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Common Zeros of Two Bessel Functions
Part II. Approximations and Tables

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Abstract. In [1] it was shown that two Bessel functions $J_\nu(x)$, $J_\mu(x)$ could have two zeros which were common to both functions, and a computer program was made which takes approximate values of ν, μ and $j_{\nu,k} = j_{\mu,h}$ and $j_{\nu,k+n} = j_{\mu,h+m}$ and from them computes the exact values. Here it will be shown how to find the necessary approximate values to initiate the computation. A table of the smaller ratios $m : n$ with the orders of the functions less than one hundred is given.

1. Introduction. In our paper [1] on common zeros of two Bessel functions we developed a computer program which takes rough approximations to the orders μ, ν ($\nu > \mu$) of the Bessel functions of the first kind $J_\nu(x)$ and $J_\mu(x)$ and to the positions of the two common zeros and derives the exact values. In this paper methods of obtaining the initial approximations will be discussed and a table of values given.

2. Notation. It is understood that the orders μ, ν are real numbers $\geqslant 0$. The zeros of $J_\nu(x)$ will be denoted by $j_{\nu,s}$, and $j_{\nu,1}$ is the first positive zero with $j_{\nu,s+1} > j_{\nu,s}$, s is referred to as the rank of the zero. If the functions $J_\nu(x)$, $J_\mu(x)$ have a pair of common zeros $j_{\nu,k} = j_{\mu,h}$ and $j_{\nu,k+n} = j_{\mu,h+m}$, it will be convenient to speak of n intervals of $J_\nu(x)$ covering m intervals of $J_\mu(x)$, it being understood that the intervals are the segments between two successive zeros and the numbers h, k, m, n are all integers.

3. Properties of the Zeros. The sequence of zeros of $J_\nu(x)$, call it $[j_{\nu,s}]$, is an increasing sequence. If $0 < \nu < \frac{1}{2}$, the intervals $j_{\nu,s+1} - j_{\nu,s}$ are less than π and approach π as $s \to \infty$, but for $\nu > \frac{1}{2}$ they are greater than π and decrease toward π as $s \to \infty$; for $\nu = \frac{1}{2}$ they are all exactly π in length. If s is fixed and $j_{\nu,s}$ is considered as a function of the order ν, $j_{\nu,s}$ is a continuous increasing function of ν with

$$(1) \qquad \frac{\partial j_{\nu,s}}{\partial \nu} = 2 j_{\nu,s} \int_0^\infty K_0(2 j_{\nu,s} \sinh t) e^{-2\nu t}\, dt.$$

Proofs of these facts are contained in Watson's treatise [2], particularly in Chapter 15. Some other properties are contained in the following theorems.

THEOREM 1. *If s, t are fixed integers, $t > s$, and $j_{\nu,s}$, $j_{\nu,t}$ are zeros of $J_\nu(x)$ where ν is real and $\geqslant 0$, then $j_{\nu,t} - j_{\nu,s}$ is an increasing continuous function of ν.*

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Proof. $\partial j_{\nu,s}/\partial\nu$ is given by (1) and

$$\text{(2)}\qquad \frac{\partial j_{\nu,t}}{\partial\nu} = 2j_{\nu,t}\int_0^\infty K_0(2j_{\nu,t}\sinh\phi)e^{-2\nu\phi}\,d\phi.$$

Now $j_{\nu,s}$ and $j_{\nu,t}$, are not functions of the variable of integration ϕ, so if $j_{\nu,t}\sinh\phi = j_{\nu,s}\sinh\theta$ and $j_{\nu,t}\cosh\phi\,d\phi = j_{\nu,s}\cosh\theta\,d\theta$, (2) becomes

$$\frac{\partial j_{\nu,t}}{\partial\nu} = 2j_{\nu,t}\int_0^\infty K_0(2j_{\nu,s}\sinh\theta)e^{-2\nu\operatorname{arc\,sinh}((j_{\nu,s}/j_{\nu,t})\sinh\theta)}$$
$$\cdot\frac{j_{\nu,s}}{j_{\nu,t}}\left[1+\left(\frac{j_{\nu,s}}{j_{\nu,t}}\sinh\theta\right)^2\right]^{-1/2}\cosh\theta\,d\theta.$$

