Quadratic Pencils and Least-Squares Piecewise-Polynomial Approximation*

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Abstract. For a partition $\xi = (0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1)$ of the unit interval, S_ξ^{km}, $k > m$, denotes the space of piecewise-polynomials of order k and of smoothness $m - 1$; this space can be represented as the graph of the appropriate linear operator between two finite-dimensional Hilbert spaces. It gives an approach to the C. de Boor problem, 1972, on uniform boundedness (with respect to ξ) in the L_∞-norm of the orthogonal projections onto S_ξ^{km}, and we give the detailed analysis of a quadratic pencil (matrix-valued polynomial of the second degree) which appears in the case of a geometric mesh ξ if $2m \leqslant k$. The explicit calculations and estimates of zeros of the "characteristic" polynomial show that in the case $S_{\xi(x)}^{63}$, $\xi(x)$ the geometric mesh with the parameter x, $0 < x < \infty$, the orthogonal projectors are uniformly bounded.

0. Introduction. For any pair of integers k, m, $k > m > 0$, and any partition $\xi = (\xi_\alpha)_0^{n+1}$ of the unit interval $[0, 1]$,

$$(0.1) \qquad 0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1,$$

we define the piecewise-polynomial subspace

$$(0.2) \qquad S \equiv S_\xi^{km} = \mathbf{P}_{k,\xi} \cap C^{(m-1)}[0, 1],$$

where

$$\mathbf{P}_{k,\xi} = \left\{ f \in L[0,1] : f| \, \Delta_\alpha \text{ is a polynomial } \sum_0^{k-1} c_i^\alpha x^i \text{ of order } k, 0 \leqslant \alpha \leqslant n \right\},$$

$$\Delta_\alpha = [\xi_\alpha, \xi_{\alpha+1}].$$

Each interior breakpoint ξ_α, $1 \leqslant \alpha \leqslant n$, generates m continuity conditions

$$(0.3) \qquad (f| \, \Delta_{\alpha-1})^{(i)}(\xi_\alpha) = (f| \, \Delta_\alpha)^{(i)}(\xi_\alpha), \qquad 0 \leqslant i < m,$$

to make a function $f \in \mathbf{P}_{k\xi}$ be an element of S_ξ^{km}.

Let $Q = Q_S$ be the orthogonal projection onto S in $L^2[0, 1]$, i.e. with respect to the inner product

$$(f, g) = \int_0^1 f(x) \, \overline{g(x)} \, dx.$$

We are interested in Q as a map in $C[0, 1]$ or $L^\infty[0, 1]$ and we would like to get the estimates of its norm

$$\|Q\|_\infty = \sup_f \|Qf\|_\infty / \|f\|_\infty,$$

where $\| f \|_\infty = \operatorname{ess\,sup}_{0 \leqslant x \leqslant 1} |f(x)|$, $f \in L^\infty[0, 1]$.

The general problem (C. de Boor, 1972) on uniform boundedness of these projectors in L^∞ over any partition ξ (for fixed k and m) remains unsolved. The series of important particular cases was treated by Z. Ciesielski [1], C. de Boor [2], J. Douglas, T. Dupont and L. Wahlbin [5], S. Demko [6] and others [11].

We now consider the case $k \geqslant 2m$. It seems that with respect to the smoothness conditions this case is a degenerate one; however, for our approach (see Section 1 below) this is the only nondegenerate case. We represent the space $S = S_\xi^{km}$ as the graph of a certain linear operator l between two Hilbert (finite-dimensional) spaces so the projections can be written explicitly in terms of l and l^*. In the case of the geometric mesh, i.e.

$$(0.4) \qquad \mu_\alpha := \frac{\Delta_{\alpha+1}}{\Delta_\alpha} = \frac{\xi_{\alpha+2} - \xi_{\alpha+1}}{\xi_{\alpha+1} - \xi_\alpha} \equiv \mu \quad \text{for any } \alpha, \, 0 \leqslant \alpha < n,$$

this construction produces $(m \times m)$-block Toeplitz operators, and the analysis of the corresponding quadratic pencil (see [7], [8]) is the crucial point to get the estimates of norms of Q_S. In particular, we prove that least-squares approximants by quintic splines on a geometric mesh with triple knots (the case $\mathbf{P}_\xi^{6,3}$, ξ of (0.1)) are uniformly bounded in L^∞-, or C-, norm independently on the mesh-ratio μ in (0.4) or the number n of breakpoints in (0.1).

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1. Reduction of the Problem to an Analysis of a Finite-Dimensional Vector-Sequence Space. For the partition (0.1) let us define for any $\alpha, 0 \leqslant \alpha \leqslant n$

$$\varphi_\alpha(t) = \begin{cases} 0, & t = \xi_\alpha, \\ 1, & t = \xi_{\alpha+1} \\ \text{linear on } \Delta_\alpha, \end{cases}$$

so $\varphi_\alpha: \Delta_\alpha \to I$, and for any function $f \in L_1[0, 1]$ we put $\Phi_\xi f = (f_\alpha)_0^n$, $f_\alpha \in L_1[0, 1]$, $f_\alpha(x) = f(\varphi_\alpha^{-1}(x))$. This operator Φ_ξ maps $L^2[0, 1]$ isometrically onto $l_{n+1}^2(L^2(I), \Delta)$, the Hilbert space of sets $g = (g_\alpha)_0^n$, $g_\alpha \in L^2(I)$, with the inner product

$$(g, h) = \sum_{\alpha=0}^n \Delta_\alpha \cdot \int_0^1 g(x) \overline{h(x)} \, dx,$$

and the same operator Φ_ξ gives the isometrical isomorphism

$$L^\infty[0, 1] \overset{\sim}{\to} l_{n+1}^\infty(L^\infty[0, 1])$$

$$= \text{the Banach space of } (n + 1)\text{-triples } g = (g_\alpha)_0^n, \, g_\alpha \in L^\infty(I)$$

with the norm

$$\|g\| = \max_{0 \leqslant \alpha \leqslant n} \operatorname*{ess\,sup}_{x \in I} -|g_\alpha(x)|.$$

Remark. This cutting does not work if we consider the spaces of continuous functions, i.e. $\Phi_\xi(C(I))$ is a proper subspace of $l_{n+1}^\infty(C(I))$, of codimension n. But we do not need the special analysis in C because the norm of the projection Q_S is the same in L_∞ and C by the general argument: S is a finite-dimensional subspace of continuous functions so Q_S is an integral operator $f \to \int_0^1 (\Sigma_1^d h_i(x)h_i(y))f(y)\,dy$, where $(h_i)_1^d$ is an orthonormal basis in S and the continuity of these function h's and the Lusin theorem imply that

$$\|Q: L_\infty \to L_\infty\| = \|Q: C \to C\|.$$

What is the image of S_ξ (see (0.2) and (0.3)) under the mapping Φ_ξ? The continuity conditions (0.3) after the isomorphism Φ_ξ have the following form:

$$(1.1) \qquad \Delta_\alpha^{-i} g_\alpha^{(i)}(1) = \Delta_{\alpha+1}^{-i} g_{\alpha+1}^{(i)}(0), \qquad 0 \leqslant i < m, 0 \leqslant \alpha \leqslant n.$$

(Recall that $g_\alpha(x) = (f \mid \Delta_\alpha)(\xi_\alpha + x(\xi_{\alpha+1} - \xi_\alpha))$.) Instead of the projection $Q = Q_S$ in the original spaces $L^2(I)$ or $L^\infty(I)$ we have to analyze the orthogonal projection $\tilde{Q}: l_{n+1}^2(L^2(I), \Delta) \to \tilde{S} = \{g = (g_\alpha)_0^n: g_\alpha \text{ are polynomials of } P_k \text{ and } (1.1) \text{ holds}\}$ as an operator in $l_{n+1}^\infty(L^\infty(I))$.

