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On the Linear Independence of Multivariate B-Splines. II: Complete Configurations

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Abstract. The first part of this paper is concerned with global characterizations of both the multivariate B-spline and the multivariate truncated power function as smooth piecewise polynomials. In the second part of the paper we establish combinatorial criteria for the linear independence of multivariate B-splines corresponding to certain configurations of knot sets.

1. Introduction. A characteristic feature of the univariate B-spline is the well-known fact that it has minimal support among all splines of a fixed degree having the requisite smoothness conditions determined by the multiplicities of its corresponding knots. In particular, for any set of distinct knots $x_0 < x_1 < \cdots < x_n$, there is (up to a multiplicative constant) a unique function (B-spline) which is supported on (x_0, x_n), has $n - 2$ continuous derivatives everywhere and on each interval (x_j, x_{j+1}), $0 \leqslant j \leqslant n - 1$, is a polynomial of degree $\leqslant n - 1$. This function has an explicit representation as a divided difference

$$M(t \mid x_0, \ldots, x_n) = \frac{1}{(n-1)!}[x_0, \ldots, x_n](\circ - t)_+^{n-1} \tag{1.1}$$

and can be seen to have exactly $n - 1$ knots $x_1, \ldots, x_{n-1}$ interior to its interval of support (x_0, x_n). It is this latter fact that characterizes $M(t)$ as having minimal support. For there is no (nontrivial) function sharing the same properties as M but having one less knot in its support. These remarkable properties of the univariate B-spline (minimal support and the representation (1.1)) are due to Curry and Schoenberg [4].

One of us has already provided a multivariate version of (1.1) by using multivariate B-splines and truncated powers [5]. However, the analog of the minimal support property of the univariate B-spline has yet to be settled in higher dimensions. The piecewise polynomial nature of the multivariate B-spline is known to be much more complicated than in the univariate case. Thus it may be somewhat surprising that a global characterization of the multivariate B-spline is indeed valid. One of our objectives is to prove such a result. Furthermore, by characterizing the restriction of the multivariate truncated power function to hyperplanes, we are able to establish a similar result for this density function as well.

The second and main objective of this paper is to use these results to establish some combinatorial criteria for the linear independence of multivariate B-splines corresponding to certain configurations of knots. We have already treated this

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question to some degree in [8] (cf. also [13]), where certain collections of B-splines were studied. However, the arguments which insure the linear independence of the corresponding B-splines involve, among other things, some restrictions on the knot positions, which, although being necessary for the stability of the basis, do not seem to be essential for their linear independence.

Our objective is therefore to explore the connections between linear dependencies of B-splines and purely combinatorial properties of the corresponding configurations of knot sets. To this end, we shall, for instance, determine the dimension of the span of B-splines associated with what we call a complete configuration. (This terminology comes from the fact that the complete configuration for a set of affinely independent points in the plane is the complete graph whose vertices are these points.) Furthermore, our results will allow us to remove the above mentioned restrictions on the knot configurations considered in [8], [13].

Finally, we mention that there is a direct relationship between linearly independent B-splines and certain classes of multivariate rational functions. This fact will be explained fully later. We also mention in this context that we are able to give a simpler proof of some recent results of Hakopian [12] on multivariate divided differences and interpolation.

2. A Global Characterization of the Multivariate *B*-Spline. A central objective of this section is to point out in what sense the multivariate B-spline and related functions have minimal support.

Let us start by fixing some notation and summarizing a few known facts for later reference.

Elements of the Euclidean space R^s will be denoted by x, u, z, $x = (x_1, \ldots, x_s)$. $\chi_A(x)$, $[A]$, $\mathrm{vol}_s(A)$, $|A|$ will denote the indicator function, the (closed) convex hull, the s-dimensional Lebesgue measure and, in case A is finite, the cardinality (counting multiplicities) of a given set A, respectively. The restriction of a given function f: $R^s \to R$ to some subset $A \subset R^s$ will be indicated by $f|_A$. In particular, when A is an affine subspace of R^s, $f|_A$ will be always considered as a function of correspondingly fewer variables with respect to a suitable coordinate system for A. Adopting the standard multi-index notation for $\alpha, \beta \in Z^s_+$, i.e. $|\alpha| = \alpha_1 + \cdots + \alpha_s$, $x^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, we define the s-variate polynomials of (total) degree $\leqslant k$ as usual by

$$\Pi_{k,s} = \left\{ \sum_{|\alpha| \leqslant k} c_\alpha x^\alpha : c_\alpha \in R, x \in R^s \right\}.$$

Let Γ be any collection of $(s-1)$-dimensional sets in R^s. $\Pi_{k,s}(\Gamma)$ will then mean the class of all (s-variate) functions f such that $f|_D \in \Pi_{k,s}$ for any region D which is not intersected by any element of Γ.

We shall sometimes refer to any $(s-1)$-dimensional set ρ in R^s such that in a neighborhood of some point of ρ, f is in $\Pi_{k,s}$ on each side of ρ (but not in $\Pi_{k,s}$ in the neighborhood) as a *cut region* for f. The set of all cut regions of a given function f will be denoted by $\Gamma(f)$, and clearly $\Gamma(f) \subset \Gamma$ whenever $f \in \Pi_{k,s}(\Gamma)$.

A knot set $K = \{x^0, \ldots, x^n\} \subset R^s$, $n \geqslant s$, will always be assumed to satisfy $\mathrm{vol}_s([K]) > 0$. Let us emphasize here that in our notation $|K| = n + 1$, even though K may contain fewer than $n + 1$ distinct vectors.

The B-spline $M(x \mid x^0,\ldots,x^n)$, or briefly $M(x \mid K)$, is then conveniently defined as the unique density of the functional [5], [14]

$$(2.1)\quad \int_{R^s} M(x \mid x^0,\ldots,x^n) f(x)\,dx = n! \int_{S^n} f(t_0 x^0 + \cdots + t_n x^n)\,dt_1 \cdots dt_n,$$

where f is any continuous function and

$$S^n = \left\{(t_1,\ldots,t_n):\ \sum_{i=1}^{n} t_i = 1 - t_0,\, t_i \geq 0,\, i = 0,\ldots,n\right\}$$

is the standard n-simplex.

It is well known [5], [14] that $M(x \mid K)$ is a piecewise polynomial of degree $k = |K| - s - 1$ supported on $[K]$ and that its cut region

$$\Gamma_s(K) = \Gamma_s(x^0,\ldots,x^n) = \left\{[x^{i_1},\ldots,x^{i_s}]: 0 \leq i_1 < \cdots < i_s \leq n\right\}$$

consists of all $(s-1)$-simplices spanned by subsets of s elements of K. Furthermore, it will be convenient to denote by H_ρ the $(s-1)$-flat spanned by $\rho \in \Gamma_s(K)$.

As for various representations and properties of the B-spline the reader is referred to [5], [6], [14], [15].

Here we need the following formula for the derivatives of M. Let $D_z f = \sum_{1 \leq i \leq s} z_i \partial f / \partial x_i$ denote the directional derivative of f along z. Then

$$(2.2)\qquad D_z M(x \mid x^0,\ldots,x^n) = n \sum_{j=0}^{n} \mu_j M(x \mid x^0,\ldots,x^{j-1}, x^{j+1},\ldots,x^n),$$

whenever $z = \sum_{0 \leq i \leq n} \mu_i x^i$, $\sum_{0 \leq i \leq n} \mu_i = 0$.

This allows us to show that $M(x \mid K) \in C^{n-s-d}(R^s)$, whenever the convex hull of any $s + d$ knots in K has a nonvanishing s-dimensional volume [5], [14]. Recently, Hakopian [11] pointed out that these conditions are actually sharp. The resulting local characterization of the smoothness properties of $M(\circ \mid K)$ in terms of the knot positions may be reformulated as follows

PROPOSITION 2.1 (HAKOPIAN). *Let $H = H_\rho$ for some $\rho \in \Gamma_s(K)$ and*

$$l = l_H = |K| - 1 - |K \cap H|.$$

Then for any x in the interior of $[H \cap K]$ relative to H and any $\lambda \perp H$,

$$0 < \lim_{t \to 0^+} \left|\left((D_\lambda)^l M(x + t\lambda \mid K) - (D_\lambda)^l M(x - t\lambda \mid K)\right)\right| < \infty.$$

So, we may associate with any knot set K the class C_K of functions having at least the same smoothness properties as $M(\circ \mid K)$ with respect to $\Gamma_s(K)$ stated in Proposition 2.1 and vanishing outside $[K]$.

In particular, we are interested in the class

$$\Pi_{k,s,K} = \Pi_{k,s}(\Gamma_s(K)) \cap C_K,$$

which contains by definition at least $M(x \mid K)$. In fact, we will show that $M(x \mid K)$ is essentially the only nontrivial element of $\Pi_{k,s,K}$.

