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Chebyshev Expansions for the Bessel Function $J_n(z)$ in the Complex Plane

By J. P. Coleman and A. J. Monaghan

Abstract. Polynomial-based approximations for $J_0(z)$ and $J_1(z)$ are presented. The first quadrant of the complex plane is divided into six sectors, and separate approximations are given for $|z| \leqslant 8$ and for $|z| \geqslant 8$ on each sector. Each approximation is based on a Chebyshev expansion in which the argument of the Chebyshev polynomials is real on the central ray of the sector. The errors involved in extrapolation off the central ray are discussed. The approximation obtained for $|z| \geqslant 8$ can also be used to evaluate the Bessel functions $Y_0(z)$ and $Y_1(z)$ and the Hankel functions of the first and second kinds.

1. Introduction. Many polynomial and rational approximations are available for mathematical functions of real argument [1], [10], [11]. Practical approximation in the complex plane is less well developed. Our main concern here is the Bessel function of the first kind of integer order, but the ideas used can also be applied to other functions.

For the Bessel function $J_n(z)$ it is possible in principle to use the Taylor expansion within a disc $|z| \leqslant R$ and an asymptotic expansion for $|z| \geqslant R$. This approach suffers from cancellation errors and from the need for an increasingly large number of terms in the Taylor series as $|z|$ increases, as well as from the limitation on the accuracy achievable from an asymptotic expansion for a given value of R.

By truncating a Chebyshev series for a function on the real interval $[-1, 1]$ we obtain, in many cases, a close approximation to the minimax polynomial of the same degree. The error in truncating a rapidly convergent Chebyshev series $\sum_{k=0}^{\infty} a_k T_k(x)$ after $(n + 1)$ terms is dominated by the first neglected term; this, being proportional to $T_{n+1}(x)$, has the equioscillation property characteristic of the error in the minimax polynomial approximation of degree n. Approximations, based on Chebyshev series, for Bessel functions of real argument have been obtained by Clenshaw [3], Luke [10], and Coleman [4].

For a domain D of the complex plane we can ask what monic polynomial of degree n will minimize

$$\max_{z \in D} |z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| .$$

The polynomial $\mathfrak{T}_n^D(z)$ which satisfies this condition is called the Chebyshev polynomial of degree n for the domain D. The success of Chebyshev expansions on the real axis suggests that by truncating an expansion of the form

$$f(z) = \sum_{k=0}^{\infty} b_k \mathfrak{T}_k^D(z) \tag{1}$$

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we could hope to obtain a near-best polynomial approximation for a function f in the domain D. When D is the real interval $[-1, 1]$, $\mathfrak{T}_n^D$ is 2^{1-n} times the classical Chebyshev polynomial T_n, whereas if D is the unit disc, $\mathfrak{T}_n^D(z) = z^n$, in which case the right-hand side of (1) is simply the Taylor series for f. For other domains the properties of the polynomials $\mathfrak{T}_n^D(z)$ are not in general known. Explicit expressions for a few polynomials $\mathfrak{T}_n^D(z)$ of low degree for specific domains have been found [8], and algorithms exist [7], [13] which will produce numerical values of the coefficients for a given domain D and degree n, but without a better understanding of the properties of the polynomials $\mathfrak{T}_n^D(z)$ it is not feasible to obtain the coefficients of the expansion (1).

Our approach in this paper is to divide the complex plane into a number of sectors and, in the absence of an expansion of the form (1), we use in each sector an approximation

$$f(z) \simeq \sum_{k=0}^{n} a_k T_k(z/\gamma),$$

the complex constant γ being chosen so that z/γ is real on the central ray of the sector. Our expansions for $J_n(z)$ and the generation of their coefficients are described in Section 2. The study, in Section 3, of the truncation error requires an upper bound for $|T_n(z)|$ on a sector; such a bound is established in Appendix 1. Section 4 describes the coefficient tables and provides a guide to the accuracy achievable.

2. Expansions for $J_n(z)$. Throughout this work we shall concentrate on values of z in the first quadrant, $0 \leqslant \arg z \leqslant \pi/2$, only, since symmetry relations can be used to deduce the values of $J_n(z)$ elsewhere. It is convenient to consider separately two regions of the complex plane, an inner region $|z| \leqslant R$ and an outer region $|z| \geqslant R$, each of which will be mapped into the unit disc. It will be assumed that n is a positive integer but the extension to nonintegral orders is straightforward.

2.1. *The Inner Region.* Clenshaw [3] tabulated coefficients for expansions of the form

$$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{2r} T_{2r}(x/8) \tag{2}$$

on the interval $[-8, 8]$, for $n = 0$ and 1. The prime on the summation sign indicates that the first term is to be halved. Coefficients of the corresponding expansions for $n = 2, 3, \ldots, 10$ were calculated by Coleman [4] and are incorporated in a subroutine which calculates $J_n(x)$ to a specified accuracy.

If x is replaced by the complex variable z the expansion (2) still converges. Since $z^{-n}J_n(z)$ is a regular function of z it can be shown [14, p. 143] that, for any $\rho > 1$,

$$|a_{2r}| \leqslant \frac{2M(\rho)}{\rho^{2r}},$$

where $M(\rho)$ is an upper bound on $|2^n n! z^{-n} J_n(z)|$ for $|z| = 8\rho$. Furthermore,

$$|T_{2r}(\zeta)| \leqslant \tfrac{1}{2}\left[\left(1 + \sqrt{2}\right)^{2r} + \left(1 - \sqrt{2}\right)^{2r}\right]$$

for $|\zeta| \leq 1$, the bound being attained when $\zeta = \pm i$. It follows that

$$|a_{2r}T_{2r}(z/8)| \leq \frac{2M(\rho)\left(1+\sqrt{2}\right)^{2r}}{\rho^{2r}},$$

and since we may take $\rho > (1+\sqrt{2})$, the convergence of the series is established. There is nevertheless a substantial deterioration in the rate of convergence as $\arg z$ increases. For example, if the series is truncated to give a polynomial of degree 20, the first neglected term increases by a factor of 1.3×10^8 as $\arg z$ increases from 0° to 90°, while $|z| = 8$.

It is shown in Appendix 1 that in the sector $|\zeta| \leq 1$, $|\arg \zeta| \leq \Theta$

$$|T_n(\zeta)| \leq |T_n(e^{i\Theta})|.$$

The right-hand side of this inequality is a monotonically increasing function of Θ for $0° < \Theta < 90°$ so the smaller the angular range about the real axis in which we use a given number of terms of the expansion (2) the less significant the loss of accuracy.

The advantages of a Chebyshev expansion on the real interval $[-1, 1]$ may be retained for any particular ray in the complex plane by writing

$$J_n(z) = \frac{1}{n!}(z/2)^n \sum_{r=0}^{\infty}{}' a_{2r}T_{2r}(z/\gamma) = \frac{1}{n!}(z/2)^n v(z/\gamma), \tag{3}$$

where γ is a complex number having the same argument as z, and such that $0 \leq z/\gamma \leq 1$. The function v in Eq. (3) satisfies the differential equation

$$tv'' + (2n+1)v' + \gamma^2 tv = 0, \tag{4}$$

with $v(0) = 1$ and $v'(0) = 0$, where the prime denotes differentiation with respect to $t = z/\gamma$. Clenshaw's method [2] applied to this equation yields recurrence relations for the coefficients of the Chebyshev expansions for $v(t)$ and $v'(t)$; these may be solved by backward recurrence. Since

$$T_{2r}(t) = T_r(2t^2 - 1) = T_r^*(t^2),$$

where T_r^* is a shifted Chebyshev polynomial, Eq. (3) may be expressed as

$$J_n(z) = \frac{1}{n!}(z/2)^n \sum_{r=0}^{\infty}{}' a_{2r}T_r^*(z^2/\gamma^2).$$

We have used this method for $|\gamma| = 8$ and $\arg \gamma = 7.5°$ (15°) 82.5°, thus providing a separate expansion for each 15° sector in the first quadrant. The accuracy of these expansions is discussed in Section 3.

The general theory of the expansion of generalized hypergeometric functions in series of functions of the same kind is described by Luke [9]. From this he obtains, in [11], recurrence relations for the Chebyshev coefficients a_{2r} which are equivalent to those given by Clenshaw's method for this problem (see [10, p. 501]).

