

On the Integral $\int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx$

By K. S. Kölbig

Abstract. A formula is derived for the integral in the title which allows easy evaluation by formula manipulation on a computer.

1. Introduction. In a monograph on generalized polylogarithms, and in a paper on series of reciprocal powers, Nielsen [7], [8] remarked that

$$(1) \quad r_{np} = \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx \quad (n \geq 0, p \geq 0)$$

can be expressed as π times a polynomial in $\eta(q)$ ($1 \leq q \leq n + p$), with rational coefficients, where

$$(2) \quad \eta(q) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^q} = \begin{cases} \log 2 & (q = 1), \\ (1 - 2^{1-q})\zeta(q) & (q > 1). \end{cases}$$

This polynomial is homogeneous of degree $n + p$ if one considers $\eta(q)$ to be of degree q , and if the degree of a product is the sum of the degrees of its factors. $\zeta(q)$ is the Riemann zeta function for integer arguments. Since $r_{np} = r_{pn}$, it is sufficient to consider $n \geq p \geq 0$.

In an earlier paper [4], the following complicated closed expression for r_{np} was derived:

$$(3) \quad r_{np} = \frac{\pi n! p!}{2^{n+p+1}} \sum_{k=1}^{n+p} \frac{1}{k!} \sum_{\{p_i\}} \sum_{\{n_i\}} f(p_1, n_1) \cdots f(p_k, n_k),$$

where the innermost sums run over all sets $\{p_i\}, \{n_i\}$, which satisfy the conditions

$$(4) \quad p_i \geq 0, \quad \sum_{i=1}^k p_i = p; \quad n_i \geq 0, \quad \sum_{i=1}^k n_i = n,$$

and where the function $f(r, s)$ is given by

$$(5) \quad f(r, s) = (1 - \delta_{0r})(1 - \delta_{0s}) \frac{(-1)^{r+s+1}}{r+s} \binom{r+s}{r} \zeta(r+s) + |\delta_{0r} - \delta_{0s}| \zeta(r+s),$$

where

$$(6) \quad \xi(q) = \begin{cases} -2 \log 2, & q = 1, \\ (-1)^q \frac{2^q - 2}{2} \zeta(q), & q > 1, \end{cases}$$

and δ_{ij} is the Kronecker symbol.

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The actual computation of r_{np} from (3), even for small n and p , is quite complicated. Explicit expressions for r_{np} ($1 \leq n \leq 4, 0 \leq p \leq n$) in terms of $\eta(q)$, computed from (3), have been given in [4]. It may be noted that, in the relevant handbooks, only expressions for r_{n0} or r_{0n} with $n = 1, 2$ are listed.*

2. An Expression for r_{np} . It is the purpose of this note to derive another expression for r_{np} which is well-adapted to evaluation by a formula-manipulation system such as REDUCE [2]. As in the derivation of (3), we start from the definition (1) and, after substituting $t = \cos^2 x$ in (1), express r_{np} as a derivative of Euler's beta function:

$$(7) \quad r_{np} = \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \int_0^1 t^{\beta-1/2} (1-t)^{\alpha-1/2} dt \Big|_{\alpha=\beta=0}$$

$$= \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \frac{\Gamma(\frac{1}{2} + \alpha) \Gamma(\frac{1}{2} + \beta)}{\Gamma(1 + \alpha + \beta)} \Big|_{\alpha=\beta=0}$$

We introduce the power series [6]

$$(8) \quad \Gamma(1 + x) = \sum_{k=0}^{\infty} b_k x^k \quad (|x| < 1),$$

$$1/\Gamma(1 + x) = \sum_{k=0}^{\infty} a_k x^k \quad (|x| < \infty),$$

where $a_0 = b_0 = 1$,

$$(9) \quad a_k = -\frac{1}{k} \sum_{m=1}^k (-1)^m \zeta(m) a_{k-m},$$

$$b_k = \frac{1}{k} \sum_{m=1}^k (-1)^m \zeta(m) b_{k-m} \quad (k > 0),$$

and $\zeta(1) = \gamma$ (Euler's constant), $\zeta(m) = \zeta(m)$ for $m \geq 2$. A direct approach by setting $x = -\frac{1}{2} + \alpha$ and $x = -\frac{1}{2} + \beta$ in (8) and then carrying out the differentiations in (7) does not lead to a satisfactory result; the resulting expression would contain infinite series, and the expected quantities π and $\log 2$ would appear only implicitly. We therefore take a different route and apply the duplication formula [5]

$$(10) \quad \Gamma(2x) = \frac{1}{\sqrt{2\pi}} 2^{2x-1/2} \Gamma(x) \Gamma\left(\frac{1}{2} + x\right),$$

so that from (7)

$$(11) \quad r_{np} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \frac{2^{-2\alpha-2\beta}}{\Gamma(1 + \alpha + \beta)} \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)} \frac{\Gamma(1 + 2\beta)}{\Gamma(1 + \beta)} \Big|_{\alpha=\beta=0}$$

We have separated out the factor π , and the terms involving $\log 2$ will arise from the differentiation of $2^{-2\alpha-2\beta}$.