To this end, let us collect a few more preliminary facts. Any collection Γ of $(s-1)$-simplices is called (n, s)-complete if $\Gamma = \Gamma_s(K)$ for some knot set $K \subset R^s$ consisting of $n + 1$ vectors (counting multiplicities). Γ is called (n, s)-incomplete when Γ is a proper subset of some $\Gamma_s(K)$.

The geometrical definition of the multivariate B-spline provides a useful way to view $\Gamma_s(K)$. Defining for any n-simplex $\sigma = [v^0, \ldots, v^n]$ the function

$$M_\sigma(x) = \mathrm{vol}_{n-s}(\{u \in \sigma : u|_{R^s} = x\}),$$

one may easily conclude from (2.1) that

$$M(x \mid K) = M_\sigma(x)/\mathrm{vol}_n(\sigma), \tag{2.3}$$

whenever $K = \{v^0|_{R^s}, \ldots, v^n|_{R^s}\}$. Hence $\Gamma_s(K)$ is the collection of all projected $(s-1)$-faces of the simplex $\sigma = \sigma(K)$. This is used in

LEMMA 2.1. *Let x^0 be an exposed knot of $K = \{x^0, \ldots, x^n\} \subset R^s$ (i.e. a vertex of the convex s-polytope $[K]$), and let H be some $(s-1)$-flat separating x^0 from the remaining knots x^i, $i > 0$, Then*

(i) *$\{\rho \cap H : \rho \in \Gamma_s(K)\}$ can be identified with some $(n-1, s-1)$-complete cut region $\Gamma_{s-1}(K')$ for some set $K' \subseteq H$, and*

(ii) *any $g \in \Pi_{k,s,K}$ satisfies*

$$g|_H \in \Pi_{k,s-1,K'}.$$

Proof. Choose $\sigma = [v^0, \ldots, v^n]$, $\mathrm{vol}_n(\sigma) = 1$, such that $K = \{v^0|_{R^s}, \ldots, v^n|_{R^s}\}$. Then the $(n-1)$-flat

$$\overline{H} = H \times R^{n-s}$$

clearly separates the vertex v^0, $v^0|_{R^s} = x^0$, from the remaining vertices. Setting $u^i = [v^i, v^0] \cap \overline{H}$, $i = 1, \ldots, n$, we observe that

$$\overline{H} \cap \sigma = [u^1, \ldots, u^n]$$

is an $(n-1)$-simplex and

$$M(x \mid K)|_H = M_\sigma(x)|_H = \mathrm{vol}_{n-1-(s-1)}(\{u \in [u^1, \ldots, u^n] : u|_H = x\}),$$

so that (i) follows with

$$K' = \{z^1, \ldots, z^n\}, \qquad z^i = u^i|_H, i = 1, \ldots, n. \tag{2.4}$$

As for (ii) we note that $g \in \Pi_{k,s,K}$ implies $\Gamma(g) \subset \Gamma_s(K)$ and hence $\Gamma(g|_H) \subset \Gamma(M(\circ \mid K)|_H) = \Gamma_{s-1}(K')$. To finish the proof we observe that $g \in C_K$ implies $g|_H \in C_{K'}$.

We are now ready to state

THEOREM 2.1. *Suppose $K = \{x^0, \ldots, x^n\} \subset R^s$ is in general position. Then for $k = n - s$,*

$$g \in \Pi_{k,s,K} \quad \textit{iff} \quad g(x) = cM(x \mid K)$$

for some constant $c \in R$.

Since obviously, $M(x \mid K) > 0$, for x in the interior of $[K]$, as an immediate consequence of Theorem 2.1 we state

COROLLARY 2.1. *Let $K \subset R^s$ be in general position. Then $M(\circ \mid K)$ has minimal support, i.e. there exists no (nontrivial) element in $\Pi_{k,s,K}$ which vanishes somewhere in the interior of $[K]$.*

Proof of Theorem 2.1. We will proceed by induction on the spatial dimension s, recalling that the assertion is well known for $s = 1$ (cf. e.g. [1]). So let us assume that the statements in Theorem 2.1 and Corollary 2.1 are valid for $s - 1 \geqslant 1$. Let us denote by $R(K)$ the collection of all those regions in $[K]$ which are bounded by but not intersected by any element of $\Gamma_s(K)$.

LEMMA 2.2. *Suppose that under the above assumptions $F \in \Pi_{k,s,K}$ vanishes on some $B \in R(K)$, where (the closure of) B contains some exposed knot $x^0 \in K$. Then F has to vanish identically on $[K]$.*

Proof. Pick any $(s - 1)$-flat H separating x^0 from the remaining knots, so that Lemma 2.1 assures $F|_H \in \Pi_{k,s-1,K'}$, $K' = \{z^1,\dots,z^n\}$ (cf. (2.4)). Since $H \cap B$ has dimension $s - 1$ and $F|_{H\cap B} = 0$, we conclude that $\operatorname{supp}(F|_H)$ is strictly contained in $[K']$. Since H was any hyperplane separating x^0 from x^i, $i > 0$, our induction assumption implies $F|_H = 0$ because K' is in general position with respect to H. Hence

$$\operatorname{supp}(F) \subset [x^1,\dots,x^n].$$

Note that we have clearly produced at least one further region $B_1 \in R(K)$, $B_1 \subset [x^0,\dots,x^n]$, adjacent to some other exposed knot x^1, say. Repeating the above arguments, we remove step-by-step all the exposed knots of K, thereby restricting the support of F to some polyhedral domain P strictly contained in $[K]$. P satisfies one of the following conditions:

(i) $\operatorname{vol}_s(P) = 0$;

(ii) no vertex of P coincides with some knot of K;

(iii) some of the vertices of P are knots of K.

As to (i) there is nothing further to show. In the case that (ii) occurs, again let H be some $(s - 1)$-flat which is sufficiently close to some vertex u of P. Since there are certainly less than n edges $[x^i, x^j]$ intersecting $H \cap P$ (since otherwise u would have to be a knot), $\Gamma(F|_{H\cap P})$ is $(n - 1, s - 1)$-incomplete. So, again using our induction assumption, another finite number of reduction steps would lead us to (i). So, let us assume (iii) holds. Then at least $s + 1$ exposed knots $x^0,\dots,x^s$, say, have been removed before, and we let $x^i \in K$ be a vertex of the remaining polytope P. We may again choose a hyperplane H separating x^i from the remaining vertices of P. We observe that at least one of the n edges $[x^i, x^j]$, $j \neq i$, does not intersect H, since x^i was originally not an exposed knot. Therefore $\Gamma(F|_H)$ is again $(n - 1, s - 1)$-incomplete, and so the support of F can be reduced further, which proves Lemma 2.2.

In order to finish the proof of Theorem 2.1, let $F \in \Pi_{k,s,K}$ and x^0 be an exposed knot in K. Assume that the theorem was proved for $s - 1$. Let H_1, H_2 be two distinct $(s - 1)$-flats separating x^0 from the remaining vertices. Furthermore suppose H_1, H_2 have a nonempty intersection with the interior of $[K]$. By the induction hypothesis, there are constants c_i, d_i, $i = 1, 2$, such that

$$F|_{H_i} = c_i M(\circ \mid K_i), \tag{2.5}$$

$$M(\circ \mid K)|_{H_i} = d_i M(\circ \mid K_i), \qquad i = 1, 2, \tag{2.6}$$

where $K_i = H_i \cap \{[x^0, x^1], \ldots, [x^0, x^n]\}$, $i = 1, 2$, and d_1, d_2 are nonzero. Choosing a constant a such that $ad_1 = c_1$, then $F - aM(\circ \mid K)$ clearly vanishes on H_1. On the other hand, since H_2 intersects H_1 and

$$(2.7) \qquad (F - aM(\circ \mid K))\,|_{H_2} = (c_2 - ad_2)M(\circ \mid K_2),$$

$F - aM(\circ \mid K)$ also vanishes on H_2. Since a actually did not depend on H_2, we infer from (2.7) that

$$(F - aM(\circ \mid K))\,|_B = 0$$

for some region $B \in R(K)$ where B is adjacent to x^0. Clearly $F - aM(\circ \mid K) \in \Pi_{k,s,K}$, and so Lemma 2.2 confirms

$$F - aM(\circ \mid K) = 0 \quad \text{on } [K],$$

which finishes the proof of Theorem 2.1.

In order to derive analogous results for multivariate truncated powers as well, it is useful to introduce the following more general class of density functions.

For a given ω: $R \to R$ and $x^1, \ldots, x^s \in R^s$ we define $G_\omega(x \mid x^1, \ldots, x^m)$ formally by requiring that [7]

$$(2.8) \qquad \int_0^\infty \cdots \int_0^\infty \omega(t_1 + \cdots + t_m) f(t_1 x^1 + \cdots + t_m x^m)\, dt_1 \cdots dt_m$$
$$= \int_{R^s} G_\omega(x \mid x^1, \ldots, x^m) f(x)\, dx$$

holds for any locally supported continuous function f on R^s.

