MATHEMATICS OF COMPUTATION
VOLUME 40, NUMBER 161
JANUARY 1983, PAGES 337-342

On the Convergence Behavior of Continued Fractions with Real Elements*

By Walter Gautschi

Abstract. We define the notion of transient (geometric) convergence rate for infinite series and continued fractions. For a class of continued fractions with real elements we prove a monotonicity property for such convergence rates which helps explain the effectiveness of certain continued fractions known to converge "only" sublinearly. This is illustrated in the case of Legendre's continued fraction for the incomplete gamma function.

1. Introduction. Continued fractions, as is well known, can be viewed in terms of infinite series. To describe the convergence behavior of a series it is useful to consider the notion of *transient* (geometric) *convergence rate*. Given a convergent series $\Sigma_{n=0}^{\infty} t_n$, the nth transient convergence rate is the quantity $|\rho_n|$, $n = 1, 2, \ldots$, where $t_n = \rho_n t_{n-1}$ (assuming $t_{n-1} \neq 0$). If $\lim_{n\to\infty} |\rho_n| = r$, $0 \leqslant r \leqslant 1$, convergence is *linear* (geometric) with convergence rate r, if $0 < r < 1$, *superlinear*, if $r = 0$, and *sublinear* if $r = 1$. It is important to note, however, that these concepts are asymptotic in nature, hence not necessarily relevant for numerical (finite!) computation. Thus, a series need not be dismissed as useless, simply because it converges only sublinearly. The approach of $|\rho_n|$ to the limit 1 indeed may be so slow that the series has "converged to machine precision" long before $|\rho_n|$ reaches the neighborhood of 1. For this reason, convergence of a series ought to be judged on the basis of the complete sequence $\{\rho_n\}$ of convergence rates, and not just on the basis of asymptotic properties of ρ_n. In this connection, properties of monotone behavior significantly add to the understanding of the quality of convergence.

The purpose of this note is to prove a criterion for the sequence $\{|\rho_n|\}$ to be (ultimately) monotonically increasing, in the case where the partial sums of the series are convergents of a continued fraction with real elements. We illustrate the result with Legendre's continued fraction for the incomplete gamma function, which, though sublinearly convergent, provides an effective tool of numerical computation.

2. Continued Fractions and Infinite Series. We consider continued fractions of the form

$$c = \frac{1}{1+}\,\frac{a_1}{1+}\,\frac{a_2}{1+}\cdots, \tag{2.1}$$

where, for some integer $k_0 \geqslant 1$,

$$\begin{aligned} &a_k > 0 \quad \text{for } 1 \leqslant k \leqslant k_0 - 1, \\ &a_k < 0 \quad \text{and} \quad |a_k| \leqslant \tfrac{1}{4} \quad \text{for } k \geqslant k_0. \end{aligned} \tag{2.2}$$

Received February 16, 1982.

1980 *Mathematics Subject Classification.* Primary 40A15; Secondary 33A70.

Key words and phrases. Convergence of real continued fractions, Legendre's continued fraction for the incomplete gamma function.

*Sponsored in part by the National Science Foundation under Grant MCS-7927158.

0025-5718/82/0000-0715/$02.25

It can be seen from Worpitzky's theorem (Henrici [3, p. 506]) that the tail of the continued fraction (2.1) beginning with the element a_{k_0}, hence also the complete continued fraction, converges. The infinite series

$$s = \sum_{k=0}^{\infty} t_k \qquad (t_0 = 1) \tag{2.3}$$

is *equivalent* to the continued fraction (2.1) if its nth partial sum

$$s_n = 1 + \sum_{k=1}^{n-1} t_k \tag{2.4}$$

is equal to the nth convergent of c, for each $n = 1, 2, 3, \ldots$. According to Euler,

$$s_1 = 1, \quad s_{k+1} = s_k + t_k, \qquad k = 1, 2, 3, \ldots, \tag{2.5}$$

where

$$\left.\begin{aligned} &\rho_0 = 0, \qquad t_0 = 1, \\ &\rho_k = \frac{-a_k(1 + \rho_{k-1})}{1 + a_k(1 + \rho_{k-1})} \\ &t_k = \rho_k t_{k-1} \end{aligned}\right\} \qquad k = 1, 2, 3, \ldots . \tag{2.6}$$

This represents a convenient algorithm for evaluating the continued fraction c, and is also useful for analyzing qualitative properties of convergence. Note indeed that the quantities ρ_n in (2.6) yield the transient convergence rates $|\rho_n|$ of the series (2.3).