These spaces are infinite-dimensional but instead of the entire space $L^2(I)$, or $L^\infty(I)$, as a coordinate one, we can consider only polynomials P_k; more precisely, for (any) fixed α

$$(1.2) \qquad L^2(I) = P_k \oplus P_k^\perp,$$

so

$$(1.3) \qquad l_{n+1}^2(L^2(I); \Delta) = l_{n+1}^2(P_k; \Delta) \oplus l_{n+1}^2(P_k^\perp; \Delta)$$

and if $p_k: L^2(I) \to P_k$ is the orthogonal projection onto the subspace of polynomials of order k and

$$\tilde{p}_k^n: l_{n+1}^2(L^2(I); \Delta) \to l_{n+1}^2(P_k; \Delta)$$

is the orthogonal projection generated by the decomposition (1.2), (1.3), then

$$(1.4) \qquad \|\tilde{p}_k^n\|_\infty = \|\tilde{p}_k^n: l_{n+1}^\infty(L^\infty(I)) \to l_{n+1}^\infty(P_k)\|$$

$$= \|p_k: L^\infty(I) \to P_k\| = \|p_k\|_\infty = C_1(k) < \infty,$$

and this constant C_1 does not depend on the partition ξ, i.e. on the number n of breakpoints and weights $\Delta = (\Delta_\alpha)_0^n$. The space $P_k = \{\Sigma_0^{k-1} c_i x^i\}$ is finite-dimensional and the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent:

$$(1.5) \qquad \|f\|_2 \leqslant \|f\|_\infty \leqslant k^{1/2}\|f\|_2 \quad \forall f \in P_k.$$

For this reason we can consider the L^2-norm in the coordinate space P_k in both the $l_{n+1}^\infty(P_k)$- and $l_{n+1}^2(P_k; \Delta)$-cases; it will change the L^∞-norms of operators under consideration by some factors which are uniformly bounded with respect to n and ξ. The piecewise-polynomial subspace $\tilde{S}_\xi$ is a subspace of $L = l_{n+1}^2(\mathbf{P}_k; \Delta)$, or, what is the same, $l_{n+1}^\infty(\mathbf{P}_k)$, so $\tilde{Q} = (\tilde{Q} \mid L) \cdot \tilde{p}_k^n$ and

$$(1.6) \qquad \|\tilde{Q}\|_\infty \leqslant \|\tilde{Q} \mid L\|_\infty \cdot \|\tilde{p}_k^n\|_\infty = c_1(k)\|q'\|_\infty$$

by (1.4), where $q' = \tilde{Q} \mid L$.

And we can consider the finite-dimensional space L with two norms $l_{n+1}^2(\mathbf{P}_k; \Delta)$ and $l_{n+1}^\infty(\mathbf{P}_k)$, and the projection q' only.

2. Representation of $\tilde{S}_\xi$ as a Graph of a Linear Operator.

For a given m, $0 \le m < k$, $2m \le k$, we split $\mathbf{P}_k$ by

$$(2.1) \qquad \mathbf{P}_k = A \oplus B, \qquad B = \text{linear span}\{x^j; \, k - m \le j < k\}$$

and $A = \{f \in \mathbf{P}_k : f \perp B\}$.

We choose the special basis $\{e_i\}_0^{k-m-1}$ in A:

$$(2.2) \qquad e_i = x^i/i! + \beta_i(x), \qquad \beta_i \in B, \, 0 \le i < k - m.$$

Then

$$(2.3) \qquad e_i^{(j)}(0) = \delta_{ij}, \qquad 0 \le i, j < k - m.$$

Put

$$(2.4) \qquad s_{ji} = e_i^{(j)}(1)$$

so

$$(2.5) \qquad S: \alpha \to c, \qquad c_j = \sum_0^{k-m-1} s_{ji} a_i, \qquad c = \sum_0^{m-1} c_j e_j$$

is a mapping $A \to A$. For any $a = \sum \tilde{a}_i e_i$ and $a' = \sum \tilde{a}'_i e_i$ in A their inner product is

$$(2.6) \qquad (a, a') = \langle T\tilde{a}, \tilde{a}' \rangle, \qquad \text{where } T = \{(e_i, e_j)\}_0^{k-m-1}$$

is the Gramian of the basis (2.2) and $\langle \cdot, \cdot \rangle$ is the canonical inner product in the coordinate l^2-space.

In the subspace B we choose the basis $(v_i)_0^{m-1}$ in such a way that

$$(2.7) \qquad v_i^{(j)}(1) = \delta_{ij}, \qquad 0 \le i, j < m,$$

and then B is isometrically isomorphic to $l_m^2[V]$, i.e. for $b, b' \in B$, $b = \sum_0^{m-1} \tilde{b}_i v_i(x)$,

$$(2.8) \qquad (b, b') = \langle V\tilde{b}, \tilde{b}' \rangle, \qquad \text{where } V = \{(v_i, v_j)\}_0^{m-1}.$$

The results of explicit computation of these polynomials and matrices are presented below, $k = 2m$, for $m = 1, 2, 3$.

(2.9.1) $\quad m = 1, e_0 = 1 - 3/2x; \quad v_0 = x, S = -1/2; \quad T = 1/4; \quad V = 1/3.$

(2.9.2) $\quad m = 2,$

$$e_0 = 1 - \frac{15}{2}x^2 + 7x^3; \qquad v_0 = 3x^2 - 2x^3;$$

$$e_1 = x - 3x^2 + \frac{21}{10}x^3; \qquad v_1 = -x^2 + x^3;$$

$$S = \begin{bmatrix} 1/2 & 1/10 \\ 6 & 13/10 \end{bmatrix}, \quad T = \begin{bmatrix} 1/4 & 1/40 \\ 1/40 & 1/300 \end{bmatrix}, \quad V = \begin{bmatrix} 13/35 & -11/210 \\ -11/210 & 1/105 \end{bmatrix}.$$

(2.9.3) $\quad m = 3,$

$$e_0 = 1 - 42x^3 + 90x^4 - \frac{99}{2}x^5;$$

$$e_1 = x - \frac{72}{5}x^3 + \frac{192}{7}x^4 - \frac{99}{7}x^5;$$

$$e_2 = \frac{1}{2}x^2 - \frac{9}{4}x^3 + \frac{45}{14}x^4 - \frac{165}{112}x^5;$$

$$S = \begin{bmatrix} -1/2 & -4/35 & -1/112 \\ -27/2 & -16/5 & -29/112 \\ -162 & -1404/35 & -95/28 \end{bmatrix}, \quad T = \begin{bmatrix} 1/4 & 1/35 & 1/672 \\ 1/35 & 16/3675 & 1/3920 \\ 1/672 & 1/3920 & 1/62720 \end{bmatrix},$$

$$v_0 = 10x^3 - 15x^4 + 6x^5,$$

$$v_1 = -4x^3 + 7x^4 - 3x^5,$$

$$v_2 = \frac{1}{2}x^3 - x^4 + \frac{1}{2}x^5,$$

$$V = \begin{bmatrix} \dfrac{181}{462} & -\dfrac{311}{4620} & \dfrac{281}{55440} \\[2mm] -\dfrac{311}{4620} & \dfrac{52}{3465} & -\dfrac{23}{18480} \\[2mm] \dfrac{281}{55440} & -\dfrac{23}{18480} & \dfrac{1}{9240} \end{bmatrix}.$$

Any element (or function) $f = (f_\alpha)_0^n \in L = l_{n+1}^2(\mathbf{P}_k; \Delta)$ can be decomposed as $f = \{(a_\alpha; b_\alpha)\}_0^n$, where $a_\alpha = \mathrm{Pr}_A f_\alpha$, $b_\alpha = \mathrm{Pr}_B f_\alpha$, $0 \le \alpha \le n$, and Pr_A, Pr_B are the orthogonal projectors to the first and second components in (2.1), correspondingly. Therefore,

$$(2.10) \quad L = \mathfrak{A} \oplus \mathfrak{B}', \quad \mathfrak{A} = \bigoplus_{\alpha=0}^n A_\alpha; \quad \mathfrak{B}' = \bigoplus_{\alpha=0}^n B_\alpha; \quad A_\alpha \equiv A, \quad B_\alpha \equiv B.$$