The close connection between G_ω and the B-spline is revealed by the following relation [7]

$$(2.9) \qquad G_\omega(x \mid x^1, \ldots, x^m) = \int_0^\infty \omega(h) h^{m-s-1} M(h^{-1}x \mid x^1, \ldots, x^m)\, dh.$$

We are mainly interested in the following choices of ω [7]: $\omega_1(t) = \chi_{[0,1]}(t)$, $\omega_2(t) = 1$, $\omega_3(t) = e^{-t}$. In fact, in the first case we simply have [7] $G_{\omega_1}(\circ \mid x^1, \ldots, x^m) = (1/m!)M(\circ \mid 0, x^1, \ldots, x^m)$, whereas ω_2 gives rise to the multivariate truncated powers [5], [15] which will be henceforth briefly denoted by $G(\circ \mid x^1, \ldots, x^m)$. However, in order to make sure that (2.8) makes sense in this case, we have to impose the following conditions on the directions $\{x^i\}$:

$$(2.10) \qquad 0 \notin [x^1, \ldots, x^m].$$

It is known [5] that $G(\circ \mid x^1, \ldots, x^m)$ is a piecewise polynomial of degree $m - s$ with support

$$\langle x^1, \ldots, x^m \rangle_+ = \left\{ \sum_{j=1}^m t_j x^j : t_j \geq 0, j = 1, \ldots, m \right\}$$

and cut regions

$$\Gamma_s^0(x^1, \ldots, x^m) = \left\{ \langle x^{i_1}, \ldots, x^{i_{s-1}} \rangle_+ : 1 \leq i_1 < \cdots < i_{s-1} \leq m \right\}.$$

Furthermore, let H be any $(s-1)$-flat in $\Gamma_s^0(x^1, \ldots, x^m)$. Setting

$$(2.11) \qquad l = m - 1 - |H \cap \{x^1, \ldots, x^m\}|,$$

the lth order derivatives of $G(\circ \mid x^1, \ldots, x^m)$ are known to have finite jumps across H [5]. The same cut regions and smoothness properties are obtained for the third choice $\omega_3(t)$ [7]. In this case

(2.12) $$\Lambda(x \mid x^1, \ldots, x^m) = G_{\omega_3}(x \mid x^1, \ldots, x^m)$$

turns out to be a *piecewise exponential* function (where no restrictions on the directions $\{x^i\}$ are required).

A useful link between functions of the above type and their univariate counterparts is conveniently expressed by means of the *Radon transform* (cf. [10]) which is defined for $f \in L_1(R^s)$ by

$$(Rf)(\lambda, t) = \int_{x \cdot \lambda = t} f(x)\, dx, \qquad \lambda \in \Omega_s = \{y \in R^s \colon \|y\| = 1\},$$

where $\|y\| = (y \cdot y)^{1/2}$ and $x \cdot y$ denotes the standard inner product on R^s. Consequently one has the identity [14]

$$\bigl(RM(\circ \mid x^0, \ldots, x^n)\bigr)(\lambda, t) = M\bigl(t \mid \lambda \cdot x^0, \ldots, \lambda \cdot x^n\bigr),$$

i.e. the Radon transform maps multivariate B-splines into univariate ones.

Similarly we obtain

LEMMA 2.3. *For any $\lambda \in \Omega_s$ let $H_{\lambda,t} = \{x \in R^s\colon x \cdot \lambda = t\}$, and let G and Λ be defined as above.*

(i) *Suppose $x^1, \ldots, x^m$ satisfy (2.10) and $\lambda \in \Omega_s$ is chosen so that $H_{\lambda,t} \cap \langle x^1, \ldots, x^m \rangle_+$ is bounded for any $t > 0$. Then one has*

$$\bigl(RG(\circ \mid x^1, \ldots, x^m)\bigr)(\lambda, t) = \left(\prod_{j=1}^{m} \lambda \cdot x^j\right)^{-1} t_+^{m-1}.$$

(ii) $(R\Lambda(\circ \mid x^1, \ldots, x^m))(\lambda, t) = (E_1 * \cdots * E_m)(t)$, $E_i(t) = t_+^0 e^{-\lambda \cdot x^i t}$, *where* $t_+^d = \chi_{R_+}(t) t^d$ *and "$*$" denotes convolution.*

Proof. (i): Suppose $g\colon R \to R$ has local support. Note that then $\operatorname{supp}(g(\lambda \cdot x)) \cap \langle x^1, \ldots, x^m \rangle_+$ is bounded by assumption, so that we may write (cf. (2.8))

$$\int_0^\infty \cdots \int_0^\infty g\bigl(t_1 \lambda \cdot x^1 + \cdots + t_m \lambda \cdot x^m\bigr)\, dt_1 \cdots dt_m$$
$$= \int_{R^s} G\bigl(x \mid x^1, \ldots, x^m\bigr) g(\lambda \cdot x)\, dx.$$

Since [2], [7]

(2.13) $$\int_{R^s} f(x) g(\lambda \cdot x)\, dx = \int_{-\infty}^{\infty} (Rf)(\lambda, t) g(t)\, dt,$$

the right-hand side of the previous equation reads

$$\int_R \bigl(RG(\circ \mid x^1, \ldots, x^m)\bigr)(\lambda, t) g(t)\, dt.$$

Assertion (i) follows then from the equality

$$\int_0^\infty \cdots \int_0^\infty g\left(t_1\lambda\cdot x^1+\cdots+t_m\lambda\cdot x^m\right)dt_1\cdots dt_m$$

$$=\left(\prod_{j=1}^m \lambda\cdot x^j\right)^{-1}\int_R t_+^{m-1}g(t)\,dt.$$

As to (ii) it is not hard to prove that

$$\int_{R^s}\Lambda\left(x\,|\,x^1,\ldots,x^m\right)e^{-\lambda\cdot x}\,dx=\left(\prod_{j=1}^m\left(1+\lambda\cdot x^j\right)\right)^{-1} \tag{2.14}$$

holds for $\mathrm{Re}(\lambda\cdot x^j)>-1$. Thus (2.13) again yields

$$\int_R\left(R\Lambda\left(\circ\,|\,x^1,\ldots,x^m\right)\right)(\lambda,t)e^{-t}\,dt=\left(\prod_{j=1}^m\left(1+\lambda\cdot x^j\right)\right)^{-1}.$$

On the other hand we know that $(1+\lambda\cdot x^j)^{-1}=\int_R t_+^0\, e^{-\lambda\cdot x^j t}e^{-t}\,dt$, whence the assertion readily follows.

We are now in the position to state the precise relation between truncated powers and lower-dimensional B-splines.

THEOREM 2.2. *Suppose* $0\notin[x^1,\ldots,x^m]$ *and every* s *of the vectors* x^i *span* R^s. *Furthermore suppose that* $\lambda\in\Omega_s$ *has been chosen so that* $H_{\lambda,t}\cap\langle x^1,\ldots,x^m\rangle_+$ *is bounded. Then*

$$G\left(\circ\,|\,x^1,\ldots,x^m\right)|_{H_{\lambda,t}}=\left(\prod_{j=1}^m\lambda\cdot x^j\right)^{-1}t_+^{m-1}M\left(\circ\,|\,z^1,\ldots,z^m\right),$$

where $z^i=H_{\lambda,t}\cap\{rx^i\colon r\geqslant 0\}$, $i=1,\ldots,m$.

Proof. Choosing z^i as above, we clearly have for $F=G(\circ\,|\,x^1,\ldots,x^m)|_{H_{\lambda,t}}$

$$\mathrm{supp}(F)=H_{\lambda,t}\cap\langle x^1,\ldots,x^m\rangle_+=\left[z^1,\ldots,z^m\right].$$

Moreover, $K'=\{z^1,\ldots,z^m\}$ is in general position (with respect to $H_{\lambda,t}$) and

$$\Gamma_s^0\left(x^1,\ldots,x^m\right)|_{H_{\lambda,t}}=\Gamma(F)=\Gamma_{s-1}(K').$$

Recalling the smoothness properties of the B-spline (Proposition 2.1) and those of the truncated powers (2.11) we conclude

$$F\in\Pi_{m-s,s-1,K'}.$$

Hence Theorem 2.1 implies

$$F(x)=cM(x\,|\,K'),\qquad x\in H_{\lambda,t},$$

for some $c\in R$. In order to determine the constant c, we recall that $\int_{R^s}M(x\,|\,K)\,dx=1$, so that one obtains, in view of the definition of the Radon transform,

$$c=\int_{H_{\lambda,t}}cM\left(z\,|\,z^1,\ldots,z^m\right)dz=\int_{H_{\lambda,t}}G\left(z\,|\,x^1,\ldots,x^m\right)dz$$

$$=\left(RG\left(\circ\,|\,x^1,\ldots,x^m\right)\right)(\lambda,t)=\left(\prod_{j=1}^m\lambda\cdot x^j\right)^{-1}t_+^{m-1},$$

where we have used Lemma 2.3(i) in the last equality.