2.2. *The Outer Region*. The Bessel function $J_n(z)$ may be expressed as

$$J_n(z) = \tfrac{1}{2}\left[H_n^{(1)}(z) + H_n^{(2)}(z)\right],$$

in terms of the Hankel functions of the first and second kinds which have the asymptotic forms

$$H_n^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-n\pi/2-\pi/4)},$$

$$H_n^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-n\pi/2-\pi/4)},$$

as $z \to \infty$. Any ray in the region $|z| \geqslant |\gamma|$ may be mapped onto $[0, 1]$ by the transformation $t = \gamma/z$ and the function

$$u^{(2)}(t) = \left(\frac{\pi z}{2}\right)^{1/2} \exp\left[i\left(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right)\right] H_n^{(2)}(z)$$

satisfies the differential equation

$$t^2 u'' + 2(t + i\gamma)u' + \left(\tfrac{1}{4} - n^2\right)u = 0, \tag{5}$$

where the prime denotes differentiation with respect to the real variable t, and $u^{(2)}(0) = 1$. Similarly, extraction of the asymptotically dominant term in $H_n^{(1)}(z)$ yields a function $u^{(1)}(t)$ which satisfies a differential equation differing from (5) only in the sign of $i\gamma$.

We used Clenshaw's method to calculate the coefficients of the expansions

$$u^{(i)}(t) = \sum_{r=0}^{\infty}{}' a_r^{(i)} T_r^*(t), \qquad i = 1, 2, \tag{6}$$

from which we can evaluate the Bessel functions of the first and second kinds, since

$$J_n(z) = (2\pi z)^{-1/2}\left[u^{(1)}(t)\exp\{i\alpha(z)\} + u^{(2)}(t)\exp\{-i\alpha(z)\}\right] \tag{7}$$

and

$$Y_n(z) = -i(2\pi z)^{-1/2}\left[u^{(1)}(t)\exp\{i\alpha(z)\} - u^{(2)}(t)\exp\{-i\alpha(z)\}\right], \tag{8}$$

where

$$\alpha(z) = z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi.$$

The expansions in (6) are particular cases of a series discussed by Luke [11, p. 88].

When z is real the differential equation (5) has only one solution of the required form and the coefficients can be found by backward recurrence from an arbitrary starting point and use of the initial condition for the differential equation. When $\operatorname{Im}\gamma > 0$ there is no longer just one solution since (5) is also satisfied by

$$w(t) = e^{2i\gamma/t}u(-t) \quad \text{for } t > 0.$$

It follows that any solution of the recurrence relations corresponds to a linear combination of $u(t)$ and $w(t)$. The unwanted solution $w(t)$ may be removed by solving a boundary-value problem; the recurrence relations are solved twice with independent initial values, and the known value of $J_n(\gamma)$ is used in addition to the condition $u(0) = 1$ to obtain the desired solution. In practice the extent to which the unwanted solution $w(t)$ enters will depend on the value of $\operatorname{Im}\gamma$. In producing the tables we solved the recurrence systems with two different starting values and decided by comparison of the results whether to solve an initial- or boundary-value problem.

3. Error Bounds. A real- or complex-valued function f of a real variable x may, under suitable conditions, be expressed on the interval $[-1, 1]$ as a Chebyshev series

$$f(x) = \sum_{r=0}^{\infty}{}' b_r T_r(x),$$

where

$$b_r = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_r(x)}{\sqrt{1-x^2}}\,dx.$$

In particular, if f is differentiable to all orders on $[-1, 1]$ and $|f^{(r)}(x)| \leqslant Q_r$ on that interval, the work of Elliott [6] shows that

$$|b_r| \leqslant \frac{Q_r}{2^{r-1}r!}.$$

To apply this result it is convenient to write Eq. (3) as

$$2^n n! z^{-n} J_n(z) = f(x) = \sum_{r=0}^{\infty}{}' a_{2r} T_r(x),$$

where

$$x = 2t^2 - 1 = 2(z/\gamma)^2 - 1.$$

Then

$$\frac{d^r f}{dx^r} = 2^n n!\left(\frac{\gamma}{2}\right)^{2r}\left(\frac{1}{z}\frac{d}{dz}\right)^r\left(\frac{J_n(z)}{z^n}\right)$$
$$= \frac{(-1)^r 2^n n!}{z^{n+r}}\left(\frac{\gamma}{2}\right)^{2r} J_{n+r}(z)$$

(see [15, p. 18]). Watson [15, p. 49] establishes the bound

(9) $$|J_m(z)| \leqslant \frac{|z|^m}{2^m m!}\exp|\operatorname{Im} z|.$$

Thus if $|z| \leqslant |\gamma|$ and $\arg z = \arg \gamma = \phi$, then

(10) $$|a_{2r}| \leqslant \frac{2n!|\gamma|^{2r}}{2^{4r}r!(n+r)!}\exp(|\gamma|\sin\phi).$$

An alternative bound on the coefficients is obtained by noting that, for $r \geqslant 1$,

$$a_{2r} = \frac{2(-1)^r \gamma^{2r} n!}{2^{4r} r!(n+r)!}\, {}_1F_2\left(r + \frac{1}{2}; n + r + 1, 2r + 1; -\frac{\gamma^2}{4}\right)$$

(see [11, p. 77]) so

$$|a_{2r}| \leqslant \frac{2n!|\gamma|^{2r}}{2^{4r}r!(n+r)!}\sum_{k=0}^{\infty}\frac{(n+r)!r!(2r+2k)!|\gamma|^{2k}}{(r+k)!(r+n+k)!(2r+k)!k!16^k}.$$

The ratio of successive terms in this series is

$$\frac{(2r+2k)(2r+2k-1)|\gamma|^2}{16(r+k)(r+n+k)(2r+k)k} < \frac{|\gamma|^2}{4(2r+k)k},$$

since $n > -\frac{1}{2}$, and therefore

$$(11) \qquad |a_{2r}| < \frac{2n!|\gamma|^{2r}}{2^{4r}r!(n+r)!}\left[1 + \sum_{k=1}^{\infty}\left\{\frac{|\gamma^2|}{4(2r+1)}\right\}^k \frac{1}{k!}\right]$$
$$= \frac{2n!|\gamma|^{2r}}{2^{4r}r!(n+r)!}\exp\left(\frac{1}{4}\frac{|\gamma|^2}{2r+1}\right).$$

Note that this bound, unlike (10), is independent of the angle ϕ. Table 1 gives some comparisons of these bounds with the values of $|a_{2r}|$ when $n = 1$.

TABLE 1
Values of $|a_{2r}|$ *for* $J_1(z)$ *when* $\gamma = 8\exp(i\phi)$, *and the upper bounds* (10) *and* (11).

ϕ	r	$\lvert a_{2r}\rvert$	Bounds from (10)	Bounds from (11)
0	5	7.3 (−3)	2.4 (−2)	1.0 (−1)
	10	7.4 (−9)	1.4 (−8)	3.1 (−8)
	20	1.2 (−26)	1.8 (−26)	2.6 (−26)
	30	8.2 (−49)	1.1 (−48)	1.4 (−48)
	35	4.9 (−61)	6.1 (−61)	7.7 (−61)
$\frac{\pi}{24}$	5	7.6 (−3)	6.7 (−2)	1.0 (−1)
	10	7.6 (−9)	4.1 (−8)	3.1 (−8)
	20	1.2 (−26)	5.0 (−26)	2.6 (−26)
	30	8.2 (−49)	3.0 (−48)	1.4 (−48)
	35	5.0 (−61)	1.7 (−60)	7.7 (−61)
$\frac{11}{24}\pi$	5	6.9 (−2)	66	1.0 (−1)
	10	2.7 (−8)	4.0 (−5)	3.1 (−8)
	20	2.5 (−26)	4.9 (−23)	2.6 (−26)
	30	1.4 (−48)	2.9 (−45)	1.4 (−48)
	35	7.6 (−61)	1.7 (−57)	7.7 (−61)

Suppose that the Chebyshev series for $2^n n! z^{-n} J_n(z)$ on the ray $\arg\gamma = \phi$ is truncated to give a polynomial of degree $2m$ and the resulting approximation is used in the domain

$$D = \{z; |z| \leqslant |\gamma|, \phi - \Theta \leqslant \arg z \leqslant \phi + \Theta\},$$

where Θ is given. Then the truncation error is

$$E_{m,n}(z) = \sum_{r=m+1}^{\infty} a_{2r}T_{2r}(z/\gamma),$$

and its magnitude is bounded by

$$2n!A \sum_{r=m+1}^{\infty}\left(\frac{|\gamma|}{4}\right)^{2r}\frac{|T_{2r}(e^{i\Theta})|}{r!(n+r)!},$$

with

$$A = \exp(|\gamma|\sin\phi) \quad \text{or} \quad \exp\left(\frac{1}{4}\frac{|\gamma|^2}{2m+3}\right).$$

Since (see Appendix 1)

$$|T_{2r}(e^{i\Theta})| \leq \tfrac{1}{2}\left(\rho^{2m+2} + \rho^{-(2m+2)}\right)\rho^{2(r-m-1)},$$

with

$$\rho = |\exp(i\Theta) + \sqrt{\exp(2i\Theta) - 1}|,$$

and

$$\frac{(m+1)!\,(n+m+1)!}{(m+1+k)!\,(n+m+1+k)!} \leq \frac{1}{(m+2)^k(n+m+2)^k},$$

it follows that

$$(12)\qquad |E_{m,n}(z)| \leq \frac{n!A}{(m+1)!\,(n+m+1)!}\left(\frac{|\gamma|}{4}\right)^{2m+2}\frac{\left(\rho^{2m+2} + \rho^{-(2m+2)}\right)}{1 - R_{m,n}},$$

where

$$R_{m,n} = \frac{|\gamma|^2\rho^2}{16(m+2)(m+n+2)}.$$

When $R_{m,n} \ll 1$ the truncation error is dominated by the first neglected term. For example, when $|\gamma| = 8$ and $\Theta = 7.5°$, $R_{22,0} \simeq 0.014$ and $R_{32,0} \simeq 0.007$.