Slightly more convenient for analytical purposes are the quantities $\sigma_k = 1 + \rho_k$, which satisfy

$$\sigma_0 = 1, \quad \sigma_k = \frac{1}{1 + a_k \sigma_{k-1}}, \qquad k = 1, 2, 3, \ldots . \tag{2.7}$$

3. Convergence Behavior. Some first insights into the convergence behavior of the continued fraction (2.1) can be gained from the following lemma.

LEMMA 3.1. *If the partial numerators* a_k *in* (2.1) *satisfy* (2.2), *then the quantities* σ_k *in* (2.7) *satisfy*

$$0 < \sigma_k < 1 \quad \text{for } 1 \leq k \leq k_0 - 1, \tag{3.1}$$

and

$$1 < \sigma_k \leq \frac{2(k - k_0 + 2)}{k - k_0 + 3} \quad \text{for } k \geq k_0. \tag{3.2}$$

Proof. The inequalities (3.1) follow immediately from the positivity of a_k and (2.7). To prove (3.2), we use induction. Since $-\frac{1}{4} \leq a_{k_0} < 0$ and $0 < \sigma_{k_0 - 1} \leq 1$, we have $1 < \sigma_{k_0} \leq 4/3$, so that (3.2) is true for $k = k_0$. Assuming its truth for some $k \geq k_0$, we obtain

$$1 < \sigma_{k+1} = \frac{1}{1 + a_{k+1}\sigma_k} \leq \frac{1}{1 - \dfrac{1}{4}\,\dfrac{2(k - k_0 + 2)}{k - k_0 + 3}} = \frac{2(k - k_0 + 3)}{k - k_0 + 4},$$

which is (3.2) with k replaced by $k + 1$. □

Lemma 3.1, in particular, implies $0 < \sigma_k < 2$, hence $-1 < \rho_k < 1$, for all $k \geqslant 1$. The series (2.3), therefore, has terms that are strictly decreasing in absolute value. Furthermore, by (3.1) and (3.2),

$$(3.3) \qquad -1 < \rho_k < 0 \quad \text{for } 1 \leqslant k \leqslant k_0 - 1, \quad \text{and} \quad 0 < \rho_k < 1 \quad \text{for } k \geqslant k_0,$$

so that the series initially (if $k_0 > 1$) behaves like an alternating series and subsequently turns into a monotone series.

A more detailed description of convergence is provided by the following theorem.

THEOREM 3.1. *If the partial numerators* a_k *in* (2.1) *satisfy* (2.2), *and in addition* $-\frac{1}{4} \leqslant a_{k+1} \leqslant a_k < 0$ *for* $k \geqslant k_0$, *then*

$$(3.4) \qquad -1 < \rho_k < 0 \quad \text{for } 1 \leqslant k \leqslant k_0 - 1 \quad \text{and} \quad \rho_{k+1} > \rho_k > 0 \quad \text{for } k \geqslant k_0.$$

In particular,

$$(3.5) \qquad \lim_{k \to \infty} \rho_k = \rho, \qquad \rho = \frac{1 - \sqrt{1 + 4a}}{1 + \sqrt{1 + 4a}},$$

where $a = \lim_{k \to \infty} a_k$; *the continued fraction* (2.1) *converges linearly, with convergence rate* ρ, *if* $a > -\frac{1}{4}$, *and sublinearly if* $a = -\frac{1}{4}$.