As an immediate consequence of Theorems 2.1 and 2.2 we state

COROLLARY 2.2. *Let* $x^1,\dots,x^m \in R^s$ *satisfy the hypotheses of Theorem* 2.2. *Suppose that* F *is any piecewise polynomial of degree* $m-s$ *with cut region* $\Gamma(F) \subset \Gamma_s^0(x^1,\dots,x^m)$ *and local smoothness properties as specified by* (2.11), *i.e.* $F \in C^{m-s-1}(R^s)$. *Then, if* $\operatorname{supp}(F) \subseteq \langle x^1,\dots,x^m\rangle_+$,

$$F = cG(\circ \mid x^1,\dots,x^m)$$

for some $c \in R$.

3. Lagrange Interpolation in R^s. This section is concerned with the connection of B-splines and the construction of multivariate interpolation operators (cf. e.g. [14]).

In a recent paper [12] Hakopian proposed the following interesting extension of univariate Lagrange interpolation into a multivariate setting. Let for $x^0,\dots,x^{s-1} \in R^s$ (cf. (2.1))

$$\begin{aligned}\int_{[x^0,\dots,x^{s-1}]} f &= \int_{S^{s-1}} f\left(t_0x^0 + \cdots + t_{s-1}x^{s-1}\right) dt_1 \cdots dt_{s-1}\\ &= \frac{1}{(s-1)!}\int_{R^s} f(x)M\left(x \mid x^0,\dots,x^{s-1}\right) dx,\end{aligned}$$

where $M(\circ \mid x^0,\dots,x^{s-1})$ is to be interpreted now as a distribution.

Then Hakopian showed

THEOREM 3.1 (HAKOPIAN). *Let* $x^0,\dots,x^n$ *be in general position. Then for any continuous function* f *on* R^s *there exists a unique polynomial* P *of degree* $\leqslant n-s+1$ *such that*

$$\int_{[x^{i_0},\dots,x^{i_{s-1}}]} P = \int_{[x^{i_0},\dots,x^{i_{s-1}}]} f$$

holds for all $0 \leqslant i_0 < \cdots < i_{s-1} \leqslant n$.

The two-dimensional case was treated earlier in Part II of [2] which appeared in *Quantitative Approximation*, Eds. R. DeVore, K. Scherer, Academic Press, 1980. Theorem 3.1, as well as related results, is based on the expansion of the multivariate *divided difference* functional

$$[x^0,\dots,x^n]^\alpha f = \int_{R^s} M\left(x \mid x^0,\dots,x^n\right) D^\alpha f(x)\, dx,$$

$|\alpha| = n-s+1$, where $D^\alpha f$ denotes the partial derivatives of f of order $\alpha \in Z_+^s$ in terms of the functionals $\int_{[x^{i_0},\dots,x^{i_{s-1}}]} f$ [12, Theorem 1] (which for $s=1$ reduce, of course, to point evaluations).

The existence of such an expansion is clear from the recurrence relation (2.2), a point which was made in [14]. However, Hakopian provides an explicit formula for this representation. This result turns out to be very closely related to our study [7]. In fact, by briefly pointing out this relationship, we wish to present an alternate and, as it seems, much shorter derivation of Theorem 1 in [12].

To this end, let us state the following lemma from [3], [7] which will be also used in the subsequent section.

LEMMA 3.1. *Suppose* $0, x^0, \ldots, x^n \in R^s$ *are in general position and let for any* $I \subset \{0, \ldots, n\}$, $|I| = s$, x^I *be the unique solution of* $1 + x^j \cdot x^I = 0$, $j \in I$. *Then every polynomial of total degree* $\leqslant n + 1 - s$ *can be expanded as*

$$Q(x) = \sum_{|I|=s} Q(x^I) Q_I(x),$$

and the polynomials $Q_I(x) = \prod_{j \notin I}(1 + x \cdot x^j)/(1 + x^I \cdot x^j)$ *satisfy* $Q_I(x^J) = \delta_{IJ}$, $I, J \subset \{0, \ldots, n\}$, $|I| = |J| = s$.

Now let $q_m(x)$ be any homogeneous polynomial of degree m and $q_m(D)$ the associated differential operator. In view of (2.2) we can find coefficients μ_J, $J \subset \{0, \ldots, n\}$, such that for smooth $g\colon R \to R$ and $K = \{x^0, \ldots, x^n\} \subset R^s$

$$\text{(3.1)} \quad \int_{R^s} q_m(D) M(x \mid K) g^{(n-m)}(z \cdot x)\, dx$$
$$= \sum_{|J|=m} \mu_J \int_{R^s} g^{(n-m)}(z \cdot x) M\big(x \mid K \backslash \{x^j : j \in J\}\big)\, dx.$$

Integration by parts applied to the left-hand side of (3.1) readily yields the divided difference

$$n!\,(-1)^m q_m(z)\big[z \cdot x^0, \ldots, z \cdot x^n\big] g.$$

Similarly the right-hand side of (3.1) can be rewritten as

$$(n-m)! \sum_{|J|=m} \mu_J\big[z \cdot \big(K \backslash \{x^j : j \in J\}\big)\big] g.$$

In particular we may choose $g(t) = 1/(1+t)$, which provides

$$\frac{n!}{(n-m)!} q_m(z) \Big/ \prod_{j=0}^{n} (1 + z \cdot x^j) = \sum_{|J|=m} \mu_J \Big/ \prod_{j \notin J} (1 + z \cdot x^j)$$

and hence

$$\text{(3.2)} \qquad q_m(z) = \frac{(n-m)!}{n!} \sum_{|J|=m} \mu_J \prod_{j \in J} (1 + z \cdot x^j)$$

(which is easily seen to be equivalent to (3.1) by the denseness of the span of the $1/(1 + at)$, $a \in \mathbf{C}$). Choosing now $m = n + 1 - s$, we obtain

$$q_{n+1-s}(z) = \frac{(s-1)!}{n!} \sum_{|J|=n+1-s} \mu_J \prod_{j \in J} (1 + z \cdot x^j),$$

which we write in the equivalent form

$$= \frac{(s-1)!}{n!} \sum_{|I|=s} \eta_I \prod_{j \notin I} (1 + z \cdot x^j).$$

But Lemma 3.1 then says that the coefficients η_I are uniquely determined. In particular, when $q_{n+1-s}(x) = x^\alpha$, $|\alpha| = n + 1 - s$, combining (3.1) with Lemma 3.1 readily gives

$$\big[x^0, \ldots, x^n\big]^\alpha f = \sum_{I = \{i_1, \ldots, i_s\}} \eta_I^\alpha \int_{[x^{i_1}, \ldots, x^{i_s}]} f,$$

where

$$\eta_I^\alpha = \frac{n!}{(s-1)!}(-1)^{n-s+1}(x^I)^\alpha \Big/ \prod_{j \notin I}(1 + x^I \cdot x^j),$$

which is the desired explicit expansion.

The relationship of the polynomial $P(x) = \mathcal{P}(f)(x)$ given by Theorem 3.1 to Hermite interpolation is to be expected. In fact if $f_\lambda(x) = g(\lambda \cdot x)$, $\lambda \in R^s$, then, using the Hermite-Genocchi formula, again it easily follows that

$$\mathcal{P}(f_\lambda)(x) = L^{(s-1)}\big(g^{-s+1} \,|\, \lambda \cdot x^0, \dots, \lambda \cdot x^n\big)(\lambda \cdot x),$$

where $L(g \,|\, t_0, \dots, t_n)(t)$ denotes Hermite interpolation to $g(t)$ at $t_0, \dots, t_n$, and we use the usual convention that repeated values correspond to interpolation of successive derivatives. Thus in the terminology of [2] the map $\mathcal{P}$ lifts

$$L^{(s-1)}\big(g^{-s+1} \,|\, t_0, \dots, t_n\big)(t)$$

to R^s. The fact that this map can be lifted to *all* R^m is easily obtained by differentiating the Newton expansion of $L(g \,|\, t_0, \dots, t_n)(t)$ and using the methods employed in [2] (Hakopian's map corresponds to $m = s$). This procedure yields the remainder formula for the lifted map

$$f(x) - \mathcal{P}(f)(x) = \sum_{r=0}^{s-1} \binom{s-1}{r} \int_{[\underbrace{x, \dots, x}_{s-r}, x^0, \dots, x^n]} \sum_{|J|=r} \prod_{i \notin J} D_{x-x^i} f.$$

The case $s = 1$ corresponds to Kergin interpolation, see [2] for the formula and related references, while $s = 2$ is the lift of the area matching map discussed in Part II of [2] which appeared in *Quantitative Approximation*, Eds. R. DeVore, K. Scherer, Academic Press, 1980.

