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On the Integral $\int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t\, dt$

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Abstract. A recurrence relation is given for the integral in the title. Formulae which allow easy evaluation by formula manipulation are given for integer and half-integer values of ν. Explicit expressions for these integrals for small values of m and $\nu = n$, $\nu = n + \frac{1}{2}$, are also presented.

1. Introduction. The purpose of this note is to give expressions for the Laplace (or Mellin) integral

$$R_m(\mu, \nu) = \int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t\, dt \qquad (\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0), \tag{1}$$

in particular for integer and half-integer values of ν. This integral, with integer values of ν, occurs when establishing an asymptotic expansion, as $x \to \infty$, for the Landau density function [2], [10]

$$\phi(x) = \frac{1}{\pi} \int_0^\infty e^{-xt} t^{-t} \sin \pi t\, dt. \tag{2}$$

A contour integral related to (1) appears in the theory of heat conduction [4], [12] and in other physical problems. For $m \leqslant 3$, $R_m(\mu, \nu)$ can be found, for example, in [6, No. 4.3521, 4.3582, 4.3583]:

$$R_1(\mu, \nu) = \frac{\Gamma(\nu)}{\mu^\nu}[\psi(\nu) - \log \mu], \tag{3}$$

$$R_2(\mu, \nu) = \frac{\Gamma(\nu)}{\mu^\nu}\left\{[\psi(\nu) - \log \mu]^2 + \zeta(2, \nu)\right\}, \tag{4}$$

$$R_3(\mu, \nu) = \frac{\Gamma(\nu)}{\mu^\nu}\left\{[\psi(\nu) - \log \mu]^3 + 3[\psi(\nu) - \log \mu]\zeta(2, \nu) - 2\zeta(3, \nu)\right\},^* \tag{5}$$

where

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{6}$$

and where

$$\zeta(r, q) = \sum_{j=0}^\infty \frac{1}{(q+j)^r} \tag{7}$$

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*Note that the corresponding formula in [6], copied from McLachlan et al. [11], is incorrect, as has been pointed out in [3]. In addition, $\zeta(2, \nu)$ and $\zeta(3, \nu)$ must be replaced by $\zeta(2, \nu + 1)$ and $\zeta(3, \nu + 1)$, respectively, in the formulae corresponding to $m = 2$ and $m = 3$ in [11].

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is the generalized zeta function related to $\psi(x)$ by [6, No. 8.3638]

$$\psi^{(k)}(x) = (-1)^{k+1} k!\zeta(k+1, x). \tag{8}$$

2. A Recurrence Formula for $R_m(\mu, \nu)$. In order to give a recurrence formula for $R_m(\mu, \nu)$, we need the following

LEMMA. *Let $f(x)$ be m times differentiable. Then*

$$\left(\frac{d}{dx}\right)^m e^{f(x)} = e^{f(x)} F_m(x), \tag{9}$$

where $F_0(x) = 1$, $F_1(x) = f'(x)$, and

$$F_k(x) = F'_{k-1}(x) + F_1(x) F_{k-1}(x) \qquad (k = 2, 3, \ldots, m). \tag{10}$$

The proof by induction is easy.

We apply this lemma to $R_m(\mu, \nu)$ and obtain

$$\begin{aligned} R_m(\mu, \nu) &= \int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t\, dt \\ &= \left(\frac{d}{d\nu}\right)^m \int_0^\infty e^{-\mu t} t^{\nu-1}\, dt = \left(\frac{d}{d\nu}\right)^m \frac{\Gamma(\nu)}{\mu^\nu} = \frac{\Gamma(\nu)}{\mu^\nu} G_m(\mu, \nu), \end{aligned} \tag{11}$$

where $G_0(\mu, \nu) = 1$, $G_1(\mu, \nu) = \psi(\nu) - \log \mu$, and

$$G_k(\mu, \nu) = \frac{d}{d\nu} G_{k-1}(\mu, \nu) + G_1(\mu, \nu) G_{k-1}(\mu, \nu). \tag{12}$$

With a formula manipulation system, such as REDUCE [8], it is easy to compute $R_m(\mu, \nu)$ from (8) and (12) in terms of $\psi(\nu) - \log \mu$ and $\zeta(k, \nu)$ $(2 \leqslant k \leqslant m)$. The resulting expressions for $R_m(\mu, \nu)$ for $m = 1(1)9$ are given in Table 1.

TABLE 1

Let

$$Y_m = \frac{\mu^\nu}{\Gamma(\nu)} \int_0^\infty e^{-\mu t} t^{\nu-1} \log^m t\, dt = \frac{\mu^\nu}{\Gamma(\nu)} R_m(\mu, \nu) \qquad (\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0)$$

$$\phi = \psi(\nu) - \log \mu,$$

and

$$\beta_2 = \zeta(2, \nu)$$

$$\beta_3 = \zeta(3, \nu)$$

$$\beta_4 = \zeta^2(2, \nu) + 2\zeta(4, \nu)$$

$$\beta_5 = 5\zeta(2, \nu)\zeta(3, \nu) + 6\zeta(5, \nu)$$

$$\beta_6 = 3\zeta^3(2, \nu) + 18\zeta(2, \nu)\zeta(4, \nu) + 8\zeta^2(3, \nu) + 24\zeta(6, \nu)$$