The differentiations with respect to α and β may now be carried out in (11). The result is a complicated six-fold sum in a form suitable for evaluation by formula

*Nielsen [8] made a similar comment.

manipulation. A much simpler expression can, however, be obtained as follows. We rewrite (11) as

$$(12) \quad r_{np} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta^n \partial \alpha^p} \frac{G(\alpha)G(\beta)}{\Gamma(1 + \alpha + \beta)} \Big|_{\alpha=\beta=0},$$

where

$$(13) \quad G(x) = 2^{-2x} \frac{\Gamma(1 + 2x)}{\Gamma(1 + x)},$$

and develop $G(x)$ as a power series. Using the series [6]

$$(14) \quad \log \Gamma(1 + x) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^k / k \quad (|x| < 1),$$

we find

$$(15) \quad \log G(x) = -(\gamma + 2 \log 2)x + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{2^k - 1}{k} x^k \quad \left(|x| < \frac{1}{2} \right).$$

In order to obtain a series for $G(x)$ from (15), we make use of the theorem that if

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is a formal power series with $a_0 = 0$, then

$$e^{f(x)} = \sum_{k=0}^{\infty} c_k x^k,$$

where the coefficients c_k are given recursively by

$$(16) \quad c_k = \frac{1}{k} \sum_{m=1}^k m a_m c_{k-m} \quad (k > 0).$$

This theorem can be proved analogously to Theorem 1.6c of Henrici [3, p. 42]. Applying this result to the series (15), we find

$$(17) \quad G(x) = \sum_{k=0}^{\infty} b_k^* x^k \quad \left(|x| < \frac{1}{2} \right),$$

where $b_0^* = 1$,

$$(18) \quad b_k^* = \frac{1}{k} \sum_{m=1}^k (-1)^m \zeta^*(m) b_{k-m}^*$$

and

$$(19) \quad \zeta^*(m) = \begin{cases} \gamma + 2 \log 2, & m = 1, \\ (2^m - 1)\zeta(m), & m > 1. \end{cases}$$

We now differentiate (12) with respect to α and obtain

$$(20) \quad H(\beta) = \frac{\partial^p}{\partial \alpha^p} \frac{G(\alpha)}{\Gamma(1 + \alpha + \beta)} \Big|_{\alpha=0} = \sum_{\rho=0}^p \binom{p}{\rho} \Gamma^{-1}(1 + \alpha + \beta)^{(\rho)} G(\alpha)^{(p-\rho)} \Big|_{\alpha=0}$$

$$= p! \sum_{\rho=0}^p b_{p-\rho}^* \sum_{k=\rho}^{\infty} a_k \binom{k}{\rho} \beta^{k-\rho}.$$

Similarly,

$$(21) \quad \left. \frac{\partial^n}{\partial \beta^n} H(\beta) G(\beta) \right|_{\beta=0} = \sum_{\nu=0}^n \binom{n}{\nu} H(\beta)^{(\nu)} G(\beta)^{(n-\nu)},$$

and therefore finally

$$(22) \quad r_{np} = \frac{\pi n! p!}{2^{n+p+1}} \sum_{\nu=0}^n b_{n-\nu}^* \sum_{\rho=0}^p \binom{\nu+\rho}{\rho} b_{p-\rho}^* a_{\nu+\rho}.$$

This expression, although still complicated and revealing less of the structure of r_{np} than formula (3), is much more suitable for actual computation. Using a formula manipulation system, the evaluation of (22) is in fact straightforward once the expressions for a_k ($0 < k < n+p$) and b_k^* ($0 < k < \max(n, p)$) have been initially established. It follows from (5) that, at least, all terms involving $\zeta(1) = \gamma$ will cancel in the final expression for (22). For the case $n \geq 0, p = 0$, (22) reduces to

$$(23) \quad r_{n0} = \pi \frac{n!}{2^{n+1}} \sum_{\nu=0}^n b_{n-\nu}^* a_{\nu},$$

which is another form of Bowman's determinant [1] for r_{n0} . The result of Bowman for r_{n0} can also be found in the book by Lewin [5].