We wish to mention that Carl de Boor and Klaus Höllig pointed out to one of us that Hakopian's mapping is a lifted map in the terminology of [2]. Equation (3.2) resulted from a question they raised concerning multivariate B-splines.

4. Complete Configurations. In this section we will address questions of linear independence of B-splines. We begin with an arbitrary collection $\mathcal{C}$ of knot sets in R^s, where each member of $\mathcal{C}$ has the same cardinality. Our general objective is to find a subset $\mathcal{B} \subset \mathcal{C}$ such that $\{M(\circ \,|\, K) \colon K \in \mathcal{B}\}$ forms a basis for

$$\mathcal{S}(\mathcal{C}) = \operatorname{span}\{M(\circ \,|\, K) \colon K \in \mathcal{C}\}.$$

It is to be understood that $\mathcal{C}$ is *admissible* in the sense that any knot set in $\mathcal{C}$ gives rise to a B-spline *density*. Thus, if $K \in \mathcal{C}$, then $\operatorname{vol}_s([K]) > 0$.

Returning for a moment to the univariate case, we define for any sequence $T = \{t_1, \dots, t_n\} \subset R$, $t_i < t_{i+k+1}$, the collection $\mathcal{T} = \{K \colon K \subset T, \ |K| = k+2\}$ of knot sets and the corresponding span $\mathcal{S}(\mathcal{T})$ of B-splines of degree k with knots in T and supported in $[t_1, t_n]$. Here the smoothness of the B-splines at the knots is given as usual by their respective multiplicities. In particular, any $S(x) \in \mathcal{S}(\mathcal{T})$ is to satisfy $S^{(j)}(t_i) = 0$, $j = 0, \dots, k - d_i$, $i = 1, n$, when d_1, d_n are the respective multiplicities of t_1, t_n. It is well known that $\mathcal{S}(\mathcal{T})$ is the set of *all* splines of degree k with knots in T and supported on $[t_1, t_n]$ and that

$$\dim \mathcal{S}(\mathcal{T}) = n - k - 1. \tag{4.1}$$

This suggests the following more general question: Let $P = \{x^1, \ldots, x^n\}$ be any set of points in R^s; what is $\dim \mathbb{S}(\mathcal{P}_{n,m})$, where $\mathcal{P}_{n,m}$ is defined to be the *complete configuration*

$$\mathcal{P}_{n,m} = \{K \subset P \colon |K| = m\}?$$

As a first simple observation we state

PROPOSITION 4.1. *Let $P = \{x^1, \ldots, x^n\} \subset R^s$, $\operatorname{vol}_s([P]) > 0$. Then*

$$\dim \mathbb{S}(\mathcal{P}_{n,n-1}) = s + 1.$$

Proof. Suppose that for $K_j = P \backslash \{x^{i_j}\}$, $j = 1, \ldots, s+2$, the B-splines $M(\circ \mid K_j)$ are linearly independent. Since we can always find $c \in R^{s+2} \backslash \{0\}$ such that

$$\sum_{1 \leqslant j \leqslant s+2} c_j x^{i_j} = 0, \qquad \sum_{1 \leqslant j \leqslant s+2} c_j = 0,$$

formula (2.2) would say

$$0 = D_0 M(x \mid P) = (n-1) \sum_{j=1}^{s+2} c_j M(x \mid K_j),$$

which is a contradiction. Hence $\dim \mathbb{S}(\mathcal{P}_{n,n-1}) \leqslant s + 1$.

On the other hand we certainly have again, in view of (2.2),

$$\dim \mathbb{S}(\mathcal{P}_{n,n-1}) > \dim \operatorname{span}\{D_z M(\circ \mid P) \colon z \in R^s\} = s,$$

which finishes the proof.

Note that Proposition 4.1 remains valid even if some of the B-splines $M(x \mid K)$, $K \in \mathcal{P}_{n,n-1}$ are defined in the distributional sense.

Both the above observations, (4.1) and Proposition 4.1, are actually special cases of the following more general result.

THEOREM 4.1. *Let $P = \{x^1, \ldots, x^n\} \subset R^s$ be a set of knots such that $\mathcal{P}_{n,m}$ is admissible for some $n \geqslant m > s$. Then*

$$\dim \mathbb{S}(\mathcal{P}_{n,m}) = \binom{s+n-m}{s}.$$

It is worthwhile to note the following contrast to the univariate situation.

Remark 4.1. Let us suppose that $\overline{\mathbb{S}}(\mathcal{P}_{n,m})$ denotes the space of *all* piecewise polynomials of degree $m - s - 1$, supported in $[P]$, with cut region $\Gamma_s(P)$, and having the same smoothness properties as the elements of $\mathbb{S}(\mathcal{P}_{n,m})$. Then, one has in general

$$\dim \mathbb{S}(\mathcal{P}_{n,m}) \leqslant \dim \overline{\mathbb{S}}(\mathcal{P}_{n,m}),$$

where, contrary to the univariate case, the inequality is sometimes strict.

For instance, choose $P = \{x^1, \ldots, x^4\} \subset R^2$ such that all the knots x^i are vertices of $[P]$. It is readily seen that then $\dim \overline{\mathbb{S}}(\mathcal{P}_{4,3}) = 4$, whereas Theorem 4.1 assures that $\dim \mathbb{S}(\mathcal{P}_{4,3}) = 3$.

Combining Theorems 4.1 and 2.2 readily gives

COROLLARY 4.1. *Let $P = \{x^1, \ldots, x^n\} \subset R^s$, $0 \notin [P]$ and all s elements of P are linearly independent. Then one has for $n \geqslant m \geqslant s$,*

$$\dim \operatorname{span}\{G(\circ \mid Q) \colon Q \subset P, |Q| = m\} = \binom{s-1+n-m}{s-1}.$$

As a first basic ingredient of the proof of Theorem 4.1 we state the following result.

LEMMA 4.1. *For $s < m \leqslant n$, $P = \{x^1,\dots,x^n\} \subset R^s$ one has*

$$\dim \mathcal{S}(\mathcal{P}_{n,m}) = \dim \operatorname{span}\left\{ \prod_{j\in I} (1 + x\cdot x^j) : I \subset \{1,\dots,n\}, |I| = n-m \right\}.$$

Proof. For any finite collection $\mathcal{C}$ of knot sets $K \subset R^s$, (2.9) provides the following equality:

$$\begin{aligned}\sum_{K\in\mathcal{C}} c_K \Lambda(tx\,|\,K) &= \sum_{K\in\mathcal{C}} c_K \int_0^\infty e^{-h} h^{n-s-1} M\big(th^{-1}x\,|\,K\big)\,dh \\ &= t^{n-s}\int_0^\infty e^{-th} h^{n-s-1}\Big(\sum_{K\in\mathcal{C}} c_K M\big(h^{-1}x\,|\,K\big)\Big)\,dh,\end{aligned}$$

which holds for any $x \in R^s$, $t > 0$. Hence from the properties of the Laplace transform we infer that

(4.2) $M(\circ\,|\,K)$, $K \in \mathcal{C}$, are linearly independent if and only if $\Lambda(\circ\,|\,K)$, $K \in \mathcal{C}$, are.

But, referring again to the properties of the (multivariate) Laplace transform, (2.14) implies that $\Lambda(\circ\,|\,K)$, $K \in \mathcal{C}$, are linearly independent if and only if the rational functions

$$\prod_{u\in K} (1 + x\cdot u)^{-1}, \qquad K \in \mathcal{C},$$

are. In the special case when $K \in \mathcal{C} = \mathcal{P}_{n,m}$ this in turn is equivalent to saying that the polynomials

$$\prod_{u\notin K} (1 + u\cdot x), \qquad K \in \mathcal{P}_{n,m},$$

are linearly independent if and only if the functions $\Lambda(\circ\,|\,K)$, $K \in \mathcal{P}_{n,m}$, and, on account of (4.2), the B-splines $M(\circ\,|\,K)$, $K \in \mathcal{P}_{n,m}$ are linearly independent.