$$\beta_7 = 35\zeta^2(2, \nu)\zeta(3, \nu) + 84\zeta(2, \nu)\zeta(5, \nu) + 70\zeta(3, \nu)\zeta(4, \nu) + 120\zeta(7, \nu)$$

$$\begin{aligned} \beta_8 = {} & 15\zeta^4(2, \nu) + 180\zeta^2(2, \nu)\zeta(4, \nu) + 160\zeta(2, \nu)\zeta^2(3, \nu) + 480\zeta(2, \nu)\zeta(6, \nu) \\ & + 384\zeta(3, \nu)\zeta(5, \nu) + 180\zeta^2(4, \nu) + 720\zeta(8, \nu) \end{aligned}$$

$$\begin{aligned} \beta_9 = {} & 315\zeta^3(2, \nu)\zeta(3, \nu) + 1134\zeta^2(2, \nu)\zeta(5, \nu) + 1890\zeta(2, \nu)\zeta(3, \nu)\zeta(4, \nu) \\ & + 3240\zeta(2, \nu)\zeta(7, \nu) + 280\zeta^3(3, \nu) + 2520\zeta(3, \nu)\zeta(6, \nu) \\ & + 2268\zeta(4, \nu)\zeta(5, \nu) + 5040\zeta(9, \nu). \end{aligned}$$

Then

$$Y_0 = 1$$
$$Y_1 = \phi$$
$$Y_2 = \phi^2 + \beta_2$$
$$Y_3 = \phi^3 + 3\beta_2\phi - 2\beta_3$$
$$Y_4 = \phi^4 + 6\beta_2\phi^2 - 8\beta_3\phi + 3\beta_4$$
$$Y_5 = \phi^5 + 10\beta_2\phi^3 - 20\beta_3\phi^2 + 15\beta_4\phi - 4\beta_5$$
$$Y_6 = \phi^6 + 15\beta_2\phi^4 - 40\beta_3\phi^3 + 45\beta_4\phi^2 - 24\beta_5\phi + 5\beta_6$$
$$Y_7 = \phi^7 + 21\beta_2\phi^5 - 70\beta_3\phi^4 + 105\beta_4\phi^3 - 84\beta_5\phi^2 + 35\beta_6\phi - 6\beta_7$$
$$Y_8 = \phi^8 + 28\beta_2\phi^6 - 112\beta_3\phi^5 + 210\beta_4\phi^4 - 224\beta_5\phi^3 + 140\beta_6\phi^2 - 48\beta_7\phi + 7\beta_8$$
$$Y_9 = \phi^9 + 36\beta_2\phi^7 - 168\beta_3\phi^6 + 378\beta_4\phi^5 - 504\beta_5\phi^4 + 420\beta_6\phi^3 - 216\beta_7\phi^2 + 63\beta_8\phi - 8\beta_9.$$

Thus

$$Y_m = \phi^m + \sum_{j=2}^{m} c_{m,j}\beta_j\phi^{m-j} \qquad (m = 2, \ldots, 9)$$
$$c_{m,j} = (-1)^j (j-1) \sum_{k=j}^{m} \binom{k-1}{j-1}.$$

3. A Formula for $R_m(\mu, n+1)$, n an Integer. Expressions for $R_m(\mu, n+1)$, where n is an integer, can be obtained from (12) by setting $\nu = n+1$ in $G_m(\mu, \nu)$ and using

$$\zeta(k, n+1) = \zeta(k) - \sigma(n, k), \tag{13}$$
$$\psi(n+1) = -\gamma + \sigma(n, 1), \tag{14}$$

where

$$\sigma(n, k) = \sum_{j=1}^{n} \frac{1}{j^k}, \tag{15}$$

γ is Euler's constant and $\zeta(k)$ is the Riemann zeta function.

In order to derive another formula for this case, we set $\tilde{R}_m(\mu, n) = R_m(\mu, n+1)$ and write

$$\begin{aligned} \tilde{R}_m(\mu, n) &= \int_0^\infty e^{-\mu t} t^n \log^m t\, dt \qquad (n \geqslant 0) \\ &= \left(\frac{d}{d\alpha}\right)^m \int_0^\infty e^{-\mu t} t^{n+\alpha}\, dt \,\Big|_{\alpha=0} \\ &= \frac{1}{\mu^{n+1}} \left(\frac{d}{d\alpha}\right)^m \left[\frac{1}{\mu^\alpha}\Gamma(1+n+\alpha)\right]_{\alpha=0} \\ &= \frac{1}{\mu^{n+1}} \left(\frac{d}{d\alpha}\right)^m \left[\frac{1}{\alpha}(\alpha)_{n+1}\frac{1}{\mu^\alpha}\Gamma(1+\alpha)\right]_{\alpha=0}, \end{aligned} \tag{16}$$

where [1, No. 24.1.3]