If $f \in \tilde{S}_\xi$, then by (1.1)

$$\left(a_\alpha(x) + b_\alpha(x)\right)^{(i)}\big|_{x=1} = \left(\frac{\Delta_\alpha}{\Delta_{\alpha+1}}\right)^i \cdot \left(a_{\alpha+1}(x) + b_{\alpha+1}(x)\right)^{(i)}\big|_{x=0},$$

and by the definition of B in (2.1)

$$(2.11) \quad a_\alpha^{(i)}(1) + b_\alpha^{(i)}(1) = \left(\frac{\Delta_\alpha}{\Delta_{\alpha+1}}\right)^i a_{\alpha+1}^{(i)}(0), \qquad 0 \le i < m, 0 \le \alpha < n.$$

These conditions, in terms of canonical bases (2.2) and (2.7) in A_α and B_α, the same for any α, $0 \le \alpha \le n$, mean that

$$(2.12) \qquad S\tilde{a}_\alpha + \tilde{b}_\alpha = N_\alpha \tilde{a}_{\alpha+1}, \qquad 0 \le \alpha < n,$$

where S is the matrix (2.4) (or the mapping (2.5)) and N_α is the diagonal matrix $(\nu^i \delta_{ij})_0^{m-1}$, $\nu = \nu_\alpha = \mu_\alpha^{-1} = \Delta_\alpha/\Delta_{\alpha+1}$, $0 \le \alpha < n$. We can rewrite (2.12) as

$$(2.13) \qquad \tilde{b}_\alpha = -S\tilde{a}_\alpha + N_\alpha \tilde{a}_{\alpha+1}, \qquad 0 \le \alpha < n.$$

The coordinate $b_n = \mathrm{Pr}_B f_n$ in $\tilde{S}_\xi$ does not have any contact restriction and can be chosen in an arbitrary way. More precisely, we can forget this component of $f \in \tilde{S}_\xi$ because the projector on B_n is $(\mathrm{Pr}_B)_n \cdot \tilde{P}_k^n$ and it is of local character, so its norm in L_∞ is not bigger than $C_2(k, m) = \|\mathrm{Pr}_B\|_\infty \cdot C_1(k)$ and this estimate does not depend on ξ. So instead of the entire $\tilde{S}_\xi$ in L (see (2.10)) we are interested in Σ_ξ as a subspace of $L' = \mathfrak{A} \oplus \mathfrak{B}$, $\mathfrak{A} = \oplus_{\alpha=0}^n A_\alpha$, $\mathfrak{B} = \oplus_{\alpha=0}^{n-1} B_\alpha$. The contact conditions (2.13) can be interpreted as the fact that Σ_ξ coincides with the graph of the operator

$$(2.14) \quad l: \mathfrak{A} \to \mathfrak{B}, \quad l: (a_\alpha)_0^n \to (b_\alpha)_0^{n-1}, \quad \tilde{b}_\alpha = -S\tilde{a}_\alpha + N_\alpha \tilde{a}_{\alpha+1}, \qquad 0 \le \alpha < n.$$

And for the estimate of the norm $\|q'\|_\infty$, see (1.6), we have to analyze the orthogonal projector

$$q: L' = l^2_{n+1}(A) \oplus l^2_n(B) \to \Sigma_\xi = \text{graph } l = \{(a, la): a \in \mathfrak{A}\}.$$

Recall that the inner products

$$(2.15) \qquad (a, a')_{\mathfrak{A}} = \sum_0^n \Delta_\alpha \cdot \int_0^1 a_\alpha(x) \cdot \overline{a'_\alpha(x)}\, dx = \sum_0^n \Delta_\alpha \cdot \langle T\tilde{a}_\alpha, \tilde{a}'_\alpha \rangle$$

and

$$(2.16) \qquad (b, b')_{\mathfrak{B}} = \sum_0^{n-1} \Delta_\gamma \cdot \int_0^1 b_\gamma(x) \cdot \overline{b'_\gamma(x)}\, dx = \sum_0^{n-1} \Delta_\gamma \cdot \langle V\tilde{b}_\gamma, \tilde{b}'_\gamma \rangle$$

depend heavily on ξ, or Δ, so the notions of "orthogonality" and "adjoint operator" do depend on ξ.

3. Explicit Form of the Projector q. The projection $q: \mathfrak{A} \oplus \mathfrak{B} \to \Sigma_\xi$ can be given explicitly in terms of the operator l. First of all let us recall that

$$G^\perp = (\text{graph } l)^\perp = \text{graph}(-l^*),$$

i.e. $(\{u, v\}; \{x, y\}) = (u, x)_{\mathfrak{A}} + (v, y)_{\mathfrak{B}} = 0$ for any $\{u, v\} \in G$ iff $x = -l^*y$, where $l^*: \mathfrak{B} \to \mathfrak{A}$ and "$*$" is determined by (2.15) and (2.16).

Therefore, if

$$\{a; b\} = \{x; lx\} + \{-l^*y, y\}$$

is the (only) orthogonal decomposition of the element $f = \{a, b\} \in L'$, with respect to the splitting $L' = G \oplus G^\perp$; $G \equiv \Sigma_\xi$, then

$$x - l^*y = a; \quad lx + y = b$$

and

$$y = (1 + l^*l)^{-1}a + (1 + l^*l)^{-1}l^*b,$$
$$y = -(1 + ll^*)^{-1}la + (1 + ll^*)^{-1}b,$$

or

$$x = \left(1 - l^*(1 + ll^*)^{-1}l\right)a + l^*(1 + ll^*)^{-1}b,$$
$$lx = +(1 + ll^*)^{-1}la + \left(1 - (1 + ll^*)^{-1}\right)b.$$

Hence, the projection $q: \{a, b\} \to \{x, lx\}$ has the block representation

$$(3.1a) \qquad \begin{bmatrix} (1 + l^*l)^{-1} & (1 + l^*l)^{-1}l^* \\ l(1 + l^*l)^{-1} & l(1 + l^*l)^{-1}l^* \end{bmatrix}$$

$$(3.1b) \qquad = \begin{bmatrix} 1 - l^*(1 + ll^*)^{-1}l & l^*(1 + ll^*)^{-1} \\ (1 + ll^*)^{-1}l & 1 - (1 + ll^*)^{-1} \end{bmatrix}$$

with respect to the decomposition $\mathfrak{A} \oplus \mathfrak{B}$. The last one $(1 = \text{Pr}_{\mathfrak{A}} \oplus \text{Pr}_{\mathfrak{B}})$ is orthogonal for each coordinate α (see (2.10) and (2.1)) so $\|\text{Pr}_{\mathfrak{A}}\|_\infty$, $\|\text{Pr}_{\mathfrak{B}}\|_\infty \leqslant c_3(k, m)$, and

this constant c_3 does not depend on ξ. In particular, it means that

(3.2a) $$\|q\|_\infty \overset{C_4(k,m)}{\sim}\ \text{max of } l_\infty\text{-norms of four operators which are}$$

entries of the block-matrix (3.1a),

or

(3.2b) $$\|q\|_\infty \overset{C_4}{\sim}\ \text{max of } l_\infty\text{-norms of four operators which are entries}$$

of the block-matrix (3.1b).

($a \overset{c}{\sim} b$ means as usual that $1/c \cdot a \leqslant b \leqslant c \cdot a$.) The second representation is more preferable because $\dim \mathfrak{B} < \dim \mathfrak{A}$ and the operator ll^*: $\mathfrak{B} \to \mathfrak{B}$ is invertible itself (l^*l is not invertible).

(3.3) $$l \approx \begin{bmatrix} -S & N_0 & & & & & \bigcirc \\ 0 & -S & N_i & & & & \\ & 0 & -S & N_2 & & & \\ & & 0 & -S & N_3 & & \\ \cdots \bigcirc \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & 0 & -S & N_{n-1} \end{bmatrix}$$

in particular, if $n = 3$

$$l \approx \begin{bmatrix} -S & N_0 & 0 & 0 \\ 0 & -S & N_1 & 0 \\ 0 & 0 & -S & N_2 \end{bmatrix}.$$

The entry l_γ^α in the γth row and the αth column acting from A_α to B_γ, $\alpha = \gamma$ or $\gamma + 1$.