3. A Table for r_{np} . We give here, as examples, the expressions for r_{np} for $n = 1, 2$ and $0 \leq p \leq n$. In order to complete the table of r_{np} given in [4], we also present expressions for r_{np} for $n = 5$ and $0 \leq p \leq 5$ in Table 1. Note that Nielsen has already given the expressions for r_{20}, r_{11}, r_{30} in [9], for r_{02}, r_{11}, r_{03} in [10], and for r_{12}, r_{22}, r_{13} in [11], derived by different methods.

$$\begin{aligned} r_{10} &= -\frac{1}{2} \pi \log 2, \\ r_{11} &= \frac{\pi}{8} (-\zeta(2) + 4 \log^2 2) = \frac{\pi}{2} \left(-\frac{1}{24} \pi^2 + \log^2 2 \right), \\ r_{20} &= \frac{\pi}{4} (\zeta(2) + 2 \log^2 2) = \frac{\pi}{2} \left(\frac{1}{12} \pi^2 + \log^2 2 \right), \\ r_{21} &= \frac{\pi}{8} (\zeta(3) - 4 \log^3 2) = \frac{\pi}{2} \left(-\log^3 2 + \frac{1}{4} \zeta(3) \right), \\ r_{22} &= \frac{\pi}{16} (-3\zeta(4) - 8\zeta(3)\log 2 + 3\zeta^2(2) + 8 \log^4 2) \\ &= \frac{\pi}{2} \left(\frac{1}{160} \pi^4 + \log^4 2 - \zeta(3)\log 2 \right). \end{aligned}$$

Numerical values of r_{np} for $0 \leq n \leq 5, 0 \leq p \leq n$, with 21 digits are given in Table 2.