Obviously, the linear space $\Pi = \operatorname{span}\{\Pi_{u\notin K}(1 + u\cdot x) : K \in \mathcal{P}_{n,m}\}$ satisfies

$$\Pi \subset \Pi_{n-m,s}.$$

Since

$$\dim \Pi_{n-m,s} = \binom{n-m+s}{s},$$

the validity of Theorem 4.1 is tantamount to showing $\Pi = \Pi_{n-m,s}$. When P contains $n-m+s$ points in general position, Lemma 3.1 does confirm that $\Pi = \Pi_{n-m,s}$. In the general case this equality still holds. One approach to proving this is to extend Lemma 3.1 to its "Hermite Form". Rather than taking this approach we will make use of (3.2). Thus if $q_l(x)$ is any homogeneous polynomial of degree $l \leqslant n-m$, there are constants μ_I such that

$$q_l(z) = \sum_{|I|=l} \mu_I \prod_{j\in I} (1 + z\cdot x^j).$$

Therefore we also have $\Pi_{n-m,s} \subseteq \Pi$. This proves Theorem 4.1 as well as

PROPOSITION 4.2. *Let $\mathcal{P}_{n,m}$ be any admissible complete configuration. Then*

$$\Pi = \Pi_{n-m,s}.$$

In the remaining part of this section we will be concerned primarily with methods for generating a basis of B-splines for the space $\mathcal{S}(\mathcal{P}_{n,m})$. At the same time though, we will introduce several useful notions applicable to arbitrary collections of knot sets $\mathcal{C}$. These ideas are motivated by two obvious facts about the univariate space $\mathcal{S}(\mathcal{T})$. Although this space contains all B-splines of degree k, it is well known that the consecutive B-splines corresponding to $K_i = \{t_i,\ldots,t_{i+k+1}\}$, $i = 1,\ldots,n-k-1$, span $\mathcal{S}(\mathcal{T})$. In this case the linear independence of the B-splines $M(\circ \mid K_i)$, $i = 1,\ldots,n-k-1$, over $[t_1, t_n]$ easily follows from the fact that each K_i contains a knot which is not in any of the previous sets $K_1,\ldots,K_{i-1}$. It is important to keep in mind that this fact is dependent on the order in which these sets are enumerated.

To place these facts in a general context we introduce the following notion

Definition 4.1. *Let $\mathcal{C}$ be a collection of knot sets in R^s. We will say $K_1,\ldots,K_N \in \mathcal{C}$ form a strong chain of length N provided that for each i, $1 \leq i \leq N$, there is some $\rho \in \Gamma_s(K_{i+1})$ and a point x in the relative interior of $[H_\rho \cap K_{i+1}]$ which is not in $\bigcup_{1\leq j\leq i} \bigcup \{\gamma \in \Gamma_s(K_j)\}$.*

LEMMA 4.2. *Let $(K_1, K_2,\ldots,K_N)$ be a strong chain. Then the B-splines $M(\circ \mid K_i)$, $i = 1,\ldots,N$, are linearly independent over $\bigcup_{1\leq i\leq N}[K_i]$.*

Proof. $M(\circ \mid K_1)$ is trivially linearly independent. Suppose $M(\circ \mid K_i)$, $i = 1,\ldots, j-1$, are also for some $j \leq N$. In view of the hypothesis we can find $\rho \in \Gamma_s(K_j)$ and a point z in the relative interior of $[H_\rho \cap K_j]$ such that z does not belong to $\bigcup_{1\leq i<j} \bigcup \{\gamma \in \Gamma_s(K_i)\}$. Suppose H_ρ is m-fold, i.e. H_ρ contains exactly $s-1+m$ knots of K_j. Defining for $\lambda \perp H_\rho$

$$L_{H_\rho}(f) = \lim_{t\to 0^+} \left((D_\lambda)^{n-s-m} f(z+t\lambda) - (D_\lambda)^{n-s-m} f(z-t\lambda)\right),$$

Proposition 2.1 assures

$$L_{H_\rho}\left(M(\circ \mid K_j)\right) \neq 0.$$

So $\sum_{i=1}^{j} c_i M(\circ \mid K_i) = 0$ would imply

$$\sum_{i=1}^{j} c_i L_{H_\rho}\left(M(\circ \mid K_i)\right) = 0 = c_j L_{H_\rho}\left(M(\circ \mid K_j)\right),$$

and hence $c_j = 0$. Thus $\sum_{1\leq i\leq j-1} c_i M(\circ \mid K_i) = 0$, which by assumption means $c_i = 0$, $i = 1,\ldots,j$.

Note that in the univariate case the sequence of consecutive knot sets is obviously the longest strong chain in $\mathcal{T}$.

This gives rise to the following

Definition 4.2. *A subset $\mathcal{B}$ of a given knot configuration $\mathcal{C}$ is called an s-basis of $\mathcal{C}$ if there is a chain $(K_1,\ldots,K_N)$ so that*

$$\mathcal{B} = \{K_i : i = 1,\ldots,N = |\mathcal{B}|\},$$

and whenever $(K_1',\ldots,K_{N'}')$ is another chain in $\mathcal{C}$ then

$$N' \leq |\mathcal{B}|\,.$$

$|\mathcal{B}| = I_s(\mathcal{C})$ is called the s-index of $\mathcal{C}$ and $\{K_i:\ i = 1,\ldots,|\mathcal{B}|\}$ is called an s-generating sequence for $\mathcal{C}$.

Example 4.1. In the univariate case the consecutive knot sets form a 1-generating sequence for $\mathcal{T}_{n,k+2}$ and $I_1(\mathcal{T}_{n,k+2}) = n-k-1$.

THEOREM 4.2. *Let $n \geqslant m > s$, $P = \{x^1, \ldots, x^n\} \subseteq R^s$. Suppose P contains $n - m + s$ points in general position. Then*

$$I_s(\mathcal{P}_{n,m}) = \binom{s+n-m}{s}.$$

Proof. We can easily obtain an upper bound on the s-index of $\mathcal{P}_{n,m}$ by combining Lemma 4.2 and Theorem 4.1 to conclude that

$$I_s(\mathcal{P}_{n,m}) \leqslant \binom{s+n-m}{s}.$$

On the other hand the proof of Lemma 3.1 (cf. Lemma 2 in [7]) suggests the following s-generating sequence for $\mathcal{P}_{n,m}$. Let $Q \subset P$ denote any fixed subset consisting of $n - m + s$ points in general position. Then $V = P \setminus Q$ clearly satisfies

$$|V| = n - n + m - s = m - s > 0.$$

Now there are exactly $\binom{n-m+s}{s}$ distinct subsets

$$I_i \subset Q, \qquad i = 1, \ldots, \binom{s+n-m}{s}, |I_i| = s.$$

Consequently, the $\binom{n-m+s}{s}$ sets $K_i = V \cup I_i$ form a strong chain in $\mathcal{P}_{n,m}$ since $|K_i| = |V| + |I_i| = m$ holds by construction. Hence Definition 4.2 says that

$$I_s(\mathcal{P}_{n,m}) \geqslant \binom{n-m+s}{s},$$

which completes the proof.

Note that the s-basis constructed in the above proof forms an s-generating sequence for $\mathcal{P}_{n,m}$ even for an arbitrary ordering of the sets K_i. However, in general, the ordering will matter as is shown by the consecutive knot sets in Example 4.1.

So, when dealing with complete configurations the dimension of the corresponding s-variate span of B-splines coincides with the s-index of the configuration. However, this simple rule is unfortunately not valid for any configuration of knot sets even in one dimension. This is confirmed by the following

Example 4.2. Suppose $x_1, x_2, \ldots, x_6$ are pairwise distinct univariate knots arranged in increasing order. Let $\mathcal{C} = \{K_1, K_2, K_3, K_4\}$ where

$$K_1 = \{x_1, x_2, x_4\}, \qquad K_2 = \{x_1, x_3, x_5\},$$
$$K_3 = \{x_4, x_5, x_6\}, \qquad K_4 = \{x_2, x_3, x_6\}.$$

It is then easy to see that the dimension of the $\mathcal{S}(\mathcal{C})$ will depend on the *position* of the knots.

Two important remarks concerning Lemma 4.2 and Theorem 4.2 should be made. The first remark concerns the definition of a strong chain. This notion, being based on the smoothness properties of the multivariate B-spline, can be weakened while still insuring the validity of Lemma 4.2. We only need to require that for each i, $1 < i < N$, there is some $\rho \in \Gamma_s(K_{i+1})$ which is r-fold, i.e. H_ρ contains exactly $s + r - 1$ points in K_{i+1}, such that either $\rho \not\subseteq \bigcup_{1 \leqslant j \leqslant i} \bigcup \{\gamma \in \Gamma_s(K_j)\}$ or, if $\rho \subseteq \bigcup \{\gamma \in \Gamma_s(K_j)\}$ for some $j \leqslant i$, then ρ is at most $(r-1)$-fold relative to K_j. Thus, even though ρ may be contained in each $\bigcup \{\gamma \in \Gamma_s(K_j)\}$, $1 \leqslant j \leqslant i$, there is at least one point in H_ρ at which the B-spline $M(\circ \mid K_{i+1})$ has a different smoothness than any of the B-splines $M(\circ \mid K_j), j \leqslant i$. When $K_1, \ldots, K_N$ have this weaker property we will say $(K_1,\ldots,K_N)$ forms a *weak chain*. The same argument used in Lemma 4.2 guarantees that $M(\circ\,|\,K_i)$, $i=1,\ldots,N$, are linearly independent if $(K_1,\ldots,K_N)$ forms only a weak chain.

Finally, we wish to point out that Theorem 4.2 gives a purely *combinatorial criterion* for the linear independence of B-splines. To emphasize this point let us observe that Definition 4.2 and Theorem 4.2 can be rephrased *entirely* as a *combinatorial* fact.