To avoid misunderstanding recall that by the definition (2.5) S acts in A; however, because of the special choice (2.7) of the basis $(v_i)_0^{m-1}$ in B we can redefine this operator as acting from A to B analogously to (2.5)

(3.4) $$S: A \to B, \quad S: a \to b, \quad \tilde{b}_j = \sum_0^{k-m-1} s_{ji}a_i, \quad b = \sum_0^{m-1} \tilde{b}_j v_j.$$

The operator (2.5) gives the extension of a to the right side with the contact conditions, and by (2.7) and (2.3) the operator (3.4) gives such an element $b \in B$ that it has the same restriction at the point "1" as a, till the order $< m$. We use the same notation S for both these operators; their matrices $m \times (k - m)$ are the same under our choice of the canonical bases (2.2) and (2.7) in A and B.

By a standard argument $l^* = (m_\alpha^\gamma)$, $m_\alpha^\gamma = (l_\gamma^\alpha)^*$; however, by (2.15), (2.16) we have nonstandard inner-products in these components A_α and B_γ:

$$(a, a')_\alpha = (a, a')_{A_\alpha} = \Delta_\alpha \cdot \langle T\tilde{a}, \tilde{a}' \rangle,$$

where $\tilde{a}(i) = a^{(i)}(0)$, $0 \leqslant i < k$, and

$$(b, b')_\gamma = (b, b')_{B_\gamma} = \Delta_\gamma \cdot \langle V\tilde{b}, \tilde{b}' \rangle,$$

where $\tilde{b}(j) = b^{(j)}(1)$, $0 \leqslant j < m$.

Therefore, if $d: A_\alpha \to B_\gamma$, $a \in A_\alpha$, $b \in B_\gamma$, then

$$(da, b)_\gamma = \Delta_\gamma \langle V d\tilde{a}, \tilde{b} \rangle = \Delta_\gamma \cdot \langle \tilde{a}, d'V\tilde{b} \rangle$$

or

$$= (a, d*b)_\alpha = \Delta_\alpha \langle T\tilde{a}, d*\tilde{b} \rangle = \Delta_\alpha \langle \tilde{a}, Td*\tilde{b} \rangle,$$

and

$$\Delta_\gamma \cdot d'V\tilde{b} = \Delta_\alpha \cdot Td*\tilde{b}, \quad \forall \tilde{b}, \forall \tilde{b} \in B_\gamma$$

so

$$d* = \frac{\Delta_\gamma}{\Delta_\alpha} T^{-1} d'V,$$

where $d' = (\bar{d}_{ji})$ if $d = (d_{ij})$.

In the case of the operator l we are interested in pairs (α, α) and $(\alpha, \alpha + 1)$ only; in these cases

$$(d_\alpha^\alpha)* = T^{-1} d'V \quad \text{and} \quad (d_\alpha^{\alpha+1})* = \nu_\alpha T^{-1} (d_\alpha^{\alpha+1})' V; \quad \nu_\alpha = \frac{\Delta_\alpha}{\Delta_{\alpha+1}}.$$

These remarks explain that the block-matrix representation of the operator $l*$: $\mathfrak{B} = \oplus_0^{n-1} B_\gamma \to \mathfrak{A} = \oplus_0^n A_\alpha$ has the form

$$(3.5) \qquad l* \sim \begin{bmatrix} -T^{-1}S'V & 0 & & \bigcirc \\ \nu_0 T^{-1} N_0 V & -T^{-1}S'V & 0 & \\ & \nu_1 T^{-1} N_1 V & -T^{-1}S'V & 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots} \\ & & & -T^{-1}S'V \\ \bigcirc & & & \nu_n T^{-1}S'V \end{bmatrix};$$

in particular, if $n = 3$,

$$l* \sim \begin{bmatrix} -T^{-1}S'V & 0 & 0 \\ \nu_0 T^{-1} N_0 V & -T^{-1}S'V & 0 \\ 0 & \nu_1 T^{-1} N_1 V & -T^{-1}S'V \\ 0 & 0 & \nu_2 T^{-1} N_2 V \end{bmatrix}.$$

By the statement (3.2a,b) we need to give the estimates of l_∞-norms of the operator $H_1 = (1 + l*l)^{-1}$ and its three colleagues in (3.1a) or l_∞-norms of the operator $H = (1 + ll*)^{-1}$ and the other three entries in (3.1b). By (3.3) and (3.5) the block-matrix of the operator $C = 1 + ll*: \mathfrak{B} \to \mathfrak{B}$ has the form ($n = 3$)

$$(3.6) \qquad 1 + ll* \approx \begin{bmatrix} \begin{matrix} 1 + ST^{-1}S'V \\ + \nu_0 N_0 T^{-1} N_0 V \end{matrix} & -N_0 T^{-1} S'V & \bigcirc \\ \begin{matrix} \\ -\nu_0 ST^{-1} N_0 V \end{matrix} & \begin{matrix} 1 + ST^{-1}S'V \\ + \nu_1 N_1 T^{-1} N_1 V \end{matrix} & -N_1 T^{-1}S'V \\ \bigcirc & -\nu_1 ST^{-1} N_1 V & \begin{matrix} 1 + ST^{-1}S'V \\ + \nu_2 N_2 T^{-1} N_2 V \end{matrix} \end{bmatrix}$$

and for any n

$$
(3.7) \quad
\begin{cases}
C = \left(C_\gamma^\alpha \right)_0^{n-1}, \quad C_\gamma^\alpha = 0, \quad |\alpha - \gamma| > 1, \\
C_\alpha^\alpha = 1 + ST^{-1}S'V + \nu_\alpha N_\alpha T^{-1} N_\alpha V, \quad 0 \leqslant \alpha < n, \\
C_\alpha^{\alpha+1} = -N_\alpha T^{-1}S'V, \quad C_\alpha^{\alpha-1} = -\nu_{\alpha-1}ST^{-1}N_{\alpha-1}V.
\end{cases}
$$

Of course, the operator $H = C^{-1} = (1 + ll^*)^{-1}$ is of norm $\leqslant 1$ for any mesh ξ as an operator in the Hilbert space $\mathfrak{B} = l_2^n(B_\alpha, \{\Delta_\alpha\})$; however, we need to estimate the norm $\|H\|_\infty = \|H: l_\infty^n(B_\alpha) \to l_\infty^n(B_\alpha)\|$. Recall that this norm $\|H\|_\infty = \max_\alpha \Sigma_{\gamma=0}^{n-1} |H_\alpha^\gamma|$, where $|h|$, $h: B_\gamma \to B_\alpha$, is a norm of an operator (matrix) in m-dimensional Hilbert space. It could be useful to consider H^t as an operator in $l_1^n(B_\alpha)$; then $(H^t)_\gamma^\alpha = (H_\alpha^\gamma)^t$ and $\|H^t\|_1 = \|H\|_\infty$.

