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On the Bisection Method for Triangles

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Abstract. Let UVW be a triangle with vertices U, V, and W. It is "bisected" as follows: choose a longest edge (say VW) of UVW, and let A be the midpoint of VW. The UVW gives birth to two daughter triangles UVA and UWA. Continue this bisection process forever.

We prove that the infinite family of triangles so obtained falls into finitely many similarity classes, and we obtain sharp estimates for the longest jth generation edge.

1. Introduction. Let UVW be the triangle with vertices U, V, and W. We "bisect" triangles as follows: choose a longest edge (say VW) of UVW, and let A be the midpoint of VW. Then UVW gives birth to two daughter triangles UVA and UAW. So the generation 0 triangle UVW gives rise to two generation 1 triangles. "Bisect" these in turn, giving rise to four generation 2 triangles, and so on. So UVW through this process gives rise to an infinite family of triangles. This bisection process and a generalization to three dimensions have a number of numerical applications; see, e.g., [1], [3], [4].

Let m_j be the length of the longest jth generation edge. A bound for the rate of convergence of m_j has been obtained in [2]. Sharp estimates for certain classes of triangles have been given in [5]. In this paper we prove that $m_j \leqslant \sqrt{3}\, 2^{-j/2} m_0$, if j is even, and that $m_j \leqslant \sqrt{2}\, 2^{-j/2} m_0$, if j is odd, with equality for equilateral triangles. We prove, moreover, the following geometrically interesting fact: the (infinite) family of UVW contains only finitely many similarity types.

Definition. If Δ is a triangle, then $\phi(\Delta) = (\text{area of } \Delta)/l^2(\Delta)$, where $l(\Delta)$ is the length of the longest edge of Δ. $\mathcal{F}_0(\Delta)$ is the collection of even generation descendants of Δ, and $\mathcal{F}_1(\Delta)$ is the collection of odd generation descendants.

Since our bisection process in particular bisects areas, in order to find out about m_j, it is enough to know how the dimensionless quantity $\phi(\Delta)$ behaves under bisection of triangles. Our results will be proved by an induction on ϕ. It is necessary to deal first with acute angled triangles, then with obtuse triangles. The squares of side-lengths needed in this paper are all calculated by straightfoward use of the law of cosines.

2. Acute Triangles. Let $\Delta = UVW$ be an acute angled triangle, with VW a longest edge. Write $\|UV\|^2 = p$, $\|UW\|^2 = q$. For convenience let $\|VW\|^2 = 1$, and assume $p \leqslant q \leqslant 1$. Bisect edge VW at A. UW is then the longest edge of UAW. Bisect it at B. Then UA is the longest edge of UBA. There are now three different possibilities to consider.

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Possibility 1. UV is a longest edge of UVA. Examination of Figure 1 shows that bisection of UAB and of UVA gives rise to triangles similar to already occurring triangles, and so (up to similarity) $\mathcal{F}_0(\Delta)$ contains only UVW and UAB, while $\mathcal{F}_1(\Delta)$ only contains UVA and UAW.

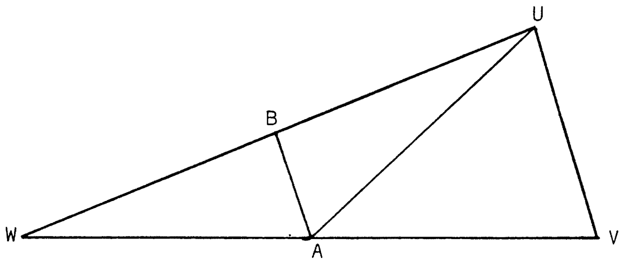

FIGURE 1

Since $\|UA\|^2 = \frac{1}{4}(2p + 2q - 1)$, $\phi(UAB) = \phi(\Delta)/2p + 2q - 1$. But since $p + q \geqslant 1$ (from the acuteness of Δ), using elementary linear programming, we find that $\phi(UAB) \geqslant \frac{1}{3}\phi(\Delta)$, with equality if Δ is equilateral. It is easy to see that $\phi(UVA)$ and $\phi(UAW)$ are both $\geqslant \frac{1}{2}\phi(\Delta)$.

Since Δ is acute, $\|AV\| \leqslant \|AU\|$, so if Possibility 1 does not hold, AU is a longest edge of UVA. Bisect AU at C. It is not hard to show that AV is a longest edge of CVA. Bisect AV at D. We have reached the position illustrated in Figure 2.

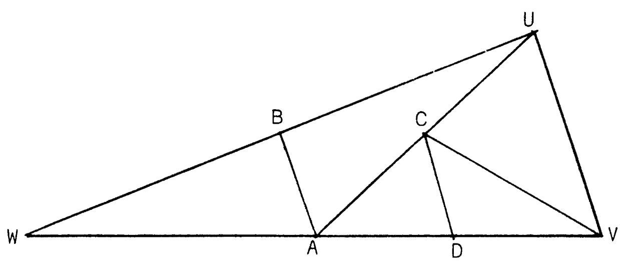

FIGURE 2

There are two possibilities now left.