Proof. The first inequalities in (3.4) have already been noted in (3.3). The others are equivalent to $\sigma_{k+1} > \sigma_k > 1$ for $k \geqslant k_0$. Since $\sigma_k > 1$, by (3.2), it suffices to prove

$$(3.6) \qquad \sigma_{k+1} > \sigma_k \quad \text{for } k \geqslant k_0.$$

We first show

$$(3.7) \qquad \sigma_{k-1} < \frac{2}{1 + \sqrt{1 - 4|a_k|}} \quad \text{for } k \geqslant k_0.$$

This is true for $k = k_0$, since by (3.1) (and (2.7), if $k_0 = 1$) $\sigma_{k_0 - 1} \leqslant 1$, while the expression on the right of (3.7) is greater than 1. Using induction, assume that (3.7) holds for some $k \geqslant k_0$. Then

$$(3.8) \qquad \begin{aligned} \sigma_k = \frac{1}{1 + a_k \sigma_{k-1}} &< \frac{1}{1 - |a_k| \dfrac{2}{1 + \sqrt{1 - 4|a_k|}}} \\ &\leqslant \frac{1}{1 - \dfrac{2|a_{k+1}|}{1 + \sqrt{1 - 4|a_{k+1}|}}}, \end{aligned}$$

where in the last inequality we have used $|a_{k+1}| \geqslant |a_k|$. Now observe that, for any $\alpha \leqslant \frac{1}{4}$,

$$\frac{1}{1 - \dfrac{2\alpha}{1 + \sqrt{1 - 4\alpha}}} = \frac{1 + \sqrt{1 - 4\alpha}}{1 + \sqrt{1 - 4\alpha} - 2\alpha}$$

$$= \frac{1 - (1 - 4\alpha)}{1 - (1 - 4\alpha) - 2\alpha\left(1 - \sqrt{1 - 4\alpha}\right)} = \frac{2}{1 + \sqrt{1 - 4\alpha}}.$$

Using this in (3.8), with $\alpha = |a_{k+1}|$, yields (3.7) with k replaced by $k + 1$, and thus establishes (3.7) for all $k \geqslant k_0$.

Now (3.6), in view of (2.7), is equivalent to

$$\frac{1}{1 + a_{k+1}\sigma_k} > \sigma_k \quad \text{for } k \geqslant k_0,$$

which in turn, since $1 + a_{k+1}\sigma_k > 0$ and $a_{k+1} < 0$ for $k \geqslant k_0$, is equivalent to

$$|a_{k+1}|\sigma_k^2 - \sigma_k + 1 > 0.$$

The quadratic function $|a_{k+1}|t^2 - t + 1$ is convex and has two real zeros $t_{1,k+1} < t_{2,k+1}$, the smaller of which is

$$t_{1,k+1} = \frac{2}{1 + \sqrt{1 - 4|a_{k+1}|}}.$$

By (3.7), $\sigma_k < t_{1,k+1}$, hence $|a_{k+1}|\sigma_k^2 - \sigma_k + 1 > 0$, which implies $\sigma_{k+1} > \sigma_k$. This proves (3.6).

Since the sequence $\{a_k\}$ is monotonically decreasing for $k \geqslant k_0$, and bounded below by $-\frac{1}{4}$, the limit $\lim_{k\to\infty} a_k = a$ exists, and $-\frac{1}{4} \leqslant a < 0$, since $a_{k_0} < 0$. Similarly, $\lim_{k\to\infty} \rho_k = \rho$, $0 < \rho \leqslant 1$, and $\lim_{k\to\infty} \sigma_k = \sigma$ with $\sigma = 1 + \rho$. Going to the limit $k \to \infty$ in (2.7) then gives

$$\sigma = \frac{1}{1 + a\sigma}, \qquad \sigma = \frac{2}{1 \pm \sqrt{1 + 4a}}.$$

Since $\sigma \leqslant 2$ and $-\frac{1}{4} \leqslant a < 0$, the minus sign in the last equation for σ cannot hold (unless $a = -\frac{1}{4}$), and we conclude that

$$\sigma = \frac{2}{1 + \sqrt{1 + 4a}}, \qquad \rho = \sigma - 1 = \frac{1 - \sqrt{1 + 4a}}{1 + \sqrt{1 + 4a}},$$

which is (3.5). The last statement of the theorem is an immediate consequence of (3.5). This completes the proof of Theorem 3.1.