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \sum_{j=1}^{k} (-1)^{j+k} S_k^{(j)} \alpha^j \tag{17}$$

is the Pochhammer symbol. The $S_k^{(j)}$ are the Stirling numbers of the first kind, defined by

$$(18)\qquad \begin{aligned} &S_{k+1}^{(j)} = S_k^{(j-1)} - kS_k^{(j)} \qquad (k \geqslant j \geqslant 1),\\ &S_k^{(1)} = (-1)^{k-1}(k-1)!,\\ &S_k^{(k)} = 1, \qquad S_k^{(0)} = \delta_{0k},\\ &S_k^{(j)} = 0 \qquad \text{for } k < j,\end{aligned}$$

$$S_k^{(j)} = \sum_{i=0}^{k-j} \frac{1}{i!} \sum_{\lambda=0}^{i} (-1)^{\lambda} \binom{i}{\lambda} \binom{k-1+i}{k-j+i} \binom{2k-j}{k-j-i} \lambda^{k-j+i}.$$

Applying the Leibniz formula, we have

$$(19)\qquad R_m(\mu, n) = \frac{1}{\mu^{n+1}} \sum_{\rho=0}^{m} \binom{m}{\rho} \left[\frac{1}{\alpha}(\alpha)_{n+1}\right]^{(\rho)} \left[\frac{1}{\mu^{\alpha}}\Gamma(1+\alpha)\right]^{(m-\rho)} \Big|_{\alpha=0}.$$

We see from (17) that

$$(20)\qquad \left(\frac{d}{d\alpha}\right)^{\rho} \frac{1}{\alpha}(\alpha)_{n+1} \Big|_{\alpha=0} = \begin{cases} (-1)^{\rho+n} \rho! S_{n+1}^{(\rho+1)} & (\rho \leqslant n),\\ 0 & (\rho > n).\end{cases}$$

We note that [6, No. 8.3421]

$$(21)\qquad \log \mu^{-\alpha}\Gamma(1+\alpha) = -(\gamma + \log \mu)\alpha + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \alpha^k / k \qquad (|\alpha| < 1),$$

and therefore, by exponentiation (see, for example, Henrici [9, Section 1.6]),

$$(22)\qquad \mu^{-\alpha}\Gamma(1+\alpha) = \sum_{k=0}^{\infty} b_k \alpha^k \qquad (|x| < 1),$$

where $b_0 = 1$,

$$(23)\qquad b_k = \frac{1}{k} \sum_{\kappa=1}^{k} (-1)^{\kappa} \tilde{\zeta}(\kappa) b_{k-\kappa},$$

and $\tilde{\zeta}(1) = \gamma + \log \mu$, $\tilde{\zeta}(\kappa) = \zeta(\kappa)$ for $\kappa > 1$.

Substituting (20) and (22) into (19), we finally have

$$(24)\qquad \begin{aligned} \tilde{R}_m(\mu, n) &= \int_0^{\infty} e^{-\mu t} t^n \log^m t \, dt\\ &= \frac{(-1)^n m!}{\mu^{n+1}} \sum_{\rho=0}^{\min(m,n)} (-1)^{\rho} S_{n+1}^{(\rho+1)} b_{m-\rho} \qquad (n \geqslant 0).\end{aligned}$$

We consider a few special cases. For $n = 0$ we obtain

$$(25)\qquad \tilde{R}_m(\mu, 0) = \frac{m!}{\mu} b_m.$$

For $m = 1$ we have

$$(26)\qquad \tilde{R}_1(\mu, n) = \frac{(-1)^n}{\mu^{n+1}} \left(S_{n+1}^{(1)} b_1 - S_{n+1}^{(2)}\right) = \frac{n!}{\mu^{n+1}} [\sigma(n, 1) - \gamma - \log \mu]$$

in accordance with (3), since from (18),

$$S_{n+1}^{(1)} = (-1)^n n!, \qquad S_{n+1}^{(2)} = (-1)^{n+1} n!\sigma(n,1). \tag{27}$$

For $m = 2$ it follows from (24) and (4) that

$$\begin{aligned}\tilde{R}_2(\mu, n) &= 2\frac{(-1)^n}{\mu^{n+1}}\left(S_{n+1}^{(1)}b_2 - S_{n+1}^{(2)}b_1 + S_{n+1}^{(3)}\right) \\ &= \frac{n!}{\mu^{n+1}}\left\{(\gamma + \log\mu)^2 - 2(\gamma + \log\mu)\sigma(n,1)\right. \\ &\qquad \left. + \frac{\pi^2}{6} + 2\frac{(-1)^n}{n!}S_{n+1}^{(3)}\right\} \\ &= \frac{n!}{\mu^{n+1}}\left\{(\gamma + \log\mu)^2 - 2(\gamma + \log\mu)\sigma(n,1)\right. \\ &\qquad \left. + \sigma^2(n,1) - \sigma(n,2) + \frac{\pi^2}{6}\right\},\end{aligned} \tag{28}$$

where we have used $\zeta(2) = \pi^2/6$. Note that $S_{n+1}^{(2)} = 0$ for $n = 0$, and $S_{n+1}^{(3)} = 0$ for $n = 0, 1$. From (28), we obtain the relation

$$S_{n+1}^{(3)} = \frac{1}{2}(-1)^n n!\left[\sigma^2(n,1) - \sigma(n,2)\right], \tag{29}$$

which is not immediately obvious from (18). In general, by comparing the result obtained from the recurrence relation (12) with that given by (24), we find that

