B_1^{(m)} = B_m^{(m)} = (1 + 0.5\sqrt{3})h_m, \\ A_k^{(m)} &= B_k^{(m)} = 2h_m, & k = 2, \dots, m - 1, \\ h_m &= (2m + \sqrt{3} - 2)^{-1}. \end{aligned}$$

Let I_m be the approximate value of I obtained by formula (2.14) with $q_m = 2m^3 + m^2$ values of $f(x, y)$, and let I'_m be the approximate value of I obtained by the optimal formula (1.5) with $k_m = m^4$ values of $f(x, y)$. We obtain for $m = 3, 6, 9$

I_m	q_m	I'_m	k_m	m
1.59480...	63	1.59484...	81	3
1.595899...	468	1.595901...	1296	6
1.595947...	1539	1.595947...	6561	9

The superiority of formula (2.14) over the optimal formula of the form (1.5) is obvious.

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