4. Truncation. In practice, the continued fraction (2.1) is evaluated by carrying out (2.5) and (2.6) for $k = 1, 2, \dots, n$ and taking s_{n+1} to approximate the value of s (or c) of the continued fraction. It is important, then, to be able to choose n in such a way that s_{n+1} approximates s to any prescribed accuracy.

Assuming first $k_0 = 1$, hence $0 < \rho_k < 1$ by (3.3) and $0 < t_k < 1$, it follows from a result of Merkes [4, Eq. (12)] that

$$|s - s_{n+1}| \leqslant \frac{1 + \rho_n}{1 - \rho_n} t_n. \tag{4.1}$$

This suggests the following *stopping rule*: Given a prescribed (relative) accuracy ε, stop the recursion (2.6) at the first integer $k = n$ for which

$$(1 + \rho_n)t_n \leqslant (1 - \rho_n)s_{n+1}\varepsilon. \tag{4.2}$$

By (4.1), this implies $|s - s_{n+1}| \leqslant s_{n+1}\varepsilon$, hence

$$\frac{|s - s_{n+1}|}{s + |s_{n+1} - s|} \leqslant \frac{|s - s_{n+1}|}{s_{n+1}} \leqslant \varepsilon,$$

from which $|s - s_{n+1}| \leqslant s\varepsilon + |s_{n+1} - s|\,\varepsilon$, that is,

$$(4.3) \qquad \left|\frac{s - s_{n+1}}{s}\right| \leqslant \frac{\varepsilon}{1-\varepsilon}.$$

Our stopping rule therefore achieves the desired accuracy, at least asymptotically for $\varepsilon \to 0$.

To avail oneself of this simple stopping rule, when $k_0 > 1$, one ought to first evaluate the "tail"

$$(4.4) \qquad c_{k_0} = \frac{1}{1+}\,\frac{a_{k_0}}{1+}\,\frac{a_{k_0+1}}{1+}\cdots$$

of the continued fraction (2.1), to which Merkes' result applies, and then compute

$$(4.5) \qquad c_k = \frac{1}{1 + a_k c_{k+1}} \quad \text{for } k = k_0 - 1, k_0 - 2, \ldots, 1,$$

to get the complete continued fraction $c = c_1$. Since $c_{k_0} > 0$ and $a_k > 0$ for $k < k_0$, the computation in (4.5) involves the addition of positive numbers and division, hence only numerically stable operations.

5. An Example. Theorem 3.1 is applicable to Legendre's continued fraction for the incomplete gamma function,

$$(5.1) \qquad \begin{gathered} (x - \alpha + 1)x^{-\alpha}e^{x}\Gamma(\alpha, x) = \frac{1}{1+}\,\frac{a_1}{1+}\,\frac{a_2}{1+}\cdots, \\ a_k = \frac{k(\alpha - k)}{(x - \alpha + 2k - 1)(x - \alpha + 2k + 1)}, \qquad k = 1, 2, 3, \ldots, \end{gathered}$$

which is used in [1, p. 475], [2] to compute the incomplete gamma function in the domain D: $x \geqslant 1.5$, $-\infty < \alpha \leqslant x + \frac{1}{4}$. Assuming α not a positive integer (otherwise, the continued fraction (5.1) would terminate and our assumption (2.2) would be violated), we have for $(x, \alpha) \in D$

$$(5.2) \qquad k_0 = \begin{cases} 1 & \text{if } \alpha < 1, \\ 1 + [\alpha] & \text{if } \alpha > 1. \end{cases}$$