The dependence of the rate of convergence on $|\gamma|$ is evident from the inequality (12). By decreasing $|\gamma|$ the rate of convergence in the inner region is increased, but at the expense of a slower rate of convergence in the outer region. Thus a compromise is necessary. We experimented with a number of different values of $|\gamma|$ and eventually chose $|\gamma| = 8$, as Clenshaw [3] did in the real case, having found no compelling reason to make a different choice.

The bound (12) applies to the inner region where $|z| \leq |\gamma|$. For $|z| \geq |\gamma|$ we do not have useful bounds for the coefficients of the expansions (6) but their asymptotic forms have been investigated by Miller [12] and by Luke [11]. The latter reference gives two slightly different asymptotic estimates. Tables 2 and 3 contain some comparisons of our calculated coefficients with these estimates; in every case to which we have applied them the estimate E_1 is better than E_2, but both are remarkably accurate.

TABLE 2

Values of $|a_r^{(1)}|$ for the Hankel function $H_0^{(1)}$, and estimates E_1 and E_2 obtained from Eqs. (2) and (3) on pp. 88–89 of [11].

ϕ	r	$a_r^{(1)}$	E_1	E_2
7.5°	20	3.39 (−22)	3.43 (−22)	3.85 (−22)
	30	5.11 (−29)	5.13 (−29)	5.49 (−29)
	40	4.49 (−35)	4.50 (−35)	4.72 (−35)
82.5°	20	1.67 (−23)	1.71 (−23)	1.53 (−23)
	30	4.80 (−31)	4.88 (−31)	4.57 (−31)
	40	8.88 (−38)	8.99 (−38)	8.60 (−38)

TABLE 3

Values of $|a_r^{(2)}|$ *for the Hankel function* $H_0^{(2)}$, *and estimates* E_1 *and* E_2 *obtained as in Table* 1.

ϕ	r	$a_r^{(2)}$	E_1	E_2
	20	1.22 (−21)	1.21 (−21)	1.33 (−21)
7.5°	30	3.69 (−28)	3.64 (−28)	3.84 (−28)
	40	6.10 (−34)	6.07 (−34)	6.29 (−34)
	20	1.24 (−16)	1.20 (−16)	1.08 (−16)
82.5°	30	8.79 (−21)	8.65 (−21)	8.14 (−21)
	40	1.84 (−24)	1.82 (−24)	1.75 (−24)

Since the functions of interest for $|z|>|\gamma|$ have a singularity at infinity the expansions (6) converge only for $0 \leqslant t \leqslant 1$. Thus, when polynomial approximations obtained by truncating these expansions are used for complex values of γ/z, the technique used to bound the truncation error when $|z| \leqslant |\gamma|$ is not applicable. Empirical evidence of the accuracy of the resulting approximation is mentioned in Section 4, where it is used in constructing Table 5.

4. The Coefficient Tables. The coefficients of the Chebyshev expansions (3) and (6), for $n = 0$ and 1 with $|\gamma| = 8$, which are tabulated in Appendix 2, were calculated in quadruple precision on an IBM 370 computer, and the results were rounded to double-precision form. For each value of n, the order of the Bessel function, there are six tables, one corresponding to each of the rays $\arg\gamma = 7.5°$ (15°) 82.5°. In each table the top set of figures corresponds to the expansion (3) and the two lower sets are the coefficients $a_r^{(1)}$ and $a_r^{(2)}$ to be used for $|z| > 8$.

The number of coefficients tabulated in each case, and the number of decimal digits to which they are quoted, were determined by the requirement that a certain accuracy be achievable in the calculation of the corresponding Bessel function in double-precision arithmetic. In the inner region the Chebyshev series converges rapidly, and the accuracy is limited only by rounding error. Table 4 shows the accuracy achievable for $|z| \leqslant 8$ with the tabulated coefficients. As $\arg\gamma$ increases the low-order coefficients increase in magnitude but

$$\sum_{r=0}^{\infty}{}' (-1)^r a_{2r} = 1$$

in all cases; the resulting loss of significant figures reduces the achievable accuracy as shown in the table.

TABLE 4

Maximum absolute error in using (3) *with the tabulated coefficients for* $|z| \leqslant 8$

arg z	$n = 0$	$n = 1$
0° − 15°	0.5, −14	0.5, −14
15° − 30°	1.0, −14	1.0, −14
30° − 45°	0.5, −13	0.5, −13
45° − 60°	1.0, −13	2.0, −13
60° − 75°	0.5, −12	0.5, −12
75° − 90°	0.5, −12	0.5, −12

In the outer region the number of terms needed to calculate the Hankel function $H_n^{(1)}(z)$ to a given accuracy is not sensitive to the value of $\arg \gamma$ in the first quadrant, but the number required in the expansion of $H_n^{(2)}(z)$ increases markedly as $\arg \gamma$ is increased. Table 5, which is based on a comparison with accurate values of $J_0(z)$ and $J_1(z)$, shows the accuracy achievable with the tabulated coefficients. For $75° < \arg z \leqslant 90°$ we have accepted a lower level of accuracy than elsewhere since the rate of decrease of $|a_r^{(2)}|$ has then become very slow; greater accuracy could be obtained in the evaluation of $H_n^{(1)}(z)$ alone.

TABLE 5

Maximum relative error in using (11) *with the tabulated coefficients for* $|z| \geqslant 8$

$\arg z$	$n = 0$	$n = 1$
$0° - 15°$	0.25, −14	0.25, −14
$15° - 30°$	0.25, −14	0.25, −14
$30° - 45°$	0.25, −14	0.25, −14
$45° - 60°$	0.5, −14	0.5, −14
$60° - 75°$	0.5, −14	0.5, −14
$75° - 90°$	0.1, −13	0.1, −13

The relative errors quoted in Table 5 are not applicable in the vicinity of the zeros of J_0 and J_1, all of which lie on the real axis. The tabulated coefficients are sufficiently accurate to permit calculations with absolute error no greater than 5×10^{-15} near the zeros.

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Appendix 1. A Bound for $|T_n(\zeta)|$ on a Sector. Here we establish a bound for the modulus of the Chebyshev polynomial $T_n(\zeta)$ on a sector D of the unit disc in which $|\arg \zeta| \leqslant \Theta \leqslant \pi/2$ radians, and we show that the bound is attained. Since $|T_n(\zeta)| = |T_n(\zeta^*)|$, it is only necessary to consider the upper half OAB of this sector, where O is the origin and A and B are the points $\zeta = 1$ and $\zeta = \exp(i\Theta)$, respectively. Furthermore the maximum modulus theorem shows that the maximum occurs on the boundary, i.e. on the radius OB or on the arc AB.

The polynomial $T_{2m}(\zeta)$ of even degree is an even function, and its zeros, all of which lie in $[-1, 1]$, may be labelled as $\pm \chi_k$ $(k = 1, 2, \ldots, m)$. Then

$$T_{2m}(\zeta) = 2^{2m-1} \prod_{k=1}^{m} (\zeta^2 - \chi_k^2)$$

and on the unit circle

$$|T_{2m}(e^{i\theta})|^2 = 2^{4m-2}\prod_{k=1}^{m}\left(1+\chi_k^4-2\chi_k^2\cos 2\theta\right).$$

Clearly this expression is a monotonically increasing function of θ for $0<\theta<\pi/2$. The maximum value on the arc AB therefore occurs at B and

$$|T_{2m}(e^{i\theta})|\geqslant 1 \quad \text{for } 0\leqslant\theta\leqslant\pi/2.$$

On the ray OB $\zeta = re^{i\Theta}$ and

$$|T_{2m}(re^{i\Theta})|^2 = 2^{4m-2}\prod_{k=1}^{m}\left(r^4+\chi_k^4-2r^2\chi_k^2\cos 2\Theta\right).$$

If $\pi/4\leqslant\Theta\leqslant\pi/2$ each term in the product is positive and monotonically increasing with r. On the other hand, if $0\leqslant\Theta<\pi/4$, the kth term in the product decreases, from the value χ_k^4 when $r=0$ to a positive minimum at $r=\chi_k(\cos 2\Theta)^{1/2}$ and thereafter increases with r. It follows that the product must take its maximum value at $r=0$ or 1. Since

$$|T_{2m}(0)| = 1 \leqslant |T_{2m}(e^{i\Theta})|,$$

we have, for the sector D, the bound

$$|T_{2m}(\zeta)|\leqslant|T_{2m}(e^{i\Theta})|.$$

The Chebyshev polynomial $T_{2m+1}(\zeta)$ of odd degree has a zero at the origin and may be written as

$$T_{2m+1}(\zeta) = 2^{2m}\zeta\prod_{k=1}^{m}\left(\zeta^2-y_k^2\right),$$

where $\pm y_k$ $(k=1,2,\dots,m)$ are the other zeros. Arguments analogous to those used above show that once again the maximum modulus on the sector D is attained at the point B.