Possibility 2. UV is a longest edge of UVC. Examination of Figure 2 will show that further bisection produces triangles similar to already occurring triangles. So (up to similarity), $\mathcal{F}_0(\Delta)$ consists of UVW, UAB, UVC, and CVA, and $\mathcal{F}_1(\Delta)$ consists of UVA, UAW, and CVD. Since UA is a longest edge of UVA, $\frac{1}{4}(2p + 2q - 1) \geqslant p$, so $q \geqslant p + \frac{1}{2}$. Elementary linear programming now gives $\phi(UAB) \geqslant \frac{1}{2}\phi(\Delta)$, and $\phi(UVC) \geqslant \frac{1}{2}\phi(\Delta)$. Of course $\phi(CVA) = \phi(\Delta)$. It turns out that $\|CV\|^2 = \frac{1}{16}(6p - 2q + 3)$. Linear programming now gives $\phi(CVD) \geqslant \frac{1}{2}\phi(\Delta)$. Similarly, we find that $\phi(UVA) \geqslant \phi(\Delta)$, and $\phi(UAW) \geqslant \frac{1}{2}\phi(\Delta)$.

If UV is not the longest edge of UVC, there remains only

Possibility 3. CV is a longest edge of UVC. So $\frac{1}{16}(6p - 2q + 3) \geqslant p$, that is $q \leqslant 3/2 - 5p$. Then (up to similarity), $\mathcal{F}_0(\Delta)$ consists of UVW, UAB, and $\mathcal{F}_1(UVA)$, while $\mathcal{F}_1(\Delta)$ consists of UAW and $\mathcal{F}_0(UVA)$. As usual, $\phi(UAW) \geqslant \frac{1}{2}\phi(\Delta)$. Since $q \leqslant 3/2 - 5p$, linear programming gives $2p + 2q - 1 \leqslant \frac{6}{5}$. So $\phi(UAB) \geqslant \frac{5}{6}\phi(\Delta)$, while $\phi(UVA) \geqslant \frac{5}{3}\phi(\Delta)$. So UVA is much "fatter" than Δ. This enables us to push through an induction.

LEMMA 1. *Let Δ be an acute triangle. Then the family of Δ contains only finitely many similarity types. If Γ is in $\mathfrak{F}_0(\Delta)$, $\phi(\Gamma) \geqslant \frac{1}{3}\phi(\Delta)$. If Γ is in $\mathfrak{F}_1(\Delta)$, $\phi(\Gamma) \geqslant \frac{1}{2}\phi(\Delta)$.*

Proof. We show that if our assertions hold whenever $\phi(\Delta) \geqslant (\frac{3}{5})^n$, they hold whenever $\phi(\Delta) \geqslant (\frac{3}{5})^{n+1}$. So suppose $\Delta = UVW$ is acute and $\phi(\Delta) \geqslant (\frac{3}{5})^{n+1}$. If Δ satisfies Possibility 1 or Possibility 2, then, by our earlier calculations, Δ certainly satisfies our lemma. So suppose that Δ falls under Possibility 3. The elements of $\mathfrak{F}_1(\Delta)$ are, up to similarity, UAW (and $\phi(UAW) \geqslant \frac{1}{2}\phi(\Delta)$) together with $\mathfrak{F}_0(UVA)$. But since $\phi(UVA) \geqslant \frac{5}{3}\phi(\Delta)$, by induction assumption UVA satisfies our lemma, so if Γ is in $\mathfrak{F}_0(UVA)$, $\phi(\Gamma) \geqslant \frac{1}{3}\phi(UVA) \geqslant \frac{5}{9}\phi(\Delta) > \frac{1}{2}\phi(\Delta)$. The same sort of calculation shows that under Possibility 3, if Γ is in $\mathfrak{F}_0(\Delta)$, $\phi(\Gamma) \geqslant \frac{1}{3}\phi(\Delta)$, indeed $\phi(\Gamma) \geqslant \frac{5}{6}\phi(\Delta)$. This completes the induction.

The inequalities for ϕ are sharp, for if Δ is equilateral, no improvement is possible. One cannot expect to make significant improvements on estimates for $\mathfrak{F}_1(\Delta)$. But our proof shows that for the "general" acute triangle (Possibility 3), if Γ is in $\mathfrak{F}_0(\Delta)$, $\phi(\Gamma) \geqslant \frac{5}{6}\phi(\Delta)$.

3. Obtuse Triangles. Suppose now we are "bisecting" a triangle $\Delta = UVW$, where as usual $\|VW\|^2 = 1$, $\|UW\|^2 = q$, $\|UV\|^2 = p$, $p \leqslant q < 1$, and where the angle VUW is $\geqslant 90°$. Bisect VW at A. Then UW is the longest edge of ΔUAW. Bisect it at B (see Figure 1).