If $k \geqslant k_0$, the condition $|a_k| \leqslant \frac{1}{4}$ is equivalent to $(x - \alpha)^2 + 4kx \geqslant 1$, hence satisfied if $x \geqslant \frac{1}{4}$ (since $k \geqslant 1$). An elementary calculation furthermore shows that $|a_{k+1}| \geqslant |a_k|$ for $k \geqslant k_0$ whenever $x \geqslant \frac{1}{2}$. It follows, in particular, that all assumptions of Theorem 3.1 are satisfied when $(x, \alpha) \in D$. Since clearly $a = \lim_{k\to\infty} a_k = -\frac{1}{4}$, we are in a case of sublinear convergence. (This is also noted by Henrici [3, p. 629] by way of a different analysis.) Nevertheless, the continued fraction is known to be quite useful as a computational tool, at least in a domain such as D. The reason for this is readily understood on the basis of Theorem 3.1: Although the transient convergence rates ρ_k eventually increase monotonically to 1, the limit is approached quite slowly. We can see this from Table 5.1 which, in the case of the continued fraction (4.4), and for selected x and α, displays the values of

$$n_\nu = \max\left(k\colon |\rho_k| \leqslant \frac{\nu}{4}\right) \quad \text{and} \quad \varepsilon_\nu = \frac{4+\nu}{4-\nu}\,t_{n_\nu}, \qquad \nu = 1, 2, 3.$$

TABLE 5.1
Convergence behavior of the continued fraction (5.1)
(*Numbers in parentheses indicate decimal exponents.*)

x	α	n_1	ε_1	n_2	ε_2	n_3	ε_3
1.5	1.75	3	1.9(−3)	13	2.9(−7)	75	6.7(−18)
	.875	3	9.2(−4)	13	1.2(−7)	75	2.7(−18)
	0.0	3	6.8(−3)	13	1.4(−6)	75	3.7(−17)
	−3.5	3	7.3(−3)	13	1.4(−6)	75	3.5(−17)
	−7.0	4	8.9(−4)	15	5.8(−8)	79	5.5(−19)
5.0	5.25	9	1.9(−8)	41	5.4(−21)	243	2.9(−56)
	2.625	10	1.0(−9)	41	7.3(−22)	243	3.0(−57)
	0.0	10	9.8(−10)	42	2.9(−22)	244	1.6(−57)
	−10.5	9	1.9(−8)	42	1.6(−21)	244	8.5(−57)
	−21.0	12	2.7(−11)	48	1.2(−25)	254	8.8(−62)
10.0	10.25	18	1.1(−16)	81	1.2(−41)	483	3.0(−112)
	5.125	18	8.6(−17)	81	7.1(−42)	483	1.5(−112)
	0.0	20	8.0(−20)	83	8.1(−45)	485	1.9(−115)
	−20.5	19	1.7(−17)	83	6.9(−43)	485	1.6(−113)
	−41.0	24	1.5(−22)	95	5.0(−51)	504	4.2(−123)
20.0	20.25	36	3.0(−33)	161	4.8(−83)	963	2.9(−224)
	10.125	36	1.2(−33)	162	5.2(−84)	964	3.1(−225)
	0.0	40	2.9(−40)	165	3.4(−90)	967	1.6(−231)
	−40.5	38	4.3(−35)	165	8.8(−86)	968	3.4(−227)
	−81.0	48	3.1(−45)	188	1.2(−101)	1004	6.8(−246)

Note that by virtue of (4.1), and the fact that $1 < s = c_{k_0} \leqslant 2$ (cf. [3, Theorem 12.3c]),

$$\left|\frac{s - s_{n_\nu+1}}{s}\right| \leqslant |s - s_{n_\nu+1}| \leqslant \varepsilon_\nu, \qquad \nu = 1, 2, 3. \tag{5.3}$$

Thus, for example, if $x = 5$, $\alpha = 0$, by the time the transient convergence rate has risen to $\frac{1}{2}$, the continued fraction has already converged to within a (relative) error of about 3×10^{-22}.

Department of Computer Sciences
Purdue University
West Lafayette, Indiana 47907

1. W. GAUTSCHI, "A computational procedure for incomplete gamma functions," *ACM Trans. Math. Software*, v. 5, 1979, pp. 466–481.

2. W. GAUTSCHI, "Algorithm 542—Incomplete gamma function," *ACM Trans. Math. Software*, v. 5, 1979, pp. 482–489.

3. P. HENRICI, *Applied and Computational Complex Analysis*, Vol. 2, Wiley, New York, 1977.

4. E. P. MERKES, "On truncation errors for continued fraction computations," *SIAM J. Numer. Anal.*, v. 3, 1966, pp. 486–496.