$$S_{n+1}^{(k+1)} = (-1)^{n+k}\frac{n!}{k!}\Omega_n^{(k)}(\sigma(n,1),\ldots,\sigma(n,k)), \tag{30}$$

where $\Omega_n^{(k)}$ is a "homogeneous" polynomial with integer coefficients in the k variables $\sigma(n,1),\ldots,\sigma(n,k)$. By a different approach, Comtet [5] has shown that

$$\begin{aligned}&\Omega_n^{(k)}(\sigma(n,1),\ldots,\sigma(n,k)) \\ &\quad = Y_k\left(\sigma(n,1), -1!\sigma(n,2), 2!\sigma(n,2),\ldots, (-1)^{k-1}(k-1)!\sigma(n,k)\right),\end{aligned} \tag{31}$$

where $Y_k(x_1,\ldots,x_k)$ is the exponential complete Bell polynomial in k variables defined by

$$\exp\left(\sum_{j=1}^{\infty} x_j\frac{t^j}{j!}\right) = 1 + \sum_{k=1}^{\infty} Y_k(x_1,\ldots,x_k)\frac{t^k}{k!}. \tag{32}$$

We give two more examples:

$$\begin{aligned}S_{n+1}^{(4)} &= \frac{1}{6}(-1)^{n+1}n!\left[\sigma^3(n,1) - 3\sigma(n,1)\sigma(n,2) + 2\sigma(n,3)\right], \\ S_{n+1}^{(5)} &= \frac{1}{24}(-1)^n n!\left[\sigma^4(n,1) + 3\sigma^2(n,2) - 6\sigma^2(n,1)\sigma(n,2)\right. \\ &\qquad \left. + 8\sigma(n,1)\sigma(n,3) - 6\sigma(n,4)\right].\end{aligned} \tag{33}$$

Note that $\Omega_n^{(k)} \equiv 0$ for $0 \leqslant n < k$. This fact is not apparent when one derives $R_m(\mu,\nu)$ by means of the recurrence (12), replacing ν by $n + 1$ in $R_m(\mu,\nu)$.

In general, formula (24) can easily be evaluated by formula manipulation. Expressions for $\tilde{R}_m(\mu, n)$, $m = 0(1)5$, $n = 0(1)5$, are given in Table 2. For special values of m and n (e.g. $n > 0$, $m = 1$, or $n = 0$, $m \leqslant 3$), these expressions can also be found in the relevant handbooks or tables of integral transforms.

TABLE 2

Let

$$I_{mn} = \mu^{n+1}\int_0^\infty e^{-\mu t} t^n \log^m t\,dt = \mu^{n+1}\tilde{R}_m(\mu, n), \quad C = \gamma + \log\mu \qquad (\operatorname{Re}\mu > 0)$$

Then

$$I_{0n} = n! \qquad (n \geqslant 0)$$

$$I_{10} = -C$$

$$I_{1n} = n!\left[\sum_{j=1}^{n} j^{-1} - C\right] \qquad (n > 0)$$

$$I_{20} = C^2 + \frac{\pi^2}{6}$$

$$I_{21} = C^2 - 2C + \frac{\pi^2}{6}$$

$$I_{22} = 2C^2 - 6C + \frac{\pi^2}{3} + 2$$

$$I_{23} = 6C^2 - 22C + \pi^2 + 12$$

$$I_{24} = 24C^2 - 100C + 4\pi^2 + 70$$

$$I_{25} = 120C^2 - 548C + 20\pi^2 + 450$$

$$I_{30} = -C^3 - \frac{\pi^2}{2}C - 2\zeta(3)$$

$$I_{31} = -C^3 + 3C^2 - \frac{\pi^2}{2}C - 2\zeta(3) + \frac{\pi^2}{2}$$

$$I_{32} = -2C^3 + 9C^2 - (\pi^2 + 6)C - 4\zeta(3) + \frac{3}{2}\pi^2$$

$$I_{33} = -6C^3 + 33C^2 - 3(\pi^2 + 12)C - 12\zeta(3) + \frac{11}{2}\pi^2 + 6$$

$$I_{34} = -24C^3 + 150C^2 - 6(2\pi^2 + 35)C - 48\zeta(3) + 25\pi^2 + 60$$

$$I_{35} = -120C^3 + 822C^2 - 30(2\pi^2 + 45)C - 240\zeta(3) + 137\pi^2 + 510$$

$$I_{40} = C^4 + \pi^2C^2 + 8\zeta(3)C + \frac{3}{20}\pi^4$$

$$I_{41} = C^4 - 4C^3 + \pi^2C^2 + 2\big(4\zeta(3) - \pi^2\big)C - 8\zeta(3) + \frac{3}{20}\pi^4$$

$$I_{42} = 2C^4 - 12C^3 + 2(\pi^2 + 6)C^2 + 2\big(8\zeta(3) - 3\pi^2\big)C - 24\zeta(3) + \frac{3}{10}\pi^4 + 2\pi^2$$

$$I_{43} = 6C^4 - 44C^3 + 6(\pi^2 + 12)C^2 + 2\big(24\zeta(3) - 11\pi^2 - 12\big)C - 88\zeta(3) + \frac{9}{10}\pi^4 + 12\pi^2$$