Since $j_{\nu,s}/j_{\nu,t} < 1$,

$$2\nu\operatorname{arc\,sinh}\left(\frac{j_{\nu,s}}{j_{\nu,t}}\sinh\theta\right) < 2\nu\operatorname{arc\,sinh}(\sinh\theta) = 2\nu\theta,$$

whence

$$e^{-2\nu\operatorname{arc\,sinh}((j_{\nu,s}/j_{\nu,t})\sinh\theta)} > e^{-2\nu\theta},$$
$$\left[1+\left(\frac{j_{\nu,s}}{j_{\nu,t}}\sinh\theta\right)^2\right]^{1/2} < [1+\sinh^2\theta]^{1/2} = \cosh\theta.$$

So it follows that

$$\frac{\partial j_{\nu,t}}{\partial\nu} > 2j_{\nu,s}\int_0^\infty K_0(2j_{\nu,s}\sinh\theta)e^{-2\nu\theta}\,d\theta = \frac{\partial j_{\nu,s}}{\partial\nu},$$

and so $\partial(j_{\nu,t} - j_{\nu,s})/\partial\nu > 0$ for all s and t with $t > s$. Q.E.D.

THEOREM 2. *If $J_\nu(x)$ and $J_\mu(x)$ have two common positive zeros with n intervals of $J_\nu(x)$ covering m intervals of $J_\mu(x)$ and $\nu > \mu$ and $\nu > \frac{1}{2}$, then $m > n$.*

Proof. Let the common zeros be $j_{\nu,k} = j_{\mu,h}$ and $j_{\nu,k+n} = j_{\mu,h+m}$. By Theorem 1, $j_{\mu,h+m} - j_{\mu,h}$ is an increasing function of μ. So if μ is increased to ν, $j_{\nu,h+m} - j_{\nu,h} > j_{\mu,h+m} - j_{\mu,h} = j_{\nu,k+n} - j_{\nu,k}$. Since $\nu > \mu$, $j_{\nu,k} > j_{\mu,k}$, but $j_{\mu,h} = j_{\nu,k} > j_{\mu,k}$ therefore $h > k$. Now if k is increased to h, $j_{\nu,h+n} - j_{\nu,h} < j_{\nu,k+n} - j_{n,k}$ if $\nu > \frac{1}{2}$ (Watson [2, bottom of p. 515]). It follows that $j_{\nu,h+m} - j_{\nu,h} > j_{\nu,h+n} - j_{\nu,h}$ and, since m, n are integers, $m > n$.

If one makes a diagram plotting ν as abscissa and $j_{\nu,s}$ as ordinate, keeping s fixed, a family of curves C_s ($s = 1, 2, 3, \ldots$) is obtained. As $\nu \to \infty$ the slopes of these curves approach different values. The slopes of the chords connecting $j_{10{,}000,s}$ and $j_{10{,}001,s}$ can be easily computed and for $s = 1$, $m = 1.00133$; $s = 4$, $m = 1.00385$; $s = 50$, $m = 1.02125$. This is an obvious consequence of Theorem 1, for in order that the differences of the ordinates of $j_{\nu,t}$ and $j_{\nu,s}$ should increase the curves must diverge as ν increases. If the value of ν is fixed and s increases, quite a different situation occurs. From Watson [2, p. 508, 15.6 (2)]:

$$\frac{\partial j_{\nu,s}}{\partial\nu} = \frac{2\nu}{j_{\nu,s}J_{\nu+1}^2(j_{\nu,s})}\int_0^{j_{\nu,s}} J_\nu^2(t)\frac{dt}{t}.$$