We have therefore shown that on the sector $D=\{\zeta;\ |\zeta|\leqslant 1,\ |\arg\zeta|\leqslant\Theta\}$, where $0<\Theta\leqslant\pi/2$,

$$|T_n(\zeta)|\leqslant|T_n(e^{i\Theta})|.$$

The Chebyshev polynomial of degree n may be expressed as

$$T_n(\zeta) = \tfrac{1}{2}(w^n+w^{-n}),$$

where $w(\zeta) = \zeta + (\zeta^2 - 1)^{1/2}$. Therefore, on the sector D,

(A1) $$|T_n(\zeta)| \leqslant \tfrac{1}{2}(\rho^n + \rho^{-n}),$$

where $\rho = |e^{i\Theta} + (e^{2i\Theta} - 1)^{1/2}|$. Alternatively, it may be noted that the ellipse

$$\zeta = \tfrac{1}{2}(\rho e^{i\phi} + \rho^{-1}e^{-i\phi})$$

intersects the unit circle $|\zeta| = 1$ at the points $\zeta = \exp(\pm i\Theta)$. On this ellipse (see [5, p. 83]), and therefore throughout the sector D, the inequality (A1) holds.

It is also useful to note that since $\rho \geqslant 1$, for $0 \leqslant \Theta \leqslant \pi/2$,

$$|T_{n+k}(\zeta)| \leqslant \tfrac{1}{2}(\rho^n + \rho^{-n})\rho^k$$

in the sector OAB.

Appendix 2. Tables of Coefficients for the Chebyshev Expansions.

A: For $|z| \leqslant 8$,

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{2r}T_{2r}(t),$$

with $a_{2r} = b_{2r} + ic_{2r}$, where b_{2r} and c_{2r} are real,

$$t = \left(\frac{z}{8}\right)\exp(-i\phi), \qquad \phi = 7.5°(15°)82.5°.$$

B: For $|z| \geqslant 8$,

$$J_n(z) = \tfrac{1}{2}\left[H_n^{(1)}(z) + H_n^{(2)}(z)\right],$$

where

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp\{i\alpha(z)\} \sum_{r=0}^{\infty}{}' a_r^{(1)}T_r^*(t)$$

and

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp\{-i\alpha(z)\} \sum_{r=0}^{\infty}{}' a_r^{(2)}T_r^*(t),$$

with $a_r^{(m)} = b_r^{(m)} + ic_r^{(m)}$, where $b_r^{(m)}$ and $c_r^{(m)}$ are real, $\alpha(z) = z - \frac{1}{2}n\pi - \frac{1}{4}\pi$ and

$$t = \left(\frac{8}{z}\right)\exp(i\phi), \qquad \phi = 7.5°(15°)82.5°.$$

Coefficients for $J_0(z)$ $\phi = 7.5°$

r	b_{2r}	c_{2r}
0	0.40795869055667140D+00	-0.56442646929690940D-01
1	0.76307829250131400D-01	-0.61302046672716610D-01
2	0.29061006595818790D+00	-0.20911435561449690D+00
3	-0.41399519311709360D+00	-0.49959512745050740D-01
4	0.14405373906065260D+00	0.91255766225367240D-01
5	-0.22191868433940320D-01	-0.29516142867063310D-01
6	0.15821011678673100D-02	0.47897681004629970D-02
7	-0.59234157455040D-05	-0.47863710713778000D-03
8	-0.93656954962600D-05	0.32225706258654D-04
9	0.98007182750800D-06	-0.15271062036970D-05
10	-0.5884900586900D-07	0.51370384202D-07
11	0.2484184311700D-08	-0.11659304865D-08
12	-0.7916668530D-10	0.13002722270D-10
13	0.19713529D-11	0.2198956D-12
14	-0.3898610D-13	-0.1571364D-13
15	0.61288D-15	0.46989D-15
16	-0.749D-17	-0.991D-17
17	0.7D-19	0.16D-18

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.19972093278930000D+01	-0.15204871279421320D-01
1	-0.15163747475273600D-02	-0.75491091447177000D-02
2	-0.11839260114859D-03	0.56461738916240D-04
3	0.273922667088D-05	0.29782675602100D-05
4	0.828154997800D-07	-0.15861416551D-06
5	-0.101411905840D-07	-0.93704584100D-09
6	0.23128749800D-09	0.66236805400D-09
7	0.39197085D-10	-0.40313221D-10
8	-0.4908361D-11	-0.1388648D-11
9	0.126666D-12	0.497639D-12
10	0.393276D-13	-0.405715D-13
11	-0.65915D-14	-0.12920D-14
12	0.3370D-15	0.7854D-15
13	0.5995D-16	-0.987D-16
14	-0.1641D-16	0.154D-17
15	0.168D-17	0.181D-17
16	0.6D-19	-0.38D-18
17	-0.5D-19	0.3D-19
18	0.1D-19	0.0D+00

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.20012235425941850D+01	0.15617615943316040D-01
1	0.47484187103883D-03	0.78224985313066600D-02
2	-0.13720222605498D-03	0.92627235709000D-05
3	-0.657447136200D-07	-0.44209644988700D-05
4	0.20479952939D-06	0.19221566758D-07
5	-0.31171482170D-08	0.12009103702D-07
6	-0.81513436600D-09	-0.40600540900D-09
7	0.51121957D-10	-0.58999420D-10
8	0.4067520D-11	0.6416037D-11
9	-0.798944D-12	0.192784D-12
10	0.121137D-13	-0.965576D-13
11	0.108301D-13	0.61567D-14
12	-0.13761D-14	0.10111D-14
13	-0.4638D-16	-0.2443D-15
14	0.3671D-16	0.1065D-16
15	-0.431D-17	0.441D-17
16	-0.29D-18	-0.100D-17
17	0.18D-18	0.4D-19
18	-0.2D-19	0.2D-19

Coefficients for $J_0(z)$ $\phi = 22.5°$

r	b_{2r}	c_{2r}
0	0.1862388136525494D+01	0.1648612285982524D+00
1	0.1452638662518095D+01	-0.1254446663870422D+00
2	0.9954877760365573D+00	-0.1084422910850202D+01
3	-0.6836306791980399D+00	-0.5932359624072628D+00
4	-0.9210389403143698D-01	0.2761514578005225D+00
5	0.5432515528670650D-01	-0.1247638183934976D-01
6	-0.5312699741457737D-02	-0.4654474585672789D-02
7	-0.313387287174160D-04	0.636229272782623D-03
8	0.342873932912790D-04	-0.260135301745500D-04
9	-0.22152426452250D-05	-0.45992988672300D-06
10	0.47791149661400D-07	0.82401441947700D-07
11	0.98772468430D-09	-0.31346016043D-08
12	-0.855594184D-10	0.404926857D-10
13	0.214976920D-11	0.8482018D-12
14	-0.1804485D-13	-0.4496504D-13
15	-0.36612D-15	0.80228D-15
16	0.1328D-16	-0.464D-17
17	-0.18D-19	-0.9D-19

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.1993518105429556D+01	-0.1383201838361587D-01
1	-0.331819644425562D-02	-0.681426224169510D-02
2	-0.73441443030410D-04	0.10221388905934D-03
3	0.37371200900500D-05	0.31890016651D-06
4	-0.7508276389D-07	-0.14026220050D-06
5	-0.3998019834D-08	0.77109094080D-08
6	0.5571257220D-09	-0.98096302D-10
7	-0.32126299D-10	-0.28528229D-10
8	-0.112145D-12	0.3684397D-11
9	0.270613D-12	-0.222595D-12
10	-0.360021D-13	-0.51698D-14
11	0.23566D-14	0.33255D-14
12	0.862D-16	-0.4810D-15
13	-0.508D-16	0.359D-16
14	0.827D-17	0.122D-17
15	-0.74D-18	-0.92D-19
16	-0.1D-19	0.17D-18
17	0.2D-19	-0.2D-19
18	-0.4D-20	0.0D+00