$$I_{44} = 24C^4 - 200C^3 + 12(2\pi^2 + 35)C^2 + 4\big(48\zeta(3) - 25\pi^2 - 60\big)C$$
$$-400\zeta(3) + \frac{18}{5}\pi^4 + 70\pi^2 + 24$$

$$I_{45} = 120C^4 - 1096C^3 + 60(2\pi^2 + 45)C^2 + 4\big(240\zeta(3) - 137\pi^2 - 510\big)C$$
$$-2192\zeta(3) + 18\pi^4 + 450\pi^2 + 360$$

$$I_{50} = -C^5 - \frac{5}{3}\pi^2 C^3 - 20\zeta(3)C^2 - \frac{3}{4}\pi^4 C - \frac{10}{3}\zeta(3)\pi^2 - 24\zeta(5)$$

$$I_{51} = -C^5 + 5C^4 - \frac{5}{3}\pi^2 C^3 - 5(4\zeta(3) - \pi^2)C^2 + \left(40\zeta(3) - \frac{3}{4}\pi^4\right)C - \frac{10}{3}\zeta(3)\pi^2 - 24\zeta(5) + \frac{3}{4}\pi^4$$

$$I_{52} = -2C^5 + 15C^4 - 10\left(\frac{1}{3}\pi^2 + 2\right)C^3 - 5(8\zeta(3) - 3\pi^2)C^2 + \left(120\zeta(3) - \frac{3}{2}\pi^4 - 10\pi^2\right)C - \frac{20}{3}\zeta(3)\pi^2 - 40\zeta(3) - 48\zeta(5) + \frac{9}{4}\pi^4$$

$$I_{53} = -6C^5 + 55C^4 - 10(\pi^2 + 12)C^3 - 5(24\zeta(3) - 11\pi^2 - 12)C^2 + \left(440\zeta(3) - \frac{9}{2}\pi^4 - 60\pi^2\right)C - 20\zeta(3)\pi^2 - 240\zeta(3) - 144\zeta(5) + \frac{33}{4}\pi^4 + 10\pi^2$$

$$I_{54} = -24C^5 + 250C^4 - 20(2\pi^2 + 35)C^3 - 10(48\zeta(3) - 25\pi^2 - 60)C^2 + 2(1000\zeta(3) - 9\pi^4 - 175\pi^2 - 60)C - 80\zeta(3)\pi^2 - 1400\zeta(3) - 576\zeta(5) + \frac{75}{2}\pi^4 + 100\pi^2$$

$$I_{55} = -120C^5 + 1370C^4 - 100(2\pi^2 + 45)C^3 - 10(240\zeta(3) - 137\pi^2 - 510)C^2 + 10(1096\zeta(3) - 9\pi^4 - 225\pi^2 - 180)C - 400\zeta(3)\pi^2 - 9000\zeta(3) - 2880\zeta(5) + \frac{411}{2}\pi^4 + 850\pi^2 + 120$$

4. A Formula for $R_m(\mu, n + 1/2)$, n an Integer. Setting $\nu = n + 1/2$ and [1, No. 6.3.4, 23.2.20]

(34)
$$\zeta(k, n + 1/2) = (2^k - 1)\zeta(k) - 2^k\hat{\sigma}(n, k),$$
$$\psi(n + 1/2) = -\gamma - \log 4 + 2\hat{\sigma}(n, 1),$$

where

(35)
$$\hat{\sigma}(n, k) = \sum_{j=1}^{n} \frac{1}{(2j-1)^k},$$

in the expression for $R_m(\mu, \nu)$ obtained from (12), leads to expressions for $R_m^*(\mu, n) = R_m(\mu, n + 1/2)$. In the same way as in Section 3, we may also derive another formula in this case. We start with

(36)
$$R_m^*(\mu, n) = \int_0^\infty e^{-\mu t} t^{n-1/2} \log^m t\, dt \qquad (n \geq 0)$$
$$= \left(\frac{d}{d\alpha}\right)^m \int_0^\infty e^{-\mu t} t^{n-1/2+\alpha}\, dt \,\Big|_{\alpha=0}$$
$$= \frac{1}{\mu^{n+1/2}} \left(\frac{d}{d\alpha}\right)^m \left[\frac{1}{\mu^\alpha}\Gamma\left(n + \frac{1}{2} + \alpha\right)\right]_{\alpha=0}.$$

Using

(37)
$$\Gamma\left(\frac{1}{2} + n + \alpha\right) = \left(\alpha + \frac{1}{2}\right)_n \Gamma\left(\frac{1}{2} + \alpha\right) = \sqrt{\pi}\left(\alpha + \frac{1}{2}\right)_n 2^{-2\alpha}\frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)},$$

we obtain

(38)
$$R_m^*(\mu, \nu) = \frac{\sqrt{\pi}}{\mu^{n+1/2}} \left(\frac{d}{d\alpha}\right)^m \left[\left(\alpha + \frac{1}{2}\right)_n H(\alpha, \mu)\right]_{\alpha=0},$$

where

$$H(\alpha, \mu) = \frac{1}{(4\mu)^{\alpha}} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}. \tag{39}$$