As $s \to \infty$, $j_{\nu,s} \to \infty$, $\int_0^\infty J_\nu^2(t)dt/t = 1/2\nu$ (Watson [2, p. 405]). Now if s is large, $j_{\nu,s}$

can be made larger than ν^2, so that the first term of the asymptotic series gives the good approximation

$$\theta = 2^{1/2}\pi^{-1/2}j_{\nu,s}^{-1/2}\cos\left(j_{\nu,s} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Thus $j_{\nu,s} = \pi/2 + k\pi + \nu\pi/2 + \pi/4$, where k is a large integer. So $J_{\nu+1}(j_{\nu,s}) = 2^{1/2}\pi^{-1/2}j_{\nu,s}^{-1/2}\cos k\pi$, and $j_{\nu,s}J_{\nu+1}^2(j_{\nu,s}) \to 2/\pi$ as $j_{\nu,s} \to \infty$. Then $\partial j_{\nu,s}/\partial\nu \to \pi/2$, and this is independent of the value of ν. This diagram is of interest because finding a pair of Bessel functions with two zeros in common is equivalent to finding a rectangle with sides parallel to the axes having all four of its vertices lying on the curves C_s. The diagram suggested the proof of Theorem 2.

4. Note on the Computation of Zeros of Bessel Functions. The Royal Society Tables 7, *Bessel Functions* [3] gives a table of the zeros $j_{\nu,s}$ of $J_\nu(x)$, $0 \leqslant \nu \leqslant 20.5$ (0.5), $1 \leqslant s \leqslant 50$ (1) together with the formulae for their calculation. Olver's uniform asymptotic series [3, p. XIX and Section 8, p. XXXVI] is very powerful for the calculation of zeros when $\nu > 20$ or $s > 50$. Using a TI59 calculator, a program was constructed to avoid the complicated use of Everett's interpolation formulae suggested in [3]. This was replaced by Newton's method for solving the equation $\sqrt{z^2 - 1} - \arccos 1/z - c = 0$. The coefficient p_1 was computed so that, in $j_{\nu,s} = \nu z + p_1/\nu - p_2/\nu^3 + \cdots$, the first two terms were sufficient in most cases to give at least five places of decimals, which is enough for the needed approximations.

5. The Difference Function $D(\nu)$. Consider the expression $(j_{\nu,k+n} - j_{\nu,k}) - (j_{\mu,h+m} - j_{\mu,h})$ with $j_{\nu,k} = j_{\mu,h}$, where k, h, n and m are fixed integers. Since k and h are fixed, the value of μ will be a function of ν and for a given ν only one value of μ exists so that $j_{\nu,k} = j_{\mu,h}$. If there were two different values, say μ_1, $\mu_2 > \mu_1$ such that $j_{\mu_1,h} = j_{\mu_2 h}$, this would contradict the fact that $j_{\mu,h}$ is an increasing function of μ. This makes the given expression (with the extra equality) a function of ν alone. Call it $D(\nu)$. Since by Theorem 1 each parenthesis is a continuous increasing function of its order (ν or μ) and since the difference of two continuous functions is continuous, it follows that $D(\nu)$ is a continuous function of ν. By a well-known property of continuous functions, if values ν_1 and ν_2 exist so that $D(\nu_1) < 0$ and $D(\nu_2) > 0$, there must be a value $\bar{\nu}$ such that $\bar{\nu}$ is between ν_1 and ν_2 and $D(\bar{\nu}) = 0$. Now there will be a value $\bar{\mu}$ such that $j_{\bar{\nu},k} = j_{\bar{\mu},h}$, and because $D(\bar{\nu}) = 0$, it will be true that $j_{\bar{\nu},k+n} = j_{\bar{\mu},h+m}$ and the two Bessel Functions $J_\nu(x)$ and $J_\mu(x)$ will have two zeros in common.