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.2005317773826089D+01	0.1496870848194241D-01
1	0.254232929699090D-02	0.756832055889608D-02
2	-0.12017042742964D-03	0.80745227954180D-04
3	-0.34828585173300D-05	-0.34220986434200D-05
4	0.14370960389D-06	-0.19381878407D-06
5	0.13221804724D-07	0.830371007200D-08
6	-0.638103037D-09	0.106164402600D-08
7	-0.96834157D-10	-0.62761881D-10
8	0.7491277D-11	-0.9690533D-11
9	0.1023817D-11	0.1025303D-11
10	-0.1533926D-12	0.1084678D-12
11	-0.104736D-13	-0.241857D-13
12	0.39048D-14	-0.6596D-15
13	-0.6666D-16	0.6285D-15
14	-0.9772D-16	-0.4222D-16
15	0.1264D-16	-0.1393D-16
16	0.160D-17	0.308D-17
17	-0.66D-18	0.7D-19
18	0.39D-19	-0.127D-18
19	0.21D-19	0.18D-19
20	-0.5D-20	0.2D-20
21	0.0D+00	-0.1D-20

Coefficients for $J_0(z)$ $\phi = 37.5°$

r	b_{2r}	c_{2r}
0	0.6711927325390700D+01	0.8371270281818707D+01
1	0.7150083950901492 7D+01	0.5993532078953587D+01
2	0.6215984505542261D+01	-0.1948640063429697D+00
3	0.9176694981253303D+00	-0.2406514077822298D+01
4	-0.5677735659896193D+00	-0.3171937608557583D+00
5	-0.5298496757710103D-01	0.9281112758429071D-01
6	0.1099498309164805D-01	0.5300698477787309D-02
7	0.331554410809857D-03	-0.966000094346427D-03
8	-0.64158539295280D-04	-0.11644140366370D-04
9	-0.19262278496D-07	0.3279240906735D-05
10	0.130984600823D-06	-0.24389700433D-07
11	-0.16816077796D-08	-0.41353051819D-08
12	-0.10371152882D-09	0.7059725786D-10
13	0.2185934690D-11	0.2054503640D-11
14	0.312224550D-13	-0.533025000D-13
15	-0.10573080D-14	-0.3278130D-15
16	-0.11760D-17	0.17379D-16
17	0.239D-18	-0.41D-19

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.1990345351455046D+01	-0.1163923411861265D-01
1	-0.4845809321898998D-02	-0.569687679112094D-02
2	-0.1551745229249D-04	0.12064685290610D-03
3	0.281911293225D-05	-0.211764244359D-05
4	-0.14360704559D-06	-0.1795036670D-07
5	0.4906628595D-08	0.5773310055D-08
6	0.6628038D-11	-0.476793673D-09
7	-0.22669834D-10	0.26323062D-10
8	0.2907401D-11	-0.516505D-12
9	-0.230439D-12	-0.119535D-12
10	0.111511D-13	0.232199D-13
11	0.4999D-15	-0.27122D-14
12	-0.2263D-15	0.2203D-15
13	0.379D-16	-0.641D-17
14	-0.449D-17	-0.203D-17
15	0.36D-18	0.56D-18
16	0.0D+00	-0.9D-19
17	-0.1D-19	0.1D-19
18	0.2D-20	-0.1D-20

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.2009203666687719D+01	0.1321490775411509D-01
1	0.454197899406149D-02	0.674511798041522D-02
2	-0.6514986365456D-04	0.13806779265349D-03
3	-0.542811293493D-05	0.15640526452D-06
4	-0.11938798791D-06	-0.26193682934D-06
5	0.14755561664D-07	-0.13890379393D-07
6	0.1500431342D-08	0.935095139D-09
7	-0.65065841D-10	0.170509568D-09
8	-0.20990882D-10	-0.4971750D-11
9	0.4439555D-12	-0.2815298D-11
10	0.4098006D-12	0.540714D-13
11	-0.98505D-14	0.641445D-13
12	-0.106568D-13	-0.23368D-14
13	0.60873D-15	-0.18494D-14
14	0.32894D-15	0.16112D-15
15	-0.4228D-16	0.5855D-16
16	-0.1007D-16	-0.1091D-16
17	0.276D-17	-0.157D-17
18	0.183D-18	0.678D-18
19	-0.160D-18	-0.2D-20
20	0.12D-19	-0.36D-19
21	0.7D-20	0.5D-20
22	-0.2D-20	0.1D-20

Coefficients for $J_0(z)$ $\phi = 52.5°$

r	b_{2r}		c_{2r}
0	-0.2461560768007210D+02	,	0.4070818451974168D+02
1	-0.1367468043326338D+02	,	0.3675086860178412D+02
2	0.4205032398658339D+01	,	0.2036773969152743D+02
3	0.5862275976644791D+01	,	0.3554833613819136D+01
4	0.1320403453490864D+01	,	-0.6155346174253676D+00
5	0.1201136204349612D-01	,	-0.2107148576478635D+00
6	-0.1937249066303771D-01	,	-0.1031807319065724D-01
7	-0.131937658867523D-02	,	0.108686857430741D-02
8	0.32882094123651D-04	,	0.97769240370193D-04
9	0.4955120002488D-05	,	0.1008907163270D-06
10	0.66621081844D-07	,	-0.182232882643D-06
11	-0.494119407BD-08	,	-0.391067555 9D-08
12	-0.1422313056D-09	,	0.960901744D-10
13	0.11426229D-11	,	0.38663761D-11
14	0.81399050D-13	,	0.87296D-15
15	0.42964D-15	,	-0.136794D-14
16	-0.1941D-16	,	-0.1244D-16
17	-0.236D-18	,	0.194D-18

r	$b_r^{(1)}$		$c_r^{(1)}$
0	0.1987841124393791D+01	,	-0.8791119172615750D-02
1	-0.603832318930778D-02	,	-0.4281323482052180D-02
2	0.419038436955 2D-04	,	0.110843187326070D-03
3	0.6900399478 4D-06	,	-0.3292755752 14D-05
4	-0.92176460190D-07	,	0.983130865D-07
5	0.65807466430D-08	,	-0.1962225078D-08
6	-0.4095719820D-09	,	-0.92172009D-10
7	0.223506830D-10	,	0.19507579D-10
8	-0.855058D-12	,	-0.2201708D-11
9	-0.207947D-13	,	0.206915D-12
10	0.105995D-13	,	-0.169201D-13
11	-0.17497D-14	,	0.11015D-14
12	0.2272D-15	,	-0.279D-16
13	-0.256D-16	,	-0.819D-17
14	0.248D-17	,	0.223D-17
15	-0.18D-18	,	-0.40D-18
16	0.0D+00	,	0.6D-19

r	$b_r^{(2)}$		$c_r^{(2)}$
0	0.2012558266173131D+01	,	0.1039629190115090D-01
1	0.630090516560396D-02	,	0.535312537605316D-02
2	0.18188557607200D-04	,	0.15957531152138D-03
3	-0.3915033136280D-05	,	0.4607235879250D-05
4	-0.34706012464D-06	,	-0.90777364410D-08
5	-0.14600772838D-07	,	-0.22751745700D-07
6	0.1241443215D-08	,	-0.2342919618D-08
7	0.310669679D-09	,	0.17396328D-10
8	0.12382404D-10	,	0.40226535D-10
9	-0.5331182D-11	,	0.3513204D-11
10	-0.7654575D-12	,	-0.7392643D-12
11	0.1092308D-12	,	-0.1577829D-12
12	0.325606D-13	,	0.175800D-13
13	-0.316230D-14	,	0.686631D-14
14	-0.148926D-14	,	-0.64858D-15
15	0.15171D-15	,	-0.33195D-15
16	0.7562D-16	,	0.3957D-16
17	-0.1113D-16	,	0.1744D-16
18	-0.401D-17	,	-0.327D-17
19	0.984D-18	,	-0.904D-18
20	0.192D-18	,	0.298D-18
21	-0.89D-19	,	0.36D-19
22	-0.43D-20	,	-0.26D-19
23	0.76D-20	,	0.6D-21
24	-0.7D-21	,	0.21D-20
25	-0.5D-21	,	-0.4D-21
26	0.2D-21	,	-0.1D-21

Coefficients for $J_0(z)$ $\phi = 67.5°$

r	b_{2r}	c_{2r}
0	-0.1202323400400358D+03	-0.6669101823326376D+02
1	-0.9917022788744129D+02	-0.3914782653135659D+02
2	-0.4948255169843832D+02	-0.1480097867855856D+01
3	-0.1283463533243932D+02	0.6335161983868003D+01
4	-0.1392414563482231D+01	0.2349733484079579D+01
5	0.3587344263122892D-01	0.3618455517508258D+00
6	0.2462329695525135D-01	0.2540768358499780D-01
7	0.259890294040409D-02	0.27328614896885D-03
8	0.12385474204658D-03	-0.8647870606653D-04
9	0.1623381877430D-05	-0.6816403368627D-05
10	-0.128481885902D-06	-0.233223932872D-06
11	-0.7999500111D-08	-0.2671821660D-08
12	-0.2055159904D-09	0.938837691D-10
13	-0.19551295D-11	0.48015363D-11
14	0.375073D-13	0.959208D-13
15	0.16215D-14	0.7544D-15
16	0.259D-16	-0.89D-17