4. Blocks of q and Their Invertibility. Now we would like to use all the advantages of the representation (3.6)–(3.7) to get the estimates of $\|H\|_\infty$, $\|Hl\|_\infty$, $\|l^*H\|_\infty$ and $\|l^*Hl\|_\infty$. First of all, instead of H itself we will analyze $\tilde{V} \cdot H$, where $\tilde{D}: (b_\alpha)_0^{n-1} \to (Db_\alpha)_0^{n-1}$ denotes the block-diagonal operator for any $D: B \to B$. So we are interested in the block-matrix $\check{C} = C \cdot \widetilde{V^{-1}}$ ($n = 3$)

$$
(4.1) \quad
\check{C} =
\begin{bmatrix}
V^{-1} + ST^{-1}S' & -N_0 T^{-1}St & 0 \\
+\nu_0 N_0 T^{-1}N_0 & & \\
-\nu_0 ST^{-1}N_0 & V^{-1} + ST^{-1}S' & -N_1 T^{-1}S' \\
& +\nu_1 N_1 T^{-1}N_1 & \\
0 & -\nu_1 ST^{-1}N_1 & V^{-1} + ST^{-1}S' \\
& & +\nu_2 N_2 T^{-1}N_2
\end{bmatrix}.
$$

The general case (3.7) needs the appropriate adjustment. So

$$
(4.2) \quad H = C^{-1} = (\check{C}\tilde{V})^{-1} = \tilde{V}^{-1} \cdot \check{C}^{-1},
$$

$$
(4.3) \quad l^*H = \left(\tilde{T}^{-1}(\nu\tilde{N} - \tilde{S}')\tilde{V} \right)(\tilde{V}^{-1} \cdot \check{C}^{-1}) = \tilde{T}^{-1} \cdot (\nu\tilde{N} - \tilde{S}')\check{C}^{-1},
$$

where

$$
(4.4) \quad
\nu\tilde{N} =
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
\nu_0 N_0 & & & \bigcirc \\
& \nu_1 N_1 & & \\
& & \ddots & \\
\bigcirc & & & \nu_{n-1}N_{n-1}
\end{bmatrix}
$$

and if the matrix does not depend on ξ the symbol $\tilde{P}$ means the corresponding diagonal matrix (n by n, or $n \times (n+1)$, or $(n+1) \times n$) with entries P on the diagonal.

$$
(4.5) \quad Hl = \tilde{V}^{-1} \cdot \check{C}^{-1}(-\tilde{S} + \tilde{N}), \quad \text{where (compare (3.3))}
$$

$$
(4.6) \quad
\tilde{N} =
\begin{bmatrix}
0 & N_0 & & \\
& 0 & N_1 & \bigcirc \\
& \ddots & \ddots & \ddots \\
\bigcirc & & 0 & N_{n-1}
\end{bmatrix}
$$

and l^*Hl can be decomposed by the same way:

$$(4.7) \qquad l^* = \tilde{T}^{-1}(\nu\tilde{N} - \tilde{S}^t)\check{C}^{-1}(-\tilde{S} + \tilde{N}).$$

Many terms of these operators (4.2)–(4.7) do not depend on ξ; these factors or terms have the same norm as their generators T^{-1}, V^{-1}, or S and S^t in the block-space A or B. (T^{-1} and V^{-1} do exist as inverses of Gramians.) It means that their estimates do not depend on the mesh ξ but on k and m only.

Therefore, if we know a good estimate of the norm $\rho = \|\check{C}^{-1}\|_\infty$ then we can (have to) analyze instead of the three operators

$$l^*(1 + ll^*)^{-1}, \quad (1 + ll^*)^{-1}l, \quad l^*(1 + ll^*)^{-1}l$$

only their "parts" (see (4.3), (4.5), (4.7))

$$(4.8) \qquad \nu\tilde{N}\cdot\check{C}^{-1}, \quad \check{C}^{-1}\cdot\tilde{N}, \quad \nu\tilde{N}\cdot C^{-1}\cdot\tilde{N}.$$

Now we can formulate the main problem. Let T, V and S be matrices (2.6), (2.8) and (2.4), correspondingly. For any mesh (0.1) they produce the block-matrix $\check{C}$ by (4.1) and (3.7) and we consider it as an operator in $l_n^\infty(\mathbf{C}^n) \to l_n^\infty(\mathbf{C}^m)$.

Problem. Find good estimates for the norm $\rho = \|\check{C}^{-1}\|_\infty$ and norms of its relatives (4.8). Does there exist such a constant $C(k, m)$ that these norms do not exceed it for any choice $\nu = (\nu_0, \nu_1, \ldots, \nu_{n-1})$?

In this question ν_i could be "zero" too. It is not a more general question than the initial one. Indeed if these operators are uniformly bounded for any ν, $0 < \nu_i < \infty$, $0 \leqslant i < n$, then by the compactness argument, for any fixed n, the operator $\check{C}$ has to be invertible if some $\nu_i = 0$ and $\|\check{C}^{-1}\|_\infty$ has the same bound.

For further explicit calculation we need to know the matrices $V^{-1} + ST^{-1}S^t$ and ST^{-1}. We give now the results of this computation in the cases $k = 2, 4, 6$; $m = k/2$. Recall (2.9.1)–(2.9.3) in Section 2.

(4.9.1) $k = 2$; $V^{-1} = 3$; $T^{-1} = 4$, $V^{-1} + ST^{-1}S^t = 4$; $ST^{-1} = -2$.

(4.9.2) $k = 4$; $\det V = 1/1260$; $\det T = 1/4800$,

$$V^{-1} = \begin{bmatrix} 12 & 66 \\ 66 & 468 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 16 & -120 \\ -120 & 1200 \end{bmatrix},$$

$$ST^{-1} = \begin{bmatrix} -4 & 60 \\ -60 & 840 \end{bmatrix}, \qquad V^{-1} + ST^{-1}S^t = \begin{bmatrix} 16 & 120 \\ 120 & 1200 \end{bmatrix}.$$

(4.9.3) $k = 6$; $\det V = 474320 \cdot (55440)^{-3}$; $\det T = (62720 \cdot 27 \cdot 4900)^{-1}$,

$$V^{-1} = \begin{bmatrix} 27 & 351 & 2722 \\ 351 & 6003 & 52596 \\ 2772 & 52596 & 484272 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} 36 & -630 & 6720 \\ -630 & 14700 & -176400 \\ 6720 & -176400 & 2257920 \end{bmatrix},$$

$$-ST^{-1} = \begin{bmatrix} 6 & -210 & 3360 \\ 210 & -7140 & -110880 \\ 3360 & -110880 & 1673280 \end{bmatrix},$$

$$ST^{-1}S^t = \begin{bmatrix} 9 & 279 & 3948 \\ 279 & 8697 & 123804 \\ 3948 & 123804 & 1773648 \end{bmatrix},$$

$$V^{-1} + ST^{-1}S^t = \begin{bmatrix} 36 & 630 & 6720 \\ 630 & 14700 & 176400 \\ 6720 & 176400 & 2257920 \end{bmatrix}.$$

Remark. Put $J: \mathbf{R}^m \to \mathbf{R}^m$, $J: (x_i)_0^{m-1} \to ((-1)^i x_i)_0^{m-1}$. Our numerical data show that

(4.10.a) $$V^{-1} + ST^{-1}S^t = JT^{-1}J$$

and

(4.10.b) $$T + S^t VS = JVJ,$$

if $k = 2m = 2, 4$, or 6. I conjecture that (4.10a, b) hold for any $m > 3$, $k = 2m$, also (see (5.7) below).

5. The Case of the Geometric Mesh. If the partition $\xi = (\xi_0 = 0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1)$ is a geometric mesh, i.e. $\mu_\alpha = (\xi_{\alpha+2} - \xi_{\alpha+1})/(\xi_{\alpha+1} - \xi_\alpha) \equiv \mu$ for any α, $0 \leqslant \alpha < n$, then $\nu_\alpha = \nu = 1/\mu$ in (4.1) and all the operators

$$N_\alpha \equiv N = \left\{ \nu^i \delta_{ij} \right\}_0^{m-1} = \begin{pmatrix} 1 & & & & \\ & \nu & & & \\ & & \nu^2 & & \\ & & & \ddots & \\ & & & & \nu^{m-1} \end{pmatrix}$$

are the same. The matrix $\check{C}$ becomes a Toeplitz block-matrix with three diagonals, and its common row is

$$0 \cdots 0 - \nu ST^{-1}N \quad V^{-1} + ST^{-1}S^t + \nu NT^{-1}N \quad -NT^{-1}S^t 0 \cdots 0$$

with $(m \times m)$-matrix entries.