6. Some Approximations. (a) Consider the case $m = 2$, $n = 1$, $k = 1$ so that $[j_{\nu,2}, j_{\nu,1}]$ must cover $[j_{\nu,h+2}, j_{\nu,h}]$. Now two intervals for μ small will have $j_{\nu,h+2} - j_{\nu,h}$ approximately equal to 2π. From formulae developed by Olver [3, p. XVIII]

$$j_{\nu,2} - j_{\nu,1} = 1.388505\nu^{1/3} + 2.125093\nu^{-1/3} - 0.79333\nu^{-1} - 0.75291\nu^{-5/3} + \cdots.$$

If three terms are kept this equation becomes

$$1.389x^4 - 6.283x^3 + 2.125x^2 - 0.079 = 0, \qquad x = \nu^{1/3}.$$

This has a root $x = 4.156$, and so $\nu = 71.78$.

Suppose that $\nu = 72$ is used as an approximation, $j_{72,1} = 79.96646$. From tables [3] $J_0(x)$ has $j_{0,25} = 77.75603$ as its largest zero not greater than $j_{72,1}$ and $j_{1,25} = 79.32049 < j_{72,1} = 79.966464 < j_{1.5,25} = 80.09813$. By calculating $j_{\mu,25}$ with different values of

μ it was found that $j_{1.415209025} = 79.966464 = j_{72,1}$. Also

$$\begin{aligned} j_{72,2} &= 86.255443, & j_{1.415209,27} &= 86.250448, \\ j_{72,1} &= \underline{79.966464}, & j_{1.415209,25} &= \underline{79.966464}, \\ & \quad 6.288979 & & \quad 6.283984 \end{aligned}$$

$$D(72) = (j_{72,2} - j_{72,1}) - (j_{\mu,27} - j_{\mu,25}) = 6.288979 - 6.283984 = +0.004995.$$

Now use 71 for the approximation.

$$\begin{aligned} j_{71,2} &= 85.196177, & j_{0.75076,27} &= 85.215054, \\ j_{71,1} &= \underline{78.931722}, & j_{0.75076,25} &= \underline{78.931722}, \\ & \quad 6.264455 & & \quad 6.283332 \end{aligned}$$

$$D(71) = 6.264455 - 6.283332 = -0.018887.$$

These calculations show that $\bar{\nu}$ lies between 71 and 72, $\bar{\mu}$ lies between 0.5 and 1.5 and the zeros are between 78 and 80 and 85 and 87. These approximations in the computer program gave the results $\nu = 71.87224$, $\mu = 1.33143$, $j_{\nu,1} = j_{\mu,25} = 79.83629$, $j_{\nu,2} = j_{\mu,27} = 86.12017$.

(b) In order to find more pairs of such Bessel functions, suppose that the interval $[j_{\mu,25}, j_{\mu,27}]$ is replaced by $[j_{\mu,26}, j_{\mu,28}]$. It will be shown that this does not lead to values such that $D(\nu_1)$ and $D(\nu_2)$ have opposite signs. Taking $\mu = 0$, $j_{0,26} = 80.897556$. If $j_{\nu_0,1} = j_{0,26}$, $\nu_0 = 72.900085$. Then

$$\begin{aligned} j_{\nu_0,2} &= 87.208432, & j_{0,28} &= 87.180630, \\ j_{\nu_0,1} &= \underline{80.897556}, & j_{0,26} &= \underline{80.897556}, \\ & \quad 6.310876 & & \quad 6.283074 \end{aligned}$$

$$D(\nu_0) = +0.027802.$$

Similar calculations yield $D(74) = 0.05410$. It becomes impossible to find a negative value. Hence to obtain further values, the ranks of the zeros of $J_\mu(x)$ must be decreased so that $[j_{\mu,24}, j_{\mu,26}]$ is the next interval to be considered. In this case similar methods lead to $\nu = 72.06767$, $\mu = 3.50186$, $j_{\nu,1} = j_{\mu,24} = 80.03849$ and $j_{\nu,2} = j_{\mu,26} = 86.81724$.