As for the series (22), we can write

$$H(\alpha, \mu) = \sum_{k=0}^{\infty} b_k^* \alpha^k \qquad \left(|\alpha| < \frac{1}{2}\right), \tag{40}$$

where $b_0^* = 1$,

$$b_k^* = \frac{1}{k} \sum_{\kappa=1}^{k} (-1)^{\kappa} \zeta^*(\kappa) b_{k-\kappa}^* \tag{41}$$

and

$$\zeta^*(\kappa) = \begin{cases} \gamma + \log 4\mu & (\kappa = 1), \\ (2^{\kappa} - 1)\zeta(\kappa) & (\kappa > 1). \end{cases} \tag{42}$$

Applying the Leibniz formula to (38), we use (17) and write

$$\begin{aligned} \left(\frac{d}{d\alpha}\right)^{\rho} \left(\alpha + \frac{1}{2}\right)_n \Big|_{\alpha=0} &= \left(\frac{d}{d\alpha}\right)^{\rho} \sum_{k=1}^{n} (-1)^{k+n} S_n^{(k)} \left(\alpha + \frac{1}{2}\right)^k \\ &= \begin{cases} \rho! \sum_{k=\rho}^{n} (-1)^{k+n} S_n^{(k)} \binom{k}{\rho} 2^{\rho-k} & (\rho \leqslant n), \\ 0 & (\rho > n), \end{cases} \end{aligned} \tag{43}$$

and therefore finally

$$\begin{aligned} R_m^*(\mu, n) &= \int_0^{\infty} e^{-\mu t} t^{n-1/2} \log^m t \, dt \qquad (n \geqslant 0) \\ &= \sqrt{\frac{\pi}{\mu}} \frac{(-1)^n m!}{\mu^n} \sum_{\rho=0}^{\min(m,n)} b_{m-\rho}^* \sum_{k=\rho}^{n} (-1)^k S_n^{(k)} \binom{k}{\rho} 2^{\rho-k}. \end{aligned} \tag{44}$$

We consider two special cases. For $n = 0$, we have

$$R_m^*(\mu, 0) = \sqrt{\frac{\pi}{\mu}} \, m! b_m^*. \tag{45}$$

For $m = 1$ we obtain from (3) and (44) the expression [6, No. 4.3523]

$$\begin{aligned} R_1^*(\mu, n) &= \frac{\Gamma(n + \frac{1}{2})}{\mu^{n+1/2}} [-\gamma - \log 4\mu + 2\hat{\sigma}(n, 1)] \\ &= \sqrt{\frac{\pi}{\mu}} \frac{(2n-1)!!}{(2\mu)^n} [-\gamma - \log 4\mu + 2\hat{\sigma}(n, 1)] \\ &= \sqrt{\frac{\pi}{\mu}} \frac{1}{\mu^n} \left[-\left(\frac{1}{2}\right)_n (\gamma + \log 4\mu) + 2(-1)^n \sum_{k=1}^{n} (-1)^k S_n^{(k)} k 2^{-k} \right]. \end{aligned} \tag{46}$$

Thus, by comparison,

$$\sum_{k=1}^{n} (-1)^k S_n^{(k)} k 2^{-k} = (-1)^n \frac{(2n-1)!!}{2^n} \hat{\sigma}(n, 1). \tag{47}$$

Similarly, for $m = 2$, one finds

$$(48)\quad \sum_{k=2}^{n} (-1)^k S_n^{(k)} k(k-1)2^{-k} = (-1)^n \frac{(2n-1)!!}{2^n}\left[\hat{\sigma}^2(n,1) - \hat{\sigma}(n,2)\right].$$

In general, as in the case of $\tilde{R}_m(\mu, n)$, a comparison of the result obtained from (12) with expression (44) shows that

$$(49)\quad \sum_{k=\rho}^{n} (-1)^k S_n^{(k)} \frac{k!}{(k-\rho)!} 2^{-k} = (-1)^n \frac{(2n-1)!!}{2^n} \Omega_n^{(\rho)}(\hat{\sigma}(n,1),\ldots,\hat{\sigma}(n,\rho)),$$

where $\Omega_n^{(\rho)}$ is defined by (31), and $\Omega_n^{(\rho)} \equiv 0$ for $0 \leqslant n < \rho$.

Formula (44) can be evaluated easily by formula manipulation. Expressions for $R_m^*(\mu, n)$, $m = 0(1)5$, $n = 0(1)5$, are given in Table 3. $R_1^*(\mu, n)$ can also be found in [6, No. 4.3523].