The uniform boundedness (for a fixed ν with respect to n) of the inverses $\check{C}^{-1}$ depends heavily on the structure of singularities of the rational (or quadratic) pencil

(5.1) $$W(z) = -\nu Fz + D - F^t \frac{1}{z},$$

where

$$F = ST^{-1}N, \quad D = V^{-1} + ST^{-1}S^t + \nu NT^{-1}N.$$

For a detailed analysis see I. Gohberg and I. Feldman [8, Chapter 8]. Now we will use some general results in this connection.

LEMMA 5.1. *Let*

(5.1.0) $$V(z) = D_1 z + D_0 + D_{-1} \cdot \frac{1}{z}$$

be a pencil, where D_j, $|j| \leqslant 1$, are $(m \times m)$-matrices. The following conditions are equivalent:

(a) *there exist both left and right factorizations of special type*

(5.1.a.*l*) $$V(z) = (N_1 z + 1)N_0 \left(1 + N_{-1} \frac{1}{z} \right),$$

where spectr $N_{\pm 1} \subset D = \{ |z| < 1 \}$ *and N_0 invertible, and*

(5.1.a.*r*) $$V(z) = \left(M_{-1} \frac{1}{z} + 1 \right) M_0 (M_1 z + 1),$$

where spectr $M_{\pm 1} \subset D$ *and M_0 nonsingular;*

(b) *for n sufficiently large the system*

$$D_0 x_0 + D_{-1} x_1 = y_0,$$
$$D_1 x_0 + D_0 x_1 + D_{-1} x_2 = y_1,$$

(5.1.b) $\cdots\cdots\cdots\cdots$

$$D_1 x_{n-2} + D_0 x_{n-1} + D_{-1} x_n = y_{n-1},$$
$$D_1 x_{n-1} + D_0 x_n = y_n,$$

has a solution $x = (x_i)_0^n$, $x_i \in \mathbf{C}^m$, $0 \leqslant i \leqslant n$ *for any* $y = (y_i)_0^n$, $y_i \in \mathbf{C}^m$, $0 \leqslant i \leqslant n$, *and there exists a constant R such that*

(5.1.b.∞) $\|x\|_\infty = \sup_{0 \leqslant i \leqslant n} |x_i| \leqslant R \cdot \|y\|_\infty$ *for any* $n \geqslant n_0$,

i.e. R does not depend on $n \geqslant n_0$.

(c) *The same is true for the* l^2*-norm also, i.e. the inequality*

(5.1.b.2) $\sum_0^n |x_i|^2 \leqslant R^2 \sum_0^n |y_i|^2,$ $n \geqslant n_0,$

holds for the solution of (5.1.b). (*The constant R in* (5.1.b.2) *could be different than in* (5.1.b.∞).)

We can transform these inequalities to the case of the weight spaces l^2 or l^∞. If $\mu > 0$ and $y_i = y_i' \mu^{i/2}$, $x_i = x_i' \mu^{i/2}$, $0 \leqslant i \leqslant n$, then the system (5.1.b) becomes

$$D_0' x_0' + D_{-1}' x_1' = y_0',$$

(5.1.b') $\cdots\cdots\cdots\cdots$

$$D_1' x_{i-1}' + D_0' x_i' + D_{-1}' x_{i+1}' = y_i',$$ $0 < i < n,$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$$D_1' x_{n-1}' + D_0' x_n' = y_n',$$

where

(5.1.b'') $D_1' = D_1 \mu^{-1/2}, \quad D_0' = D_0, \quad D_{-1}' = D_{-1} \mu^{1/2}.$

Therefore, the conditions of Lemma 5.1 are equivalent to the following condition

(5.1.d) for $n \geqslant n_0$, n_0 sufficiently large, for any $\mu > 0$ the system (5.1.b') has a (unique) solution $x' = (x_i')_0^n$, $x_i' \in \mathbf{C}^m$, $0 \leqslant i \leqslant n$, and there exists a constant R such that

$$\sup_{0 \leqslant i \leqslant n} |x_i'| \mu^{i/2} \leqslant R \cdot \sup_{0 \leqslant i \leqslant n} |y_i'| \mu^{i/2},$$

and

$$\sum_0^n |x_i'|^2 \mu^i \leqslant R^2 \cdot \sum_0^n |y_i'|^2 \mu^i,$$

R does not depend on n (or on μ).

Hence, (5.1.d) implies that the pencil $D(z) = D_1 z + D_0 + D_{-1} 1/z$ has both the factorizations of type (5.1.a.l) and (5.1.a.r), and by (5.1.b'') it means that ($\zeta = \mu^{1/2} z$) the pencil $D'(\zeta) = D_i' \zeta + D_0' + D_{-1}' 1/\zeta$ has the factorizations of the same type with conditions

(5.1.d') $\begin{aligned} &\text{spectr } N_{-1}', \quad \text{spectr } M_{-1}' \subset D_{\sqrt{\mu}} = \{z: |z| < \mu^{1/2}\}, \\ &\text{spectr } N_{+1}', \quad \text{spectr } M_{+1}' \subset D_{1/\sqrt{\mu}} = \{z: |z| < \mu^{-1/2}\}. \end{aligned}$

This form (5.1.d) of Lemma 5.1 will be used to realize why the condition (5.3.b) below is too important (and maybe the only concrete fact that we do need).

LEMMA 5.2. *The pencil* (5.1.ᴄ) *has both the factorizations* (5.1.a.l) *and* (5.1.a.r) *if*

$$(5.2.a) \qquad \operatorname{Re}\langle V(z)\omega, \omega \rangle < 0, \quad \forall z, |z| = 1; \omega \in \mathbf{C}^m, \omega \neq 0.$$

These results on the discrete multi-dimensional Hopf-Wiener equation (see [8, Chapter 8]) help us right now. In [10] I give independent analysis of the pencil that appears in our problem on the piecewise-polynomial approximant and its generalization; there I get some new results on factorization of rational pencils that can be useful for other applications.

LEMMA 5.3. *The pencil* (5.1), *if* $\nu = 1$, *has the property* (5.2.a). *Indeed, in this case*

$$\langle (-ST^{-1}z + V^{-1} + ST^{-1}S^t + T^{-1} - T^{-1}S^t\bar{z})\omega, \omega \rangle$$
$$= \langle V^{-1}\omega, \omega \rangle + \langle T^{-1}S^t\omega, \omega \rangle + \langle T^{-1}\omega, \omega \rangle - 2\operatorname{Re} z\langle T^{-1}\omega, S^t\omega \rangle.$$

Both V^{-1} and T^{-1} are strictly-positive definite, so $\langle V^{-1}\omega, \omega \rangle > 0$, and by the Cauchy inequality for the inner product,

$$[u, v] = \langle T^{-1}u, v \rangle \qquad |2\operatorname{Re} z\langle T^{-1}\omega, S^t\omega \rangle| \leqslant \langle T^{-1}\omega, \omega \rangle + \langle T^{-1}S^t\omega \rangle$$

and the lemma holds.

In the case of the geometric mesh we can be interested in $\nu \in (0, 1]$ instead of $\nu \in (0, \infty)$ because the original problems for ν and $1/\nu$ ·(or μ and $1/\mu$) are equivalent; this change means only that we run over $[0, 1]$ from 1 to 0. And if $\nu \in (0, 1]$, the factors $\tilde{N}$ and $\nu\tilde{N}$ as operators in $l_n^\infty(\mathbf{C}^{k-m})$ have norms 1 so we only have to consider (see (4.8)) $\check{C}$ and $\check{C}^{-1}$. The operators depend on the parameter $\nu \in [0, 1]$; by Lemmas 5.1–5.3 it is nice if $\nu = 1$ (the uniform mesh).

We could complete the analysis of the geometric mesh if we knew that

$(5.3.a)$ $\quad \check{C}(0)$ is invertible for any n and $\|\check{C}^{-1}(0)\|_\infty \leqslant R(0) < \infty$ independently on n,

$(5.3.b)$ $\quad w_\nu(z) = \det W(z)$, W as in (5.1), does not vanish on $\{z: |z| = 1\}$ for any ν, $0 < \nu \leqslant 1$.