(c) This decreasing of the rank of the zeros of $J_\mu(x)$ leads to $[j_{\mu,4}, j_{\mu,2}]$ and stops there since $[j_{\mu,3}, j_{\mu,1}]$ must have $j_{\mu,1} = j_{\nu,1}$, which implies $\mu = \nu$ contrary to Theorem 2. However if the case $[j_{\mu,10}, j_{\mu,12}]$ is calculated, then $655 < \nu < 657$, $582 < \mu < 585$, and the zeros lie between 671 and 674 and 683 and 686. (Since $J_{100}(x)$ is the function of largest order for which the computation of Bessel functions is validated, it is not possible to carry out this determination by the computer program.) The next case $[j_{\mu,9}, j_{\mu,11}]$, however, is quite different. In this case $D(\nu) < 0$ for all ν. The following table was found:

ν	800	1,000	2,000	6,000	20,000	100,000	400,000	1,000,000
$D(\nu)$	-0.357	-0.335	-0.281	-0.231	-0.214	-0.248	-0.335	-0.432

Hence decreasing the rank fails to yield new pairs before the extreme limit is reached.

7. General Method of Approximation. Suppose n intervals of $J_\nu(x)$ cover m intervals of $J_\mu(x)$ and $j_{\nu,k}$ is the smallest zero of $J_\nu(x)$ belonging to the n intervals. The m intervals of $J_\mu(x)$ will have $j_{\mu,h+m} - j_{\nu,h}$ very nearly equal to $m\pi$ if μ is small. Determine ν so that $j_{\nu,k+n} - j_{\nu,k} < m\pi$ but $j_{\nu+1,k+n} - j_{\nu+1,k} \geqslant m\pi$. This fixes $j_{\nu,k}$. Now, from tables of $j_{0,s}$ in [3], find the nearest $j_{0,h}$ to $j_{\nu,k}$ but $j_{0,h} < j_{\nu,k}$. This fixes h. From the sequence of values of $j_{0,h}, j_{1/2,h}, j_{1,h}$, etc. in the tables [3] find the values $j_{\mu,h}$ and $j_{\mu+1/2,h}$ nearest to $j_{\nu,k}$. Then by interpolating and calculating zeros μ can be determined so that $j_{\mu,h} = j_{\nu,k}$ to a close approximation. From these results $j_{\nu,k+n}$ and $j_{\nu,h+m}$ can be found and thus $D(\nu)$ can be calculated. If $\mu \geqslant 1/2, j_{\mu,h+m} - j_{\mu,h} \geqslant m\pi$ and $j_{\nu,k+n} - j_{\nu,k}$ was fixed so as to be $\leqslant m\pi$. In this case $D(\nu)$ will always be < 0. If ν is increased, keeping m, n, h, k the same, it may be that $D(\nu + 1) > 0$, and a pair of functions with two common zeros will be determined. If $D(\nu + 1) < 0$, try $D(\nu + 2)$ etc. As example (c) in Section 5 shows there are cases where $D(\nu)$ stays < 0 no matter how large ν becomes, in which case no pair of functions is found. If $\mu < \frac{1}{2}$ it may result that $D(\nu) > 0$. In this case changing ν to $\nu - 1$ (not changing any of m, n, h, k) may give $D(\nu - 1) < 0$ and determine a pair of functions with two common zeros. As example 6(b) shows $D(\nu)$ can remain > 0 for all ν, and in this event no pair of suitable functions will be found.