TABLE 3

Let

$$J_{mn} = \sqrt{\frac{\mu}{\pi}}\,\mu^n \int_0^\infty e^{-\mu t} t^{n-1/2} \log^m t\,dt = \sqrt{\frac{\mu}{\pi}}\,\mu^n R_m^*(\mu, n) \qquad (\operatorname{Re}\mu > 0)$$

and

$$C = \gamma + \log 4\mu.$$

Then

$$J_{00} = 1$$

$$J_{0n} = \frac{(2n-1)!!}{2^n} \qquad (n > 0)$$

$$J_{10} = -C$$

$$J_{1n} = \frac{(2n-1)!!}{2^n}\left(-C + 2\sum_{j=1}^{n} \frac{1}{2j-1}\right) \qquad (n > 0)$$

$$J_{20} = C^2 + \frac{\pi^2}{2}$$

$$J_{21} = \frac{1}{2}C^2 - 2C + \frac{\pi^2}{4}$$

$$J_{22} = \frac{3}{4}C^2 - 4C + \frac{3}{8}\pi^2 + 2$$

$$J_{23} = \frac{15}{8}C^2 - \frac{23}{2}C + \frac{15}{16}\pi^2 + 9$$

$$J_{24} = \frac{105}{16}C^2 - 44C + \frac{105}{32}\pi^2 + 43$$

$$J_{25} = \frac{945}{32}C^2 - \frac{1689}{8}C + \frac{945}{64}\pi^2 + \frac{475}{2}$$

$$J_{30} = -C^3 - \frac{3}{2}\pi^2 C - 14\zeta(3)$$

$$J_{31} = -\frac{1}{2}C^3 + 3C^2 - \frac{3}{4}\pi^2 C - 7\zeta(3) + \frac{3}{2}\pi^2$$

$$J_{32} = -\frac{3}{4}C^3 + 6C^2 - 3\left(\frac{3}{8}\pi^2 + 2\right)C - \frac{21}{2}\zeta(3) + 3\pi^2$$

$$J_{33} = -\frac{15}{8}C^3 + \frac{69}{4}C^2 - 9\left(\frac{5}{16}\pi^2 + 3\right)C - \frac{105}{4}\zeta(3) + \frac{69}{8}\pi^2 + 6$$

$$J_{34} = -\frac{105}{16}C^3 + 66C^2 - 3\left(\frac{105}{32}\pi^2 + 43\right)C - \frac{735}{8}\zeta(3) + 33\pi^2 + 48$$

$$J_{35} = -\frac{945}{32}C^3 + \frac{5067}{16}C^2 - \frac{15}{2}\left(\frac{189}{32}\pi^2 + 95\right)C - \frac{6615}{16}\zeta(3) + \frac{5067}{32}\pi^2 + 345$$

$$J_{40} = C^4 + 3\pi^2C^2 + 56\zeta(3)C + \frac{7}{4}\pi^4$$

$$J_{41} = \frac{1}{2}C^4 - 4C^3 + \frac{3}{2}\pi^2C^2 + 2\left(14\zeta(3) - 3\pi^2\right)C - 56\zeta(3) + \frac{7}{8}\pi^4$$

$$J_{42} = \frac{3}{4}C^4 - 8C^3 + 3\left(\frac{3}{4}\pi^2 + 4\right)C^2 + 6\left(7\zeta(3) - 2\pi^2\right)C - 112\zeta(3) + \frac{21}{16}\pi^4 + 6\pi^2$$

$$J_{43} = \frac{15}{8}C^4 - 23C^3 + 9\left(\frac{5}{8}\pi^2 + 6\right)C^2 + 3\left(35\zeta(3) - \frac{23}{2}\pi^2 - 8\right)C$$
$$-322\zeta(3) + \frac{105}{32}\pi^4 + 27\pi^2$$

$$J_{44} = \frac{105}{16}C^4 - 88C^3 + 3\left(\frac{105}{16}\pi^2 + 86\right)C^2 + 3\left(\frac{245}{2}\zeta(3) - 44\pi^2 - 64\right)C$$
$$-1232\zeta(3) + \frac{735}{64}\pi^4 + 129\pi^2 + 24$$

$$J_{45} = \frac{945}{32}C^4 - \frac{1689}{4}C^3 + 15\left(\frac{189}{32}\pi^2 + 95\right)C^2 + 3\left(\frac{2205}{4}\zeta(3) - \frac{1689}{8}\pi^2 - 460\right)C$$
$$-\frac{11823}{2}\zeta(3) + \frac{6615}{128}\pi^4 + \frac{1425}{2}\pi^2 + 300$$

$$J_{50} = -C^5 - 5\pi^2C^3 - 140\zeta(3)C^2 - \frac{35}{4}\pi^4C - 70\zeta(3)\pi^2 - 744\zeta(5)$$

$$J_{51} = -\frac{1}{2}C^5 + 5C^4 - \frac{5}{2}\pi^2C^3 + 5\left(3\pi^2 - 14\zeta(3)\right)C^2$$
$$+35\left(8\zeta(3) - \frac{\pi^4}{8}\right)C - 35\zeta(3)\pi^2 - 372\zeta(5) + \frac{35}{4}\pi^4$$

$$J_{52} = -\frac{3}{4}C^5 + 10C^4 - 5\left(\frac{3}{4}\pi^2 + 4\right)C^3 + 15\left(2\pi^2 - 7\zeta(3)\right)C^2$$
$$+5\left(112\zeta(3) - \frac{21}{16}\pi^4 - 6\pi^2\right)C - \frac{105}{2}\zeta(3)\pi^2 - 280\zeta(3) - 558\zeta(5) + \frac{35}{2}\pi^4$$