Indeed, if ν is small enough (see (4.1)), we can consider $\check{C}$ as a perturbation of $\check{C}(0)$; more precisely, if $\nu \leqslant \frac{1}{2}(\delta(0)\cdot(\|T^{-1}\| + 2\|T^{-1}S^t\|))^{-1} = \nu_*$, then by the Neumann-series $\check{C}$ is invertible and $\|\check{C}^{-1}\|_\infty \leqslant 2\cdot R(0)$.

For $\nu \in [\nu_*, 1]$ we will use essentially the fact that the pencil (5.1) is generated by the orthogonal projection in $l_2(P_j; \Delta) = \{f = (f_\alpha)_0^{n+1}: \Sigma_0^{n+1}\|f_\alpha\|^2\mu^\alpha = \|f\|^2\}$, $\mu\cdot\nu = 1$. This projection is certainly bounded and its norm is 1. It implies that all the operators in (3.1a), (3.1b) and so on are uniformly bounded with respect to n or μ if we consider them in the Hilbert weight spaces $\Sigma_{n+1}(\mathbf{C}^m; \Delta(\mu))$. In particular it means that the system (5.1.d) for $D_1' = -\nu F$, $D_0' = D$, $D_{-1}' = -F^t$, —see (5.1) for the further concretization of these matrices, —are uniformly solvable, $n \geqslant n_0(\mu)$, in $l_{n+1}(\mathbf{C}^m; \Delta(\mu))$, i.e. (5.1.d) holds.

Therefore, by Lemma 5.1.d and the subsequent remarks there exist two factorizations

$$(5.3.c) \quad V(z) = (N_1'z + 1)N_0'\left(1 + N_{-1}'\frac{1}{z}\right) = \left(1 + M_{-1}'\frac{1}{z}\right)M_0'(M_1'z + 1)$$

with conditions (5.1.d′) where N_0', M_0' are nonsingular ($m \times m$)-matrices.

At the initial moment $\nu = 1$ the determinant $w_1(z)$ by Lemma 5.3 has m zeros inside the disc $\mathfrak{D}_1$ and m zeros outside it. When ν changes the coefficient of z^m is equal to $(-1)^m \nu^m \det F$ and is not zero (all the factors S, T^{-1} and N of F are nonsingular matrices), so the zeros of $w_\nu(z)$ depend continuously on the parameter ν and by the condition (5.3.b) m of them will stay inside $\mathfrak{D}_1$ and m others will be outside $\mathfrak{D}_1$ for any ν, $0 < \nu \leqslant 1$. By (5.3.c) we do know that m singular points of V are outside the disk $\mathfrak{D}_{\sqrt{\mu}}$, $\mu = 1/\nu > 1$, so we can be sure that V_ν has no singular points in the annulus $\{z\colon 1 \leqslant |z| \leqslant \mu^{1/2}\}$; indeed, their number is $2m$ and m of them belong to $\mathfrak{D}_1$ and m others are outside $\mathfrak{D}_{\sqrt{\mu}}$. Therefore, the factors $(1 + N_{-1}'1/z)$ and $(1 + M_{-1}'1/z)$ are invertible not only for $|z| \geqslant \sqrt{\mu}$ but for any z, $|z| \geqslant 1$; indeed, by (5.3.c) $(1 + N_{-1}'1/z)^{-1} = V(z)^{-1}(N_1'z + 1)N_0'$ for $1 \leqslant |z| \leqslant \sqrt{\mu}$ because $\det V_\nu(z) \neq 0$ in this annulus. (The analogous formula could be written for the factor of the second factorization.) It shows that the first line of (5.1.d′) could be changed to the stronger form

$$\text{spectr } N_-' , \quad \text{spectr } M_-' \subset \mathfrak{D}_1 = \{z\colon |z| < 1\}$$

and together with the second line ($\nu < 1$ so $\mu > 1$, $\mu^{-1/2} = \nu^{1/2} < 1$) it means that the spectra of all four matrices $N_{\pm 1}$, $M_{\pm 1}$ satisfy the conditions (a) of Lemma 5.1. Hence, by this lemma the condition (b) holds also for $D_1 = \nu ST^{-1}N$, $D_{-1} = -NT^{-1}S^t$, $D_0 = V^{-1} + ST^{-1}S^t + \nu NT^{-1}N$ and it is precisely what we do need. So all the operators $\check{C}$, $\nu_* \leqslant \nu \leqslant 1$, are invertible, $\|C^{-1}\|_\infty \leqslant R(\nu)$, $n \geqslant n_*$, and the Neumann-series argument shows that for the closed interval $[\nu_*, 1]$ the upper bound $R_* \geqslant R(\nu)$, $\nu_* \leqslant \nu \leqslant 1$, could be chosen as finite and independent on ν and n. Now $C(k) = \max\{2R(0); R_*\}$ gives the bound of L_∞-norms of orthogonal projectors in the case of the geometric mesh.

However this conclusion depends heavily on the assumptions (5.3.a) and (5.3.b). And now we check these conditions in the case $m = 1, 2, 3$; $k = 2m$, by explicit analysis of our concrete matrices.

If $\nu = 0$, the matrix $\check{C}$ in (4.1) loses its lower diagonal, the entry on the main diagonal is $D_0 = V^{-1} + ST^{-1}S^t$, and on the upper diagonal the entry matrix

$$G = -N_0 T^{-1}S^t = -\begin{pmatrix} 1 & & & \bigcirc \\ & 0 & & \\ & & 0 & \ddots \\ \bigcirc & & & \ddots & 0 \end{pmatrix}; \quad T^{-1}S^t = \begin{pmatrix} * & * \cdots * \\ & \bigcirc \end{pmatrix}$$

has one nonzero row only. D_0 does not depend on ν or n, so

$$\check{C}(0) = \begin{bmatrix} D_0 & G & & & \\ 0 & D_0 & G & & \\ 0 & 0 & D_0 & G & \\ \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & & & D_0 \end{bmatrix} = \begin{bmatrix} 1 & GD_0^{-1} & & \\ 0 & 1 & GD_0^{-1} & \\ & & & GD_0^{-1} \\ & & & & 1 \end{bmatrix} \cdot \tilde{D}_0$$

and these operators are certainly uniformly invertible if the spectrum $\sigma(GD_0^{-1}) < 1$. This operator GD_0^{-1} is one-dimensional, its matrix is

$$\left[\begin{array}{c} \gamma_0, \gamma_1, \ldots, \gamma_{m-1} \\ \bigcirc \end{array} \right]$$

and the only nonzero eigenvalue is

$$\gamma_0 = \sum_{j=0}^{m-1} (T^{-1}S^t)_{0j} (D_0^{-1})_{j0}.$$

The computations show that for $k = 2m$,

(5.4.1) $k = 2, \quad \gamma_0 = -\dfrac{1}{2};$

(5.4.2) $k = 4, \quad \gamma_0 = -4 \cdot \dfrac{1}{4} + (-60) \cdot \left(-\dfrac{1}{40}\right) = \dfrac{1}{2};$

(5.4.3) $k = 6, \quad \gamma_0 = 6 \cdot \dfrac{1}{4} + 210 \cdot \left(-\dfrac{1}{35}\right) + 3360 \cdot \dfrac{1}{672} = -\dfrac{1}{2}.$

This suggests that in the general case $\gamma_0 = (-1)^{k/2} \cdot \frac{1}{2}$, and it would be interesting to check this fact.