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.1986113634790794D+01	-0.546699311818147D-02
1	-0.685436198747312D-02	-0.265339495721182D-02
2	0.8721607286041D-04	0.7711845102482D-04
3	-0.158261426440D-05	-0.284998479958D-05
4	0.2298689940D-07	0.12642389127D-06
5	0.871932156D-09	-0.6372564570D-08
6	-0.164987383D-09	0.348725545D-09
7	0.17998231D-10	-0.19751442D-10
8	-0.1784492D-11	0.1076218D-11
9	0.173556D-12	-0.46317D-13
10	-0.168763D-13	-0.2451D-15
11	0.16416D-14	0.4788D-15
12	-0.1581D-15	-0.964D-16
13	0.147D-16	0.154D-16
14	-0.126D-17	-0.225D-17
15	0.9D-19	0.32D-18
16	-0.0D+00	-0.4D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.2015049931231076D+01	0.666353096322206D-02
1	0.762973668637113D-02	0.345532575581624D-02
2	0.10605258903314D-03	0.12976477119246D-03
3	0.114288375937D-05	0.656978407586D-05
4	-0.16742326288D-06	0.38315240024D-06
5	-0.32534134114D-07	0.17113269862D-07
6	-0.4003857380D-08	-0.1211086898D-08
7	-0.221712559D-09	-0.535414949D-09
8	0.52177757D-10	-0.77367965D-10
9	0.16801458D-10	0.16985555D-11
10	0.10220395D-11	0.31926549D-11
11	-0.5720293D-12	0.4330504D-12
12	-0.1302844D-12	-0.987470D-13
13	0.164144D-13	-0.356915D-13
14	0.954190D-14	0.258140D-14
15	-0.36883D-15	0.256325D-14
16	-0.70180D-15	-0.4415D-16
17	0.399D-17	-0.19714D-15
18	0.5692D-16	0.685D-18
19	-0.597D-18	0.16862D-16
20	-0.5105D-17	-0.450D-18
21	0.266D-18	-0.1570D-17
22	0.486D-18	0.136D-18
23	-0.638D-19	0.150D-18
24	-0.458D-19	-0.282D-19
25	0.119D-19	-0.135D-19
26	0.37D-20	0.49D-20
27	-0.19D-20	0.91D-21
28	-0.16D-21	-0.72D-21
29	0.26D-21	0.1D-22
30	-0.3D-22	0.9D-22
31	-0.3D-22	-0.2D-22
32	0.1D-22	-0.1D-22

Coefficients for $J_0(z)$ $\phi = 82.5°$

r	b_{2r}	c_{2r}
0	0.147863300779034OD+03	-0.186858125235421OD+03
1	0.104157682053982 1D+03	-0.144934353767633 2D+03
2	0.377737160290048 3D+02	-0.681007263436768 7D+02
3	0.715106845782556 4D+01	-0.199421710416306 9D+02
4	0.587189474637464 9D+00	-0.379459912862737 1D+01
5	-0.273368162925240 1D-01	-0.489446001776023 5D+00
6	-0.126332983854383 0D-01	-0.443441168500557 6D-01
7	-0.161111416812855D-02	-0.289444888751854D-02
8	-0.12682388318316D-03	-0.13765284525622D-03
9	-0.71010473576560D-05	-0.47052276012430D-05
10	-0.3002448370030D-06	-0.1064111996390D-06
11	-0.9908641501D-08	-0.9276122284D-09
12	-0.2602748412D-09	0.409400746D-10
13	-0.5494550 2D-11	0.23689420D-11
14	-0.930979D-13	0.721757D-13
15	-0.12406D-14	0.16058D-14
16	-0.121D-16	0.284D-16

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.1985232771500772D+01	-0.1854509999676580D-02
1	-0.7268436398177980D-02	-0.898533357041790D-03
2	0.1120504891887 9D-03	0.27549038846570D-04
3	-0.300955145505 0D-05	-0.111295795257 0D-05
4	0.11225446153D-06	0.5604713539D-07
5	-0.5246839364D-08	-0.3349880947D-08
6	0.289888972D-09	0.229698742D-09
7	-0.18228529D-10	-0.17639901D-10
8	0.1269652D-11	0.1490587D-11
9	-0.95903D-13	-0.136739D-12
10	0.77137D-14	0.134747D-13
11	-0.6490D-15	-0.14144D-14
12	0.560D-16	0.1570D-15
13	-0.48D-17	-0.183D-16
14	0.39D-18	0.22D-17
15	-0.3D-19	-0.29D-18
16	0.0D+00	0.4D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.2016386488605358D+01	0.2297755376037110D-02
1	0.835169553995907D-02	0.119637438016248D-02
2	0.16445031237631D-03	0.5050430885251D-04
3	0.6346545763420D-05	0.3288389205410D-05
4	0.3723520636 2D-06	0.3174605313 6D-06
5	0.25928705699D-07	0.42651973225D-07
6	0.639050288D-09	0.6927994043D-08
7	-0.600617041D-09	0.1056296040D-08
8	-0.237168859D-09	0.72957770D-10
9	-0.47230421D-10	-0.31181667D-10
10	0.181992D-13	-0.140132703D-10
11	0.32628649D-11	-0.17443959D-11
12	0.8104444D-12	0.6346405D-12
13	-0.9278265D-13	0.2845294D-12
14	-0.9034806D-13	-0.21917D-14
15	-0.602460D-14	-0.2756387D-13
16	0.832652D-14	-0.360423D-14
17	0.159220D-14	0.253840D-14
18	-0.79206D-15	0.63277D-15
19	-0.239865D-15	-0.255793D-15
20	0.86190D-16	-0.88850D-16
21	0.32508D-16	0.30429D-16
22	-0.11253D-16	0.11793D-16
23	-0.42381D-17	-0.43386D-17
24	0.17310D-17	-0.15017D-17
25	0.5196D-18	0.7082D-18
26	-0.2944D-18	0.1726D-18
27	-0.532D-19	-0.1233D-18
28	0.5163D-19	-0.1406D-19
29	0.229D-20	0.2144D-19
30	-0.876D-20	-0.62D-21
31	0.94D-21	-0.349D-20
32	0.134D-20	0.68D-21

Coefficients for $J_1(z)$ $\phi = 7.5°$

r	b_{2r}	c_{2r}
0	0.33165086130654000D+00	0.48593997430256700D-02
1	-0.28020674473706020D+00	0.81346848297617300D-01
2	0.34611125276606390D+00	-0.14881387653454340D-01
3	-0.16691424961145510D+00	-0.57255339954541560D-01
4	0.36569355422691320D-01	0.29268693615916090D-01
5	-0.41502924047159 1D-02	-0.63928897467874 7D-02
6	0.22555696529757 0D-03	0.81028550263970 4D-03
7	0.1520971088304D-05	-0.6726555181285 8D-04
8	-0.1246498422240D-05	0.3894514443776D-05
9	0.109089809549D-06	-0.162227191454D-06
10	-0.57752513455D-08	0.48661474047D-08
11	0.2194574620D-09	-0.9913067638D-10
12	-0.637623153D-11	0.96680976D-12
13	0.14608098D-12	0.1785835D-13
14	-0.267651D-14	-0.110782D-14
15	0.39193D-16	0.30552D-16
16	-0.448D-18	-0.601D-18
17	0.4D-20	0.9D-20

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.2007383366905616D+01	0.4601490703152938D-01
1	0.3896804586714230D-02	0.2292296248522280D-01
2	0.201499491862720D-03	-0.890040535636 8D-04
3	-0.374737902852D-05	-0.431255309874D-05
4	-0.11305222801D-06	0.20279908120D-06
5	0.12481528476D-07	0.1487939433D-08
6	-0.255755564D-09	-0.797692249D-09
7	-0.46865891D-10	0.45860127D-10
8	0.5583867D-11	0.1726258D-11
9	-0.129957D-12	-0.565616D-12
10	-0.449815D-13	0.443428D-13
11	0.72691D-14	0.15922D-14
12	-0.3519D-15	-0.8713D-15
13	-0.6775D-16	0.1065D-15
14	0.1786D-16	-0.131D-17
15	-0.178D-17	-0.199D-17
16	-0.7D-19	0.41D-18
17	0.6D-19	-0.3D-19
18	-0.1D-19	0.0D+00