TABLE I

$$\begin{aligned}
r_{50} &= -\frac{\pi}{8} (90\zeta(5) + 105\zeta(4)\log 2 + 30\zeta(2)\zeta(3) + 60\zeta(3)\log^2 2 \\
&\quad + 15\zeta^2(2)\log 2 + 20\zeta(2)\log^3 2 + 4\log^5 2) \\
r_{51} &= \frac{\pi}{32} (-30\zeta(6) + 300\zeta(5)\log 2 - 135\zeta(2)\zeta(4) - 60\zeta^2(3) + 360\zeta(4)\log^2 2 \\
&\quad - 60\zeta(2)\zeta(3)\log 2 - 15\zeta^3(2) + 200\zeta(3)\log^3 2 + 60\zeta(2)\log^4 2 + 16\log^6 2) \\
r_{52} &= \frac{\pi}{32} (90\zeta(7) + 210\zeta(6)\log 2 - 150\zeta(2)\zeta(5) + 165\zeta(3)\zeta(4) \\
&\quad - 120\zeta(5)\log^2 2 + 90\zeta(2)\zeta(4)\log 2 + 210\zeta^2(3)\log 2 - 105\zeta^2(2)\zeta(3) \\
&\quad - 240\zeta(4)\log^3 2 + 120\zeta(2)\zeta(3)\log^2 2 - 30\zeta^3(2)\log 2 \\
&\quad - 140\zeta(3)\log^4 2 - 48\zeta(2)\log^5 2 - 16\log^7 2) \\
r_{53} &= \frac{\pi}{64} (-630\zeta(8) - 1440\zeta(7)\log 2 - 165\zeta(2)\zeta(6) + 990\zeta(3)\zeta(5) - 180\zeta^2(4) \\
&\quad - 1680\zeta(6)\log^2 2 + 720\zeta(2)\zeta(5)\log 2 + 360\zeta(3)\zeta(4)\log 2 \\
&\quad - 360\zeta^2(2)\zeta(4) + 405\zeta(2)\zeta^2(3) - 600\zeta(5)\log^3 2 + 180\zeta(2)\zeta(4)\log^2 2 \\
&\quad + 360\zeta^2(2)\zeta(3)\log 2 - 60\zeta^4(2) + 60\zeta(4)\log^4 2 + 120\zeta(2)\zeta(3)\log^3 2 \\
&\quad + 60\zeta^3(2)\log^2 2 + 168\zeta(3)\log^5 2 + 60\zeta^2(2)\log^4 2 + 88\zeta(2)\log^6 2 + 32\log^8 2) \\
r_{54} &= \frac{\pi}{64} (2520\zeta(9) + 5670\zeta(8)\log 2 + 540\zeta(2)\zeta(7) + 210\zeta(3)\zeta(6) \\
&\quad - 4230\zeta(4)\zeta(5) + 6480\zeta(7)\log^2 2 + 1260\zeta(2)\zeta(6)\log 2 \\
&\quad - 3960\zeta(3)\zeta(5)\log 2 - 4455\zeta^2(4)\log 2 - 450\zeta^2(2)\zeta(5) + 990\zeta(2)\zeta(3)\zeta(4) \\
&\quad + 630\zeta^3(3) + 5040\zeta(6)\log^3 2 - 720\zeta(2)\zeta(5)\log^2 2 \\
&\quad - 6120\zeta(3)\zeta(4)\log^2 2 + 270\zeta^2(2)\zeta(4)\log 2 + 1260\zeta(2)\zeta^2(3)\log 2 \\
&\quad - 210\zeta^3(2)\zeta(3) + 2280\zeta(5)\log^4 2 - 1440\zeta(2)\zeta(4)\log^3 2 \\
&\quad - 1680\zeta^2(3)\log^3 2 + 360\zeta^2(2)\zeta(3)\log^2 2 - 45\zeta^4(2)\log 2 \\
&\quad + 432\zeta(4)\log^5 2 - 840\zeta(2)\zeta(3)\log^4 2 - 112\zeta(3)\log^6 2 \\
&\quad - 144\zeta^2(2)\log^5 2 - 96\zeta(2)\log^7 2 - 32\log^9 2) \\
r_{55} &= \frac{\pi}{256} (-45360\zeta(10) - 100800\zeta(9)\log 2 - 9450\zeta(2)\zeta(8) - 3600\zeta(3)\zeta(7) \\
&\quad + 5400\zeta(4)\zeta(6) + 73800\zeta^2(5) - 113400\zeta(8)\log^2 2 - 21600\zeta(2)\zeta(7)\log 2 \\
&\quad - 8400\zeta(3)\zeta(6)\log 2 + 169200\zeta(4)\zeta(5)\log 2 - 1800\zeta^2(2)\zeta(6) \\
&\quad + 39600\zeta(2)\zeta(3)\zeta(5) - 33075\zeta(2)\zeta^2(4) - 33300\zeta^2(3)\zeta(4) \\
&\quad - 86400\zeta(7)\log^3 2 - 25200\zeta(2)\zeta(6)\log^2 2 + 79200\zeta(3)\zeta(5)\log^2 2 \\
&\quad + 89100\zeta^2(4)\log^2 2 + 18000\zeta^2(2)\zeta(5)\log 2 - 39600\zeta(2)\zeta(3)\zeta(4)\log 2 \\
&\quad - 25200\zeta^3(3)\log 2 - 6750\zeta^3(2)\zeta(4) + 15300\zeta^2(2)\zeta^2(3) \\
&\quad - 50400\zeta(6)\log^4 2 + 9600\zeta(2)\zeta(5)\log^3 2 + 81600\zeta(3)\zeta(4)\log^3 2 \\
&\quad - 5400\zeta^2(2)\zeta(4)\log^2 2 - 25200\zeta(2)\zeta^2(3)\log^2 2 + 8400\zeta^3(2)\zeta(3)\log 2 \\
&\quad - 765\zeta^5(2) - 18240\zeta(5)\log^5 2 + 14400\zeta(2)\zeta(4)\log^4 2 \\
&\quad + 16800\zeta^2(3)\log^4 2 - 4800\zeta^2(2)\zeta(3)\log^3 2 + 900\zeta^4(2)\log^2 2 \\
&\quad - 2880\zeta(4)\log^6 2 + 6720\zeta(2)\zeta(3)\log^5 2 + 640\zeta(3)\log^7 2 \\
&\quad + 960\zeta^2(2)\log^6 2 + 480\zeta(2)\log^8 2 + 128\log^{10} 2)
\end{aligned}$$

TABLE 2

n	p	r_{np}			
0	0	1.57079	63267	94896	61923 E + 00
1	0	-1.08879	30451	51801	06525 E + 00
1	1	1.08729	73195	40018	85202 E - 01
2	0	2.04662	20244	72740	64617 E + 00
2	1	-5.10672	58055	97432	35017 E - 02
2	2	1.01152	43866	77157	15298 E - 02
3	0	-6.04188	29097	75093	52215 E + 00
3	1	4.38609	79674	10765	45916 E - 02
3	2	-4.26758	40717	27544	48003 E - 03
3	3	1.02686	60292	42556	73947 E - 03
4	0	2.40528	86060	94622	94399 E + 01
4	1	-5.42161	86358	95971	92894 E - 02
4	2	2.78905	89366	54715	61317 E - 03
4	3	-4.11669	99312	01587	89077 E - 04
4	4	1.08779	55160	65204	48571 E - 04
5	0	-1.20085	59155	37857	31652 E + 02
5	1	8.67856	96358	57061	30428 E - 02
5	2	-2.46254	98260	35781	37595 E - 03
5	3	2.32819	02210	25192	62212 E - 04
5	4	-4.23231	23761	73142	19272 E - 05
5	5	1.18188	78861	42353	10303 E - 05

European Organization for Nuclear Research
CERN
CH-1211 Geneva 23, Switzerland

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