The last step is to examine the property (5.3.b). We present explicit computation of these rational functions ($k = 2, 4, 6$) and then it is clear that (5.3.b) holds.

$$k = 2.$$
(5.5.1)
$$w_\nu(z) = 2\nu z + 4(1 + \nu) + 2 \cdot \dfrac{1}{z}$$

and $|z| = 1 \Rightarrow |w_v(z)| \geqslant 4 - 2|\bar{z}| + 4\nu - |2\nu z| = 2(1 + \nu) \geqslant 2.$

(5.5.2) $k = 4.$
$$w_\nu(z) = \det\left\{ 2\nu \begin{pmatrix} 4 & -60\nu \\ 60 & -840\nu \end{pmatrix} + \begin{pmatrix} 16(1 + \nu) & 120(1 - \nu^2) \\ 120(1 - \nu^2) & 1200(1 + \nu^3) \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} 4 & 60 \\ -60\nu & -840\nu \end{pmatrix} \cdot \dfrac{1}{z} \right\}$$

$$= 240 \det\left\{ \begin{array}{cc} \nu \cdot z + 4(1 + \nu) + 1 \cdot \dfrac{1}{z} & -15\nu^2 \cdot z + 30(1 - \nu^2) + 15 \cdot \dfrac{1}{z} \\[2mm] \nu \cdot z + 2(1 - \nu^2) - 1 \cdot \dfrac{1}{z} & -14\nu^2 \cdot z + 20(1 - \nu^2) - 14 \cdot \dfrac{1}{z} \end{array} \right\}$$

and

$$w_\nu(z) = 240 \cdot \sum_{i=-2}^{2} p_i(\nu)z^i; \qquad p_i(0) = \sum_{k=0}^{4} p_{ik}\nu^k,$$

where p_{ik} are given in Table 1.

<div align="center">TABLE 1</div>

i \\ k	4	3	2	1	0
0	20	65	92	65	20
+1	−10	−26	−26	−10	
−1		−10	−26	−26	−10
+2		1			
−2				1	
π	10	28	40	28	10

The last row $\pi_k = p_{0k} - \Sigma_{0<|i|\leqslant 2}|p_{ik}|$ contains strictly positive scalars only, and it implies for $|z| = 1$ that

$$|w_\nu(z)| \geqslant 240\left(p_0(\nu) - \sum_{0<|i|\leqslant 2}|p_i(0)|\right) = 240 \cdot \sum_{k=0}^{4}\pi_k\nu^k \geqslant 2400 > 0.$$

(5.5.3) $k = 6$

$$w_0(z) = \det\left\{+\nu z\begin{pmatrix} 6 & -210\nu & 3360\nu^2 \\ 210 & -7140\nu & 11088\nu^2 \\ 3360 & -110880\nu & 1673280\nu^2 \end{pmatrix}\right.$$

$$+\begin{pmatrix} 36(1+\nu) & 630(1-\nu^2) & 6720(1+\nu^3) \\ 630(1-\nu^2) & 14700(1+\nu^3) & 176400(1-\nu^4) \\ 6720(1+\nu^3) & 176400(1+\nu^4) & 2257920(1+\nu^5) \end{pmatrix}$$

$$\left.+\frac{1}{z}\begin{pmatrix} 6 & 210 & 3360 \\ -210\nu & -7140\nu & -110880\nu \\ 3360\nu^2 & 110880\nu^2 & 1673280\nu^2 \end{pmatrix}\right\}$$

$$= 6 \cdot 210 \cdot 3360 \cdot \left[\begin{array}{cc} \nu z + 6(1+\nu) + \dfrac{1}{z} & -35\nu^2 z + 105(1-\nu^2) + 35\cdot\dfrac{1}{z} \\[2mm] \nu z + 3(1-\nu^2) - \nu\cdot\dfrac{1}{z} & -34\nu^2 z + 70(1+\nu^3) - 34\nu\cdot\dfrac{1}{z} \\[2mm] \nu z + 2(1+\nu^3) + \nu^2\cdot\dfrac{1}{z} & -33\nu^2 z + 52.5(1-\nu^4) + 33\nu^2\cdot\dfrac{1}{z} \end{array}\right.$$

$$\left.\begin{array}{c} 560\nu^3 z + 1120(1+\nu^3) + 560\cdot\dfrac{1}{z} \\[2mm] 528\nu^3 z + 840(1-\nu^4) - 528\nu\cdot\dfrac{1}{z} \\[2mm] 498\nu^3 z + 672(1+\nu^5) + 498\nu^2\cdot\dfrac{1}{z} \end{array}\right]$$

$$= \gamma \cdot \sum_{-3}^{3} p_i(\nu)\cdot z^i = \gamma\cdot\sum_{i=-3}^{3}\left(\sum_{k=0}^{9}p_{ik}\nu^k\right)z^i,$$

where $\gamma = 2\cdot 6\cdot 210\cdot 3360 = 8{,}467{,}200$, and the polynomials p_i, $|i|\leqslant 3$, are given in Table 2.

TABLE 2

Table of the Coefficients p_{ik} of $\Sigma_{-3}^3 P_i(x) \cdot z^i = \Sigma_{-3}^3 \left(\Sigma_0^9 p_{ik} x^k \right) z^i$

i/k	9	8	7	6	5	4	3	2	1	0
0	980	6860	22792	47058	66426	66426	47058	22792	6860	980
+1	490	3136	9401	17153	20799	17153	9401	3136	490	
−1		490	3136	9401	17153	20153	17153	9401	3136	490
+2	0	56	245	461	461	245	56			
−2				56	245	461	461	245	56	0
+3					1					
−3								1		
$(\Sigma_i p_{ik}) - p_{0k}$	490	3682	12782	27072	38651	38651	27072	12782	3682	490
$\dfrac{\Sigma_k - p_{0k}}{p_{0k}} <$	= .5	.537	.561	.576	.582	.582	.576	.561	.537	= .5

This table shows that for any z, $|z| = 1$, and $\nu \geqslant 0$,

$$|w_\nu(z)| \geqslant \gamma \cdot \left(p_0(\nu) - \sum_{0 < |i| \leqslant 3} |p_i(\nu)| \right) = \gamma \cdot \sum_{k=0}^9 \pi_k \nu^k \geqslant \gamma \pi_0 = 8{,}297{,}856{,}000.$$

It means that (5.3.b) holds. Hence, in particular, we proved that

(5.6)   The orthogonal projectors $Q(S_\xi^{6;3})$ are uniformly bounded in $L_\infty[0,1]$, i.e. $\|Q(S_\xi^{6;3})\|_\infty \leqslant A < \infty$ for any geometric mesh ξ, where A is an absolute constant.

We made all these computations to show how the approach of Sections 1–4 works. One can repeat the analogous computations (from the very beginning) to get the same results for $S_\xi^{8;4}$ or $S_\xi^{10;5}$ with geometric meshes ξ. But it would not give us an understanding of the general situation. What we really need is to consider the following series of conjectures and statements in the case $2^{2m,m}$.

(5.7) $V^{-1} + ST^{-1}S^t = JT^{-1}J$ and $T + S^t VS = JVJ$

(compare (4.10.a,b)).

(5.8) The polynomial

$$w_\nu(z) = \det\left\{ -\nu ST^{-1}N_z + (V^{-1} + ST^{-1}S^t) + \nu NT^{-1}N - NT^{-1}S^t \cdot \frac{1}{z} \right\}$$

$$= \sum_{i=-m}^m P_i(\nu) z^i = \sum_{i=-m}^m \left(\sum_{k=0}^m p_{ik} \nu^k \right) z^i$$

has coefficients such that $p_{0k} > 0$, $0 \leqslant k \leqslant m^2$, and moreover $p_{0k} > \sum_{0 < |i| \leqslant m} |p_{ik}|$. (Compare Tables 1 and 2 or (5.5.1)–(5.5.3).)

(5.9) The coefficients $p_{m^2;k} = 0$ if $k \neq 0$ or 1 (it is almost evident), and $p_{m^2;1} = (-1)^{m-1} \cdot \frac{1}{2} p_{m^2;0}$ or, equivalently, $p_{0;-1} = (-1)^{m-1} \cdot \frac{1}{2} p_{0;0}$; compare (5.4.1)–(5.4.3). (5.8) would give the analog of (5.6) for any m. But now we are interested in (5.7)–(5.9) because of the close connections with the theory of Toeplitz block-matrices rather than the analysis of piecewise-polynomial least-squares approximants.

Later we will consider these statements and conjectures, and other (particular) cases of piecewise-polynomial approximation, in the framework of the approach of Sections 1–4.

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