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.1995237786481450D+01	-0.4670794665173130D-01
1	-0.2153532661257880D-02	-0.2338268704211800D-01
2	0.228122745261270D-03	-0.225193445029 3D-04
3	0.28165972275D-06	0.619274830695D-05
4	-0.26463136935D-06	-0.16837889996D-07
5	0.3388923169D-08	-0.14837345834D-07
6	0.981138453D-09	0.454002406D-09
7	-0.57357445D-10	0.70069087D-10
8	-0.4835022D-11	-0.7185141D-11
9	0.892607D-12	-0.2406758D-12
10	-0.106927D-13	0.1077711D-12
11	-0.121157D-13	-0.64884D-14
12	0.14794D-14	-0.11442D-14
13	0.5686D-16	0.2646D-15
14	-0.4001D-16	-0.1060D-16
15	0.454D-17	-0.486D-17
16	0.34D-18	0.107D-17
17	-0.19D-18	-0.4D-19
18	0.2D-19	-0.3D-19

Coefficients for $J_1(z)$ $\phi = 22.5°$

r	b_{2r}	c_{2r}
0	0.40374947400739990D+00	0.2903058949852946D+00
1	-0.8813810817561304D-01	0.4743506827181057D+00
2	0.6394271028327639D+00	0.1027687902694894D-01
3	-0.1195814266318472D+00	-0.2610087927618932D+00
4	-0.3773369287973807D-01	0.5821591694659590D-01
5	0.1055985377013672D-01	-0.6129362732768200D-03
6	-0.7442535185312140D-03	-0.8288579228032100D-03
7	-0.1193106383697100D-04	0.8521693352784300D-04
8	0.4260341241010400D-05	-0.2784666466510200D-05
9	-0.2300544040736D-06	-0.6145026115430D-07
10	0.4167805094D-08	0.8019031430D-08
11	0.9575689391D-10	-0.2673302181D-09
12	-0.689078402D-11	0.298802975D-11
13	0.15591706D-12	0.66674770D-13
14	-0.115575D-14	-0.309846D-14
15	-0.24464D-16	0.50679D-16
16	0.801D-18	-0.263D-18
17	-0.99D-20	-0.52D-20

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.2018796704796675D+01	0.4230504848413659D-01
1	0.9532265270458230D-02	0.2098377636981585D-01
2	0.1285296782799800D-03	-0.1695262623817100D-03
3	-0.5295345734020D-05	-0.58359414058D-06
4	0.9360748744D-07	0.18527832676D-06
5	0.5052240130D-08	-0.9453333411D-08
6	-0.670794303D-09	0.103004349D-09
7	0.36916323D-10	0.34204738D-10
8	0.215557D-12	-0.4235536D-11
9	-0.311964D-12	0.246442D-12
10	0.402750D-13	0.65719D-14
11	-0.25509D-14	-0.37450D-14
12	-0.1047D-15	0.5289D-15
13	0.563D-16	-0.384D-16
14	-0.899D-17	-0.148D-17
15	0.78D-18	0.101D-17
16	0.1D-19	-0.19D-18
17	-0.2D-19	0.2D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.1983144046239353D+01	-0.4421028253377329D-01
1	-0.8238685627909040D-02	-0.2224910385820393D-01
2	0.1944418038450500D-03	-0.1396187004693700D-03
3	0.4991574028530D-05	0.4605779264900D-05
4	-0.17598086096D-06	0.25361396310700D-06
5	-0.16410604625D-07	-0.9605643249D-08
6	0.713440236D-09	-0.1274469679D-08
7	0.113756062D-09	0.68913239D-10
8	-0.8157977D-11	0.11230241D-10
9	-0.1178261D-11	-0.1112879D-11
10	0.1663028D-12	-0.1248904D-12
11	0.122363D-13	0.262171D-13
12	-0.42352D-14	0.8353D-15
13	0.5397D-16	-0.6828D-15
14	0.10651D-15	0.4275D-16
15	-0.1318D-16	0.1530D-16
16	-0.180D-17	-0.325D-17
17	0.70D-18	-0.9D-19
18	-0.4D-19	0.14D-18
19	-0.2D-19	-0.2D-19
20	0.6D-20	-0.3D-20
21	0.0D+00	0.1D-20

Coefficients for $J_1(z)$ $\phi = 37.5°$

r	b_{2r}	c_{2r}
0	-0.4381621831242275D+00	0.2377738202865119D+01
1	-0.2688522523108127D+00	0.2574730924661963D+01
2	0.1471809508094292D+01	0.1038934235972632D+01
3	0.4414432465670281D+00	-0.4525763182192840D+00
4	-0.9344331075105422D-01	-0.9300501469646787D-01
5	-0.1189301113931178D-01	0.1375504626057259D-01
6	0.1475622233424320D-02	0.9960303086135680D-03
7	0.5567967447799950D-04	-0.1175855980268730D-03
8	-0.7103492388404D-05	-0.1953024252153D-05
9	-0.2598056514500D-07	0.3317296245340D-06
10	0.1217319690450D-07	-0.1518404241300D-08
11	-0.1238957347D-09	-0.3551948597D-09
12	-0.8286130380D-11	0.5130534230D-11
13	0.151852620D-12	0.153817430D-12
14	0.221351D-14	-0.351061D-14
15	-0.65918D-16	-0.22537D-16
16	-0.96D-19	0.1027D-17
17	0.13D-19	-0.2D-20

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.2028791098155013D+01	0.3591612354884225D-01
1	0.1443051867298572D-01	0.1775074481952528D-01
2	0.3069232817234D-04	-0.2044110009761 5D-03
3	-0.4083612502260D-05	0.2940992945130D-05
4	0.1877246205200D-06	0.2713166071D-07
5	-0.5978962792D-08	-0.7302555923D-08
6	-0.18089065D-10	0.573947931D-09
7	0.27168204D-10	-0.30487287D-10
8	-0.32500090D-11	0.543316D-12
9	0.260094D-12	0.140155D-12
10	-0.121490D-13	-0.263161D-13
11	-0.6018D-15	0.30150D-14
12	0.2537D-15	-0.2401D-15
13	-0.417D-16	0.647D-17
14	0.487D-17	0.228D-17
15	-0.38D-18	-0.61D-18
16	0.0D+00	0.10D-18
17	0.0D+00	-0.1D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.1971951375310728D+01	-0.3854849627726323D-01
1	-0.1393142345701756D-01	-0.1950284067059785D-01
2	0.1001872131593 5D-03	-0.2293433297216 6D-03
3	0.747546335277D-05	-0.441571489290D-06
4	0.161687744000D-06	0.327397600030D-06
5	-0.1728385198 2D-07	0.173328798570D-07
6	-0.1789327324D-08	-0.1037522455D-08
7	0.68189938D-10	-0.197340946D-09
8	0.23787157D-10	0.4845001D-11
9	-0.3952418D-12	0.3142357D-11
10	-0.4524883D-12	-0.455607D-13
11	0.86229D-14	-0.702992D-13
12	0.116235D-13	0.21883D-14
13	-0.59772D-15	0.20121D-14
14	-0.35772D-15	-0.16269D-15
15	0.4340D-16	-0.6380D-16
16	0.1104D-16	0.1132D-16
17	-0.288D-17	0.175D-17
18	-0.213D-18	-0.712D-18
19	0.169D-18	-0.2D-20
20	-0.11D-19	0.38D-19
21	-0.8D-20	-0.5D-20
22	0.2D-20	-0.1D-20

Coefficients for $J_1(z)$ $\phi = 52.5°$

r	b_{2r}	c_{2r}
0	-0.10940927246808720D+02	0.39573159179575660D+01
1	-0.83491278991613340D+01	0.57507937113324470D+01
2	-0.11814570335990530D+01	0.48915414875917930D+01
3	0.94356376140535280D+00	0.10341395452803470D+01
4	0.25587248917750550D+00	-0.55395808501553490D-01
5	0.72574138894625100D-02	-0.31960179317434170D-01
6	-0.24321261236502400D-02	-0.17271417105202500D-02
7	-0.17137186813513000D-03	0.11417503477952000D-03
8	0.26737517545940D-05	0.10914150814442D-04
9	0.48277604122700D-06	0.42624040486600D-07
10	0.68598342639D-08	-0.15901867321800D-07
11	-0.389590098200D-09	-0.340166654300D-09
12	-0.111710540D-10	0.68394653D-11
13	0.708144D-13	0.2758958D-12
14	0.537090D-14	0.22780D-15
15	0.2899D-16	-0.8401D-16
16	-0.106D-17	-0.75D-18