$$J_{53} = -\frac{15}{8}C^5 + \frac{115}{4}C^4 - 15\left(\frac{5}{8}\pi^2 + 6\right)C^3 + 15\left(\frac{23}{4}\pi^2 - \frac{35}{2}\zeta(3) + 4\right)C^2$$
$$+5\left(322\zeta(3) - \frac{105}{32}\pi^4 - 27\pi^2\right)C - \frac{525}{4}\zeta(3)\pi^2$$
$$-1260\zeta(3) - 1395\zeta(5) + \frac{805}{16}\pi^4 + 30\pi^2$$

$$J_{54} = -\frac{105}{16}C^5 + 110C^4 - 5\left(\frac{105}{16}\pi^2 + 86\right)C^3 + 15\left(22\pi^2 - \frac{245}{4}\zeta(3) + 32\right)C^2$$
$$+5\left(1232\zeta(3) - \frac{735}{64}\pi^4 - 129\pi^2 - 24\right)C - \frac{3675}{8}\zeta(3)\pi^2$$
$$-6020\zeta(3) - \frac{9765}{2}\zeta(5) + \frac{385}{2}\pi^4 + 240\pi^2$$

$$J_{55} = -\frac{945}{32}C^5 + \frac{8445}{16}C^4 - 25\left(\frac{189}{32}\pi^2 + 95\right)C^3 + 15\left(\frac{1689}{16}\pi^2 - \frac{2205}{8}\zeta(3) + 230\right)C^2$$
$$+15\left(\frac{3941}{2}\zeta(3) - \frac{2205}{128}\pi^4 - \frac{475}{2}\pi^2 - 100\right)C$$
$$-\frac{33075}{16}\zeta(3)\pi^2 - 33250\zeta(3) - \frac{87885}{4}\zeta(5) + \frac{59115}{64}\pi^4 + 1725\pi^2 + 120$$

5. Two Related Integrals. Substituting $t = T^2$ in $\tilde{R}_m(\mu, n)$, we find that

$$\int_0^\infty e^{-\mu T^2} T^{2n+1} \log^m T\, dT = \frac{1}{2^{m+1}} \tilde{R}_m(\mu, n), \tag{50}$$

and, by the same substitution in $R_m^*(\mu, n)$,

$$\int_0^\infty e^{-\mu T^2} T^{2n} \log^m T\, dT = \frac{1}{2^{m+1}} R_m^*(\mu, n). \tag{51}$$

Three particular combinations of these integrals for $m = 1$ are given in [6, No. 4.3551, 4.3553, 4.3554].

6. A Related Contour Integral. When deriving the asymptotic form of an integral of modified Bessel functions appearing in a problem of heat conduction, Ritchie and Sakakura [12] were led to the contour integral

$$I_n^m(\mu) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{\mu z} z^{n-1} \log^m z\, dz, \tag{52}$$

where $\operatorname{Re}\mu > 0$, m, n integers. The contour of integration is one which begins at $-\infty - i\alpha$, proceeds around the origin in a counterclockwise direction and ends at $-\infty + i\alpha$, with arbitrary $\alpha > 0$. For $m > 0$, they give the expression

$$I_n^m(\mu) = \mu^{-n} \sum_{j=0}^{k} A_j^{n,m} \log^j \mu, \tag{53}$$

where the coefficients $A_j^{n,m}$ are given implicitly in [12] as derivatives of gamma functions, and can therefore be expressed by the polygamma function (8). For $n > 0$, the contour of (52) can be contracted to the cut along the negative real axis, and we can write, using (1),

$$\begin{aligned} I_n^m(\mu) &= \frac{(-1)^{n-1}}{2\pi i} \int_0^\infty e^{-\mu t} t^{n-1} \left[(\log t - i\pi)^m - (\log t + i\pi)^m \right] dt \\ &= (-1)^n \sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m}{2j+1} \pi^{2j} R_{m-2j-1}(\mu, n) \qquad (m > 0). \end{aligned} \tag{54}$$

For $m = -k$, $k > 0$, the authors of [12] derive an asymptotic series for $I_n^{-k}(\mu)$, valid for $\mu \to \infty$. The coefficients of this series are again given by derivatives of the gamma function. Under the further restriction $n \geqslant 0$, they present an alternative asymptotic series.

Wood [13] shows that $I_n^{-k}(t)$ ($k > 0$, integer) can be expressed by combinations of the Ramanujan integral $\phi(\mu) = \phi^{(0)}(\mu)$ [7] and its derivatives

$$\phi^{(k)}(\mu) = \left(\frac{d}{d\mu} \right)^k \phi(\mu) = (-1)^k \int_0^\infty e^{-\mu t} t^{k-1} (\pi^2 + \log^2 t)^{-1}\, dt. \tag{55}$$

In particular one may show, analogously to (54), that $I_0^{-1}(\mu) = -\phi(\mu)$. Wood also gives the first few coefficients of an asymptotic expansion for $\phi^{(k)}(\mu)$ ($k \geqslant 1$) as $\mu \to \infty$. (Note that the labels on two curves in Figure 3 of [13] should read $n = 2$, $k = -2$; $n = 2$, $k = -3$, instead of $n = 2$, $k = 2$; $n = 3$, $k = 3$, respectively. These curves agree with formulae (2f) and (2g) in [13]. The author was not able to identify the curve labelled $n = 1$, $k = 2$ in Figure 3.)

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