Let P be any finite set of n objects and $(P)_m$ the collection of all subsets of P consisting of m objects. Thus, while $(P)_m$ is for $m \leqslant n$ the analog of the complete configuration of knot sets $\mathcal{P}_{n,m}$ considered before, $(K)_s$ corresponds for $K \in (P)_m$, $s \leqslant m$, to the cut regions $\Gamma_s(K)$, $K \in \mathcal{P}_{n,m}$. For any collection $\mathcal{C}$ of sets of objects we say $(K_1,\ldots,K_N)$ is a chain of length N in $\mathcal{C}$ if $K_i \in \mathcal{C}$, $i=1,\ldots,N$, and

$$(K_{i+1})_s \setminus \bigcup_{1\leqslant j\leqslant i} (K_j)_s \neq \varnothing, \qquad i=1,\ldots,N-1. \tag{4.3}$$

As before the s-index $I_s(\mathcal{C})$ is the longest chain in $\mathcal{C}$.

Then it follows that for $s \leqslant m \leqslant |P|$

$$I_s((P)_m) = \binom{|P|-m+s}{s},$$

because we can always *identify* P with a set of n vectors in R^s which are in general position and $(K)_s$ with the cut region of $M(\circ\,|\,K)$. Since then every chain in the sense of (4.3) gives rise to a strong chain of the same length in the sense of Definition 4.1, this identification preserves the notion of chain and index so that Theorem 4.2 is applicable.

Conversely, any collection of knot sets in R^s, each of which is in general position, can be identified with a combinatorial structure. For instance, consider

Example 4.3. Let $P=\{1,2,\ldots,6\}$. One may check that $\{\{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,4,6\}, \{1,2,5,6\}\}$ form a 2-generating sequence for $(P)_4$ and $I_2((P)_4)=6$, whereas, $I_1((P)_4)=3$ because $\{\{1,2,3,4\}, \{2,3,4,5\}, \{3,4,5,6\}\}$ is a 1-generating sequence. Thus, if we think of P as a set of 2-vectors (four among them being in general position, say), we have a way of obtaining six linearly independent B-splines spanning $\mathcal{S}(\mathcal{P}_{6,4})$ while, if P is thought of as a set of univariate knots (at least three being pairwise distinct) we obtain three independent B-splines.

5. Regular Configurations. We have seen in the previous section that the dimension of the linear span of B-splines on the complete configuration is determined by its combinatorial structure. But, unfortunately, according to Example 4.2 this is not generally true. Nevertheless, as we shall point out next, there is another large class of configurations which has this property and is also suitable for approximation. For this purpose we will call a configuration $\mathcal{C}$ of knot sets *regular* if $\dim \mathcal{S}(\mathcal{C}) = I_s(\mathcal{C})$.

We will continue using the notions s-index, s-generating sequence, s-basis, chain in both contexts, i.e. when dealing with the purely combinatorial properties of collections of sets of arbitrary objects as well as when identifying these objects with knots in R^s and referring to the corresponding cut regions. The proper interpretation will always be clear from the context. In particular, the analogy between both concepts suggests calling the elements of $(K)_s$ (abstract) $(s-1)$-simplices.

Let us consider the following definitions taken from [9]. For any $s, k \in Z_+$ and $n = s + k$ let

$$(5.1)\quad \Delta(s,k) = \Big\{K = \{(j_i, m_i): i = 0,\dots,n\}: j_0 = m_0 = 0,$$
$$(j_r, m_r) \overset{\neq}{\leqslant} (j_{r+1}, m_{r+1}),\ (j_r, m_r) \in \{0,\dots,s\} \times \{0,\dots,k\}\Big\}.$$

Note that

$$(5.2)\qquad |\Delta(s,k)| = \binom{s+k}{s}, \qquad (j_n, m_n) = (s,k).$$

For $I = \{i_0,\dots,i_s\} \in Z_+^s$ we define

$$\Delta(I,k) = \Big\{\{(i_{j_r}, m_r): r = 0,\dots,n\}: \{(j_r, m_r): r = 0,\dots,n\} \in \Delta(s,k)\Big\}.$$

There is a simple geometrical interpretation of $\Delta(s,k)$. Let $\rho = [u^0,\dots,u^s]$, $\gamma = [v^0,\dots,v^k]$ be simplices in R^s, R^k, respectively. Then

$$(5.3)\qquad \mathfrak{T}_{\rho,\gamma} = \Big\{\sigma_K = \big[(u^{j_0}, v^{m_0}),\dots,(u^{j_n}, v^{m_n})\big]:$$
$$K = \{(j_r, m_r): r = 0,\dots,n\} \in \Delta(s,k)\Big\}$$

is a triangulation of $\rho \times \gamma \subset R^n$. Here a collection $\mathfrak{T}$ of simplices is called a triangulation of Ω when $\Omega = \bigcup\{\sigma \in \mathfrak{T}\}$ and the intersection of any two elements of $\mathfrak{T}$ is empty or a common lower-dimensional face.

Now suppose $\mathfrak{J} \subset Z_+^{s+1}$ induces an (abstract) simplicial s-complex, i.e. $\mathfrak{J}$ is combinatorially equivalent to a triangulation $\mathfrak{T}$ of some polyhedral domain Ω in R^s, which means that there is a one-to-one inclusion preserving correspondence between the sets of all faces of the elements of $\mathfrak{J}$ and $\mathfrak{T}$, respectively. The configuration

$$(5.4)\qquad \Delta(\mathfrak{J},k) = \bigcup\{\Delta(I,k): I \in \mathfrak{J}\}$$

is known to form an (abstract) simplicial n-complex (cf. [8], [9, Lemma 8.9]).

In particular, when $\mathfrak{V} = \{u^i: i = 1,\dots,N\} \subset R^s$ are chosen so that

$$(5.5)\qquad \Theta(\mathfrak{J}) = \big\{\rho(I) = [u^{i_0},\dots,u^{i_s}]: I = \{i_0,\dots,i_s\} \in \mathfrak{J}\big\}$$

is a triangulation of $\Omega = \bigcup\{\rho(I): I \in \mathfrak{J}\}$, the collection

$$(5.6)\quad \mathfrak{T}_{\mathfrak{J}} = \Big\{\big[(u^{i_{j_0}}, e^{m_0}),\dots,(u^{i_{j_n}}, e^{m_n})\big]: \{(i_{j_r}, m_r): r = 0,\dots,n\} \in \Delta(\mathfrak{J},k)\Big\}$$

is a triangulation of $\Omega \times S^s$, $S^k = [e^0,\dots,e^k]$, $(e^j)_i = \delta_{ij}$, $i, j = 0,\dots,k$ (cf. [8], [13]).

Before explaining the connection with certain spline spaces we state

THEOREM 5.1. *Let $\Delta(\mathfrak{J},k)$ be defined by* (5.4). *Then one has*

$$I_s(\Delta(\mathfrak{J},k)) = |\Delta(\mathfrak{J},k)|,$$

i.e. $\Delta(\mathfrak{J},k)$ forms an s-basis (*cf. Definition* 4.1, (4.3)).

Proof. We have to show that the elements of $\Delta(\mathfrak{J},k)$ can be ordered in such a way that condition (4.3) is satisfied. To this end, we shall show first that the assertion is true for $k = 0$.

LEMMA 5.1. *Let $\mathfrak{J} \subset Z_+^{s+1}$ induce a simplicial s-complex. Then*

$$I_s(\mathfrak{J}) = |\mathfrak{J}|.$$

Proof. Suppose $|\mathcal{J}| = m$. Without loss of generality we may assume that $\mathcal{J}$ is a triangulation of some bounded polyhedral domain Ω in R^s. Hence one can find an element of $\mathcal{J}$ such that at least one of its $(s-1)$-faces is contained in the boundary of Ω and is therefore not shared by any other element of $\mathcal{J}$. Choose I_m to be such an external element. Next let I_{m-1} be any external element of $\mathcal{J}\setminus\{I_m\}$, etc. Clearly $\{I_1,\ldots,I_m\}$ form a chain.

So, suppose $\mathcal{J} = \{I_i: i = 1,\ldots,|\mathcal{J}|\}$ is a chain so that there are $(s-1)$-simplices

$$\rho_i \in (I_i)_s, \qquad \rho_i \notin \bigcup_{1\leq j<i} (I_j)_s.$$

In order to finish the proof of Theorem 5.1, we only have to show now in view of (5.4) that the elements of each $\Delta(I_i, k)$ can be ordered as

$$\Delta(I_i, k) = \left\{K_{i,j}: j = 1,\ldots, \binom{s+k}{s}\right\},$$

say, so that there is $\bar{\rho}_j \in (K_{i,j})_s$, $\bar{\rho}_j \notin \bigcup_{1\leq r<j}(K_{i,r})_s$, and the first components of the pairs in $\bar{\rho}_j$ coincide exactly with the elements of $\rho_i \in (I_i)_s$ above.