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.20367983766087490D+01	0.27315204652010100D-01
1	0.18331858468088390D-01	0.13462549102226010D-01
2	-0.68520086250330D-04	-0.19022164360953D-03
3	-0.10584513522600D-05	0.47018247084200D-05
4	0.12302359419D-06	-0.12755829863D-06
5	-0.8254885221D-08	0.2340962569D-08
6	0.493157364D-09	0.117669603D-09
7	-0.260569410D-10	-0.233152030D-10
8	0.959168D-12	0.2559386D-11
9	0.264198D-13	-0.235753D-12
10	-0.121633D-13	0.189479D-13
11	0.19676D-14	-0.12096D-14
12	-0.2521D-15	0.283D-16
13	0.281D-16	0.930D-17
14	-0.269D-17	-0.247D-17
15	0.19D-18	0.43D-18
16	0.0D+00	-0.6D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.19625255239727340D+01	-0.29985571358873980D-01
1	-0.18777882455333770D-01	-0.15244996212026010D-01
2	-0.35879742859880D-04	-0.25862117013952D-03
3	0.518165803215D-05	-0.643560379780D-05
4	0.4362625184100D-06	-0.100977896D-08
5	0.1828276804D-07	0.26684890183D-07
6	-0.1353252808D-08	0.2759792835D-08
7	-0.352779116D-09	-0.8394561D-11
8	-0.15273691D-10	-0.44424333D-10
9	0.5749189D-11	-0.4063714D-11
10	0.86190190D-12	0.7799185D-12
11	-0.1128424D-12	0.1747626D-12
12	-0.356604D-13	-0.178139D-13
13	0.31568D-14	-0.74607D-14
14	0.160941D-14	0.64268D-15
15	-0.15048D-15	0.35751D-15
16	-0.8130D-16	-0.3951D-16
17	0.1121D-16	-0.1875D-16
18	0.4325D-17	0.3326D-17
19	-0.1007D-17	0.978D-18
20	-0.210D-18	-0.306D-18
21	0.93D-19	-0.40D-19
22	0.5D-20	0.28D-19
23	-0.8D-20	-0.4D-21
24	0.7D-21	-0.22D-20
25	0.6D-21	0.4D-21
26	-0.2D-21	0.1D-21

Coefficients for $J_1(z)$ $\phi = 67.5°$

r	b_{2r}	c_{2r}
0	-0.210621121525945 6D+02	-0.275431917019071 7D+02
1	-0.196180628959527 1D+02	-0.172879727608045 1D+02
2	-0.104515503970975 4D+02	-0.309178314189170 3D+01
3	-0.264663258735315 9D+01	0.730044786975625 0D+00
4	-0.285230139848096 4D+00	0.355083117295307 7D+00
5	-0.5343267182445D-03	0.531180864630537D-01
6	0.28912211970889D-02	0.35458698775011D-02
7	0.2976409392042D-03	0.5221804892017D-04
8	0.1341724067484D-04	-0.828279090371D-05
9	0.184524261950D-06	-0.639648070560D-06
10	-0.1037542846D-07	-0.2074430335D-07
11	-0.6352672683D-09	-0.236477436D-09
12	-0.15525173D-10	0.65476190D-11
13	-0.144468D-12	0.330266D-12
14	0.23006D-14	0.62995D-14
15	0.983D-16	0.482D-16
16	0.15D-17	-0.48D-18

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.2042382662246676D+01	0.1706677977525020D-01
1	0.2104112981445151D-01	0.8395722616881160D-02
2	-0.147938414122670D-03	-0.133378619664030D-03
3	0.22354213411 4D-05	0.411336112370D-05
4	-0.2985723897D-07	-0.1668754733 7D-06
5	-0.1157164658D-08	0.7969670301D-09
6	0.202929956D-09	-0.420369494D-09
7	-0.21424702D-10	0.23160073D-10
8	0.2077355D-11	-0.1232505D-11
9	-0.198654D-12	0.51540D-13
10	0.190571D-13	0.4045D-15
11	-0.18329D-14	-0.5475D-15
12	0.1748D-15	0.1081D-15
13	-0.161D-16	-0.170D-16
14	0.137D-17	0.248D-17
15	-0.9D-19	-0.35D-18
16	0.0D+00	0.5D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.1955677216254703D+01	-0.1904897022727918D-01
1	-0.2233415055424814D-01	-0.9722608982105 83D-02
2	-0.174643454392940D-03	-0.206532519784060D-03
3	-0.172100011196D-05	-0.886788814135D-05
4	0.19741967605D-06	-0.48123495844D-06
5	0.37970803695D-07	-0.21249665620D-07
6	0.462928255 3D-08	0.125561369 1D-08
7	0.267353409D-09	0.592851561D-09
8	-0.54857193D-10	0.87333730D-10
9	-0.18409935D-10	-0.12434736D-11
10	-0.12227410D-11	-0.34187594D-11
11	0.5992116D-12	-0.4862340D-12
12	0.1427909D-12	0.1008775D-12
13	-0.1620144D-13	0.385474 6D-13
14	-0.1019941D-13	-0.240390D-14
15	0.30210D-15	-0.271903D-14
16	0.74026D-15	0.2302D-16
17	0.227D-17	0.20711D-15
18	-0.5964D-16	0.111D-17
19	0.93D-19	-0.17645D-16
20	0.5340D-17	0.312D-18
21	-0.229D-18	0.1643D-17
22	-0.510D-18	-0.127D-18
23	0.618D-19	-0.158D-18
24	0.483D-19	0.279D-19
25	-0.120D-19	0.143D-19
26	-0.40D-20	-0.49D-20
27	0.19D-20	-0.99D-21
28	0.18D-21	0.74D-21
29	-0.27D-21	0.0D+00
30	0.3D-22	-0.9D-22
31	0.3D-22	0.2D-22
32	-0.1D-22	0.9D-23

Coefficients for $J_1(z)$ $\phi = 82.5°$

r	b_{2r}	c_{2r}
0	0.43705618725051955D+02	-0.41923771467787840+02
1	0.28604219418342190+02	-0.31584658429505430+02
2	0.91755271882928420+01	-0.13664310564945430+02
3	0.15480330031503700+01	-0.35828118036049390+01
4	0.12391565686220680+00	-0.60544156619452080+00
5	-0.1262998345261220-02	-0.69486323969685970-01
6	-0.1425105567103680-02	-0.56367882215703700-02
7	-0.174407541264000-03	-0.33202506859077D-03
8	-0.127185649761980-04	-0.14370784790881D-04
9	-0.656968879965D-06	-0.451319317131D-06
10	-0.25679302333D-07	-0.9513692255D-08
11	-0.78638699370-09	-0.8329726110-10
12	-0.1924753230-10	0.28195300D-11
13	-0.38019760-12	0.1597702D-12
14	-0.6052780-14	0.461407D-14
15	-0.7612D-16	0.9707D-16
16	-0.70D-18	0.162D-17

r	$b_r^{(1)}$	$c_r^{(1)}$
0	0.20452491167887920+01	0.58031920924712 5D-02
1	0.22428260562066170-01	0.2852078483063201-02
2	-0.19182056701972D-03	-0.47825105695510-04
3	0.43224303415 0D-05	0.16135589452 2D-05
4	-0.14789697254D-06	-0.7437988953D-07
5	0.6564564386D-08	0.4216721527D-08
6	-0.350355782D-09	-0.279114576D-09
7	0.21488523D-10	0.20899693D-10
8	-0.1468695D-11	-0.1732724D-11
9	0.109291D-12	0.156602D-12
10	-0.86837D-14	-0.152483D-13
11	0.7231D-15	0.15848D-14
12	-0.618D-16	-0.1745D-15
13	0.526D-17	0.202D-16
14	-0.42D-18	-0.246D-17
15	0.3D-19	0.31D-18
16	0.0D+00	-0.4D-19

r	$b_r^{(2)}$	$c_r^{(2)}$
0	0.19520627426271020+01	-0.65353600348163 4D-02
1	-0.24225147172738180-01	-0.33431632636719 8D-02
2	-0.26465807685359D-03	-0.79494560674590-04
3	-0.857286573398D-05	-0.435425832369D-05
4	-0.46310623549D-06	-0.38551739322D-06
5	-0.31162115952D-07	-0.49308253522D-07
6	-0.906636330D-09	-0.7815796789D-08
7	0.631557827D-09	-0.1188133760D-08
8	0.256266867D-09	-0.87827546D-10
9	0.52146566D-10	0.31958917D-10
10	0.5019330D-12	0.150191532D-10
11	-0.34103945D-11	0.19803078D-11
12	-0.8827132D-12	-0.6424836D-12
13	0.8743925D-13	-0.3037824D-12
14	0.9512234D-13	-0.98477D-15
15	0.730658D-14	0.2869353D-13
16	-0.858334D-14	0.406359D-14
17	-0.174955D-14	-0.259425D-14
18	0.80352D-15	-0.68596D-15
19	0.25790D-15	0.25796D-15
20	-0.86561D-16	0.95040D-16
21	-0.34668D-16	-0.30496D-16
22	0.11276D-16	-0.12560D-16
23	0.45152D-17	0.43543D-17
24	-0.17419D-17	0.16032D-17
25	-0.5571D-18	-0.7150D-18
26	0.2984D-18	-0.1865D-18
27	0.584D-19	0.1254D-18
28	-0.527D-19	0.159D-19
29	-0.295D-20	-0.220D-19
30	0.902D-20	0.40D-21
31	-0.87D-21	0.361D-20
32	-0.139D-20	-0.67D-21

Mathematics Department
University of Durham
Durham, England

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