LEMMA 2. *If Δ is obtuse, the family of Δ contains only finitely many similarity types. If Γ is in $\mathfrak{F}_1(\Delta)$, $\phi(\Gamma) \geqslant \frac{1}{2}\phi(\Delta)$. If Γ is in $\mathfrak{F}_0(\Delta)$, $\phi(\Gamma) \geqslant \frac{1}{3}\phi(\Delta)$.*

Proof. Let $\frac{1}{2} \leqslant \lambda < 1$. We prove that if our result holds for all obtuse triangles Δ such that the smallest angle of Δ has cosine $\leqslant \sqrt{\lambda}$ and such that $\phi(\Delta) \geqslant \lambda^n$, then the result holds for all such Δ with $\phi(\Delta) \geqslant \lambda^{n+1}$. So suppose that $\phi(\Delta) \geqslant \lambda^{n+1}$, and that Δ has smallest angle α, where $\cos^2 \alpha \leqslant \lambda$. If the angle BAU ($= AUV$) is $\geqslant 90°$, there is no problem. For it is easy to see that all angles of triangles UAB, UVA are $\geqslant \alpha$. But

$$\phi(UAB) = \frac{1}{q}\phi(\Delta) \geqslant \frac{1}{\cos^2 \alpha}\phi(\Delta) \geqslant \lambda^n.$$

Also $\phi(UVA) = 2\phi(\Delta) \geqslant 2\lambda^{n+1} \geqslant \lambda^n$ since $\lambda \geqslant \frac{1}{2}$. Now $\mathfrak{F}_1(\Delta)$ consists, up to similarity, of UAW, $\mathfrak{F}_1(UAB)$, and $\mathfrak{F}_0(UVA)$. By induction assumption, if Γ is in $\mathfrak{F}_1(UAB)$, then $\phi(\Gamma) \geqslant \phi(UAB) \geqslant \frac{1}{2}\phi(\Delta)$, while if Γ is in $\mathfrak{F}_0(UVA)$, $\phi(\Gamma) \geqslant \frac{1}{3}\phi(UVA) = \frac{2}{3}\phi(\Delta) > \frac{1}{2}\phi(\Delta)$. Elements of $\mathfrak{F}_0(\Delta)$ are dealt with in the same way.

So it remains to see what happens if the angle BAU is $\leqslant 90°$. If UV is the longest edge of UVA, the family of Δ has at most four similarity types, and a quick computation yields the result. Otherwise, $\phi(UVA) = 2\phi(\Delta)$, and of course $\phi(UAB) \geqslant \phi(\Delta)$, and our result follows quickly from Lemma 1.

The estimate for $\mathfrak{F}_1(\Delta)$ cannot be significantly improved. But by a closer analysis of the possibilities that arise when the angle UAB is acute, one can show that in fact if Γ is in $\mathfrak{F}_0(\Delta)$, $\phi(\Gamma) \geqslant \phi(\Delta)$.

4. Summary, Problems. By combining Lemma 1, Lemma 2, and the fact that area goes down by a factor of 2 each generation we obtain:

THEOREM. *Under the bisection process, the family of a triangle falls into finitely many similarity classes. If j is even, $m_j \leqslant \sqrt{3}\, 2^{-j/2} m_0$. If j is odd, $m_j \leqslant \sqrt{2}\, 2^{-j/2} m_0$.*

Both estimates are sharp, for we have equality when the triangle is equilateral. If the starting triangle is far from being equilateral, the bounds for m_j when j is even can be improved. By examining the details of the proof, one can find an upper bound for the number of similarity types in the family of Δ, say as a function of $\phi(\Delta)$. But there appears to be nothing very interesting left to do for triangles.

But one can raise similar problems in a much more general setting. Let $A_1, A_2, \ldots, A_n$ be a configuration of $n + 1$ points in d-dimensional space. Suppose $\|A_0 - A_1\| \geqslant \|A_i - A_j\|$ for all i, j. Then the configuration gives birth to two daughter configurations $A_0, (A_0 + A_1)/2, A_2, \ldots, A_n$ and $(A_0 + A_1)/2, A_1, A_2, \ldots, A_n$. One can define m_j as for triangles and ask about the behavior of m_j. It seems reasonable to conjecture that $m_j = O(2^{-j/n})$. One can make the even stronger conjecture that up to similarity any configuration has a finite family.

Already for four points in general position in 3-dimensional space, the problems seem difficult. We have a proof of the "finite family" conjecture for certain classes of tetrahedra. For example, it turns out that if a tetrahedron is nearly equilateral and the second largest edge is opposite the longest edge, then the family of the tetrahedron falls into $\leqslant 37$ similarity classes. (The condition "nearly equilateral" is a little complicated to describe briefly, but, for example, it is satisfied if all edge lengths are within 5% of each other.)

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