The existence of such an ordering for $\Delta(I_i, k)$ is affirmed by the following

LEMMA 5.2. *For any $s, k \in Z_+$ and any fixed $i \in \{0,\ldots,s\}$ there is an ordering $\{K_j: j = 1,\ldots,\binom{s+k}{s}\}$ for $\Delta(s, k)$ (5.1) such that there is*

$$(5.7)\quad \rho_j = \{(j_r, m_r): r = 0,\ldots,s, j_r \neq i\} \in (K_j)_s, \qquad \rho_j \notin \bigcup_{1\leq i\leq j-1} (K_i)_s.$$

Proof. The cases $s, k = 0$ are trivial. Let us first prove the assertion for $s = 1$, $k \in N$. For $i = 1$ the ordering

$$K_j = \{(0,0),\ldots, (0, j), (1, j),\ldots, (1, k)\}, \qquad j = 0,\ldots,k,$$

obviously works, whereas for $i = 0$ the reverse ordering will do.

So, we may assume that we have proved the assertion for $s - 1 > 1$ and all $k \in Z_+$. For the purpose of advancing the induction step, we introduce the following subsets of $\Delta(s, k)$.

$$C_m = \{K \in \Delta(s, k): (j_{n-r}, m_{n-r}) = (s, k-r),$$
$$r = 0,\ldots,m,\ j_i < s,\ i < n-m\}, \qquad m = 0,\ldots,k.$$

By definition (5.1) and (5.2) one has therefore

$$(5.8)\quad K \in C_m \quad\text{iff } K = K' \cup \{(s, k-m),\ldots, (s, k)\},\quad K' \in \Delta(s-1, k-m),$$

and hence again by (5.2)

$$|C_m| = |\Delta(s-1, k-m)| = \binom{s-1+k-m}{s-1}.$$

Since obviously $C_i \cap C_j = \varnothing$, $i \neq j$, and

$$\sum_{m=0}^{k} |C_m| = \sum_{m=0}^{k} \binom{s-1+k-m}{s-1} = \binom{s+k}{s},$$

we conclude

$$\Delta(s, k) = \bigcup_{m=0}^{k} C_m.$$

Suppose now $i \neq s$. By assumption there exists for each $m = 0,\dots,k$ an ordering

$$\left\{K_j^m : j = 1,\dots,\binom{s-1+k-m}{s-1}\right\} = \Delta(s-1, k-m),$$

such that

$$\rho_j^m = \{(j_r, m_r): r = 0,\dots,s-1,\ j_r \neq i\} \in (K_j^r)_{s-1},$$

but

$$\rho_j^m \notin \bigcup_{1 \leq q \leq j-1} (K_q^m)_{s-1}.$$

So, defining

$$K_j := \begin{cases} K_j^0 \cup \{(s,k)\}; \quad j = 1,\dots,\binom{s-1+k}{s-1}, \\ K_i^m \cup \{(s,k-m),\dots,(s,k)\}; \\ \qquad j = j(m,i) = \binom{s-1+k}{s-1} + \cdots + \binom{s-1+k-m+1}{s-1} + i, \\ \qquad\qquad m = 1,\dots,k, \end{cases}$$

the corresponding sequence of $(s-1)$-simplices

$$\rho_j = \rho_i^m \cup \{(s, k-m)\} \qquad \text{when } j = j(m,i)$$

obviously satisfies (5.7) since $(s, k-m)$ never occurs in a preceding block.

Now suppose $i = s$. Associating with each

$$K = \{(j_r, m_r): r = 0,\dots,n\} \in \Delta(s,k)$$

the set

$$\bar{K} = \{(|j_r - s|, |m_r - k|): r = 0,\dots,n\}$$

defines evidently a bijective map from $\Delta(s,k)$ onto itself where $i = s$ is mapped into $i' = 0$. So the sets $\bar{K}$ may be ordered now by the same procedure as before, thereby inducing an appropriate ordering for the original sets K as well. This finishes the proof of Lemma 5.2.

Recalling our remarks subsequent to Lemma 5.1, the construction of an s-generating sequence for $\Delta(\mathcal{J}, k)$ with length $|\Delta(\mathcal{J}, k)|$ is now an immediate consequence of Lemmas 5.1, 5.2, which completes the proof of Theorem 5.1.

As to the connection with certain spline spaces let again

$$\mathcal{V} = \{u^{i,j}: i = 1,\dots,N,\ j = 0,\dots,k\}$$

be a collection of points in R^s such that for some $\mathcal{J} \subset Z_+^{s+1}$

$$\Theta(\mathcal{J}) = \{\rho(I) = [u^{i_0,0},\dots,u^{i_s,0}]: I = \{i_0,\dots,i_s\} \in \mathcal{J}\} \tag{5.9}$$

is a triangulation of $\Omega = \bigcup\{\rho(I): I \in \mathcal{J}\}$.

Defining now for $K = \{(j_r, m_r): r = 0,\dots,n\} \in \Delta(\mathcal{J}, k)$ the knot set

$$C(K) = \{u^{j_0,m_0},\dots,u^{j_n,m_n}\}, \tag{5.10}$$

it was shown in [6], [8], [13] that the collection $\{M(\circ \mid C(K)): K \in \Delta(\mathcal{J}, k)\}$, forms a basis for $\mathcal{S}(\mathcal{C})$, $\mathcal{C} = \{C(K): K \in \Delta(\mathcal{J}, k)\}$, provided the knots $u^{i,j}$ are sufficiently close to $u^{i,0}$ for all i, j. Moreover, under these assumptions the above basis turns out to be very well conditioned. This type of a basis is of particular interest because the corresponding spline spaces exhibit very good approximation properties (cf. [6], [8], [13]).

However, one may expect that the above type of restrictions on the knot positions is not essential for the linear independence of the B-splines $M(\circ \mid C)$, $C \in \mathcal{C}$. In fact, Theorem 5.1 allows us to formulate sufficient conditions for their linear independence which do not involve distances between knots and instead are satisfied for 'almost all' knot positions.

To this end let us call any configuration of knot sets in R^s *nondegenerate* if there exists an s-generating sequence in the sense of (4.3) which at the same time forms a weak chain. Then, recalling the remarks subsequent to Example 4.2, we may rephrase Theorem 5.1 as

COROLLARY 5.1. *Let $\mathcal{C}$ be defined as above with respect to a set of knots $\mathcal{V} = \{u^{i,j}\}$ in R^s. If $\mathcal{C}$ is nondegenerate, then $\{M(\circ \mid C)\colon C \in \mathcal{C}\}$ forms a basis for $\mathcal{S}(\mathcal{C})$, i.e. $\mathcal{C}$ is regular and*

$$\dim \mathcal{S}(\mathcal{C}) = |\mathcal{C}| = I_s(\mathcal{C}).$$

The simplest concrete condition to ensure nondegeneracy of $\mathcal{C}$ is, of course, to require that $\mathcal{V}$ is in general position. But nondegeneracy certainly holds under weaker assumptions. For instance if

(i) the collections $\{(C(K_{i,j})\colon j = 1,\ldots,\binom{s+k}{s})\}$ (cf. (5.10)), $K_{i,j} \in \Delta(I_i, k)$, $j = 1,\ldots,\binom{s+k}{s}$, are weak chains in $\mathcal{C}$ for $1 \leqslant i \leqslant |\mathcal{J}|$ and if

(ii) $\mathrm{vol}_{s-1}(\rho \cap \rho') = 0$ whenever $\rho \in C(K)$, $\rho' \in (K')$, $K \cap K' = \varnothing$, $K, K' \in \Delta(\mathcal{J}, k)$,

then $\mathcal{C}$ is nondegenerate since the chain constructed in the proof of Theorem 5.1 by composing the 'local chains' in $\Delta(I_i, k)$ (cf. Lemma 5.2) induces a weak chain of the same length in $\mathcal{C}$.

Condition (i) in turn holds if for instance all $K \in \mathcal{C}$ are individually in general position or if each element of $\Theta(\mathcal{J})$ (cf. (5.9)) forms a proper s-simplex and the sets

$$\{u^{j,m}\colon j \in I,\ m = 0,\ldots,k\}$$

are for each $I \in \mathcal{J}$ in general position.

The authors believe that the above assertion is even valid in a more general sense, namely when the configuration $\mathcal{C}$ is induced by *any* triangulation (cf. (5.6), (5.9)) of $R^s \times R^k$ with the property that all the vertices of the corresponding $(s+k)$-simplices belong to $\bigcup\{R^s \times \{e^i\}\colon i = 0,\ldots,k\}$. This latter condition is known to be necessary [6]. Moreover, when S^k is replaced by any k-polytope with more than $k+1$ vertices, the induced collection of B-splines is always linearly dependent [6]. Hence one has in this case, even if the corresponding configuration $\mathcal{C}$ is nondegenerate,

$$I_s(\mathcal{C}) < |\mathcal{C}|.$$

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