Suppose that the first case mentioned in the previous paragraph occurs. Then $j_{\nu,k+n} - j_{\nu,k} < m\pi$ but $j_{\nu+1,k+n} - j_{\nu+1,k} > m\pi$. Now change the rank h of $j_{\mu,h}$ to $h + 1$. Then there will be a new value ν' so that $j_{\nu',k} = j_{\mu,h+1}$. Since $j_{\mu,h+1} - j_{\mu,h} > \pi$ $(\mu > \frac{1}{2})$ and from the properties of the curves C_s discussed in Section 3, $1 < j_{\nu+1,k} - j_{\nu,k} < \pi/2$. So it follows that $\nu' \geqslant \nu + 2$. Then $j_{\nu',k+n} - j_{\nu',k} > j_{\nu+1,k+n} - j_{\nu+1,k} > m\pi$. Also $j_{\mu,k+1+m} - j_{\mu,k+1} < j_{\mu,k+m} - j_{\mu k}$ since $\mu > \frac{1}{2}$. Then $D(\nu') > 0$. Even if $\mu \leqslant \frac{1}{2}$, $j_{\mu,h+m+1} - j_{\mu,h+1} < m\pi$ and $D(\nu') > 0$. Notice that increasing ν makes the first parentheses of $D(\nu)$ increase, while the second parentheses cannot increase beyond $m\pi$, so that $D(\nu)$ is always positive. Therefore most cases of suitable pairs of Bessel functions can be obtained only by decreasing the rank h.

If h is decreased, new cases may result until the case $h = k + 1$ is reached. The process stops here because $h = k$ would require $j_{\nu,k} = j_{\mu,k}$, and this implies $\mu = \nu$, which is not true by Theorem 2.

8. Comments on the Approximations. In order to obtain one set of approximations for determining a pair of Bessel functions $J_\nu(x)$ and $J_\mu(x)$ with two zeros in common, at least eight zeros of the functions are needed, four to find a place where $(j_{\nu,k+n} - j_{\nu,k}) - (j_{\mu,g+m} - j_{\mu,g}) < 0$ and four where this difference is > 0. The uniform asymptotic formulae of F. W. J. Olver enables one to calculate the zeros very easily and without them this work could never have been carried out.

A limitation on all the calculations here is imposed by the fact that our large computer program for calculation of Bessel functions is only guaranteed to produce correct results for orders not exceeding 100. In order to determine how many cases arise, it is necessary to have a table of zeros of $J_{100}(x)$. This has been computed using three terms of Olver's series, so the values are correct to 8 places of decimals at least.

S	$j_{100,s}$	S	$j_{100,s}$
1.	108.83616 58968	14.	169.89299 68759
2.	115.73935 12229	15.	173.75627 22487
3.	121.57533 10257	16.	177.57743 71980
4.	126.87075 61516	17.	181.36076 93720
5.	131.82393 46667	18.	185.10991 46998
6.	136.53571 82239	19.	188.82800 88671
7.	141.06584 76591	20.	192.51777 03132
8.	145.45320 90903	21.	196.18157 27018
9.	149.72480 08232	22.	199.82150 22255
10.	153.90027 12271	23.	203.43940 27048
11.	157.99444 31441	24.	207.03691 27349
12.	162.01882 07206	25.	210.61549 54381
13.	165.98245 03531	26.	214.17646 28640

Consider the case of three intervals covering four $4\pi = 12.56637$ and $j_{100,11} - j_{100,8} = 12.54123$. To cover four intervals with any $J_\mu(x)$ with $\mu > \frac{1}{2}$ it will be necessary to take $\nu > 100$. But $j_{100,10} - j_{100,7} = 12.83442$ and this will cover four intervals. So all cases from $j_{\nu,4}, j_{\nu,1}$ up to $j_{\nu,10}, j_{\nu,7}$ must be considered. By this method it is possible to find limits for any given m and n keeping $\nu \leq 100$.

9. Other Problems. Perhaps the most interesting problem raised by this work is the question as to whether or not two Bessel functions can have more than two zeros in common. If a pair with three common zeros exists, they will of course have three cases of having pairs in common, so that if a complete list of pairs could be made, there would be three cases where for the same pair of values of ν and μ there would be entries in the list. This suggests that the list be arranged so that the order ν is in ascending order, and then that one looks at the values of μ to see if any pairs are the same. This was done with the values in the table and no pairs were found. Of course this suggests that it is not possible for two Bessel functions of the first kind to have more than two zeros in common, but it is also possible that the covering ratio m to n has not been extended far enough to make such a case occur. So it seems that there is not enough evidence to make a conjecture at this time.

Another question is whether there could be a third $J_\lambda(x)$ which also has the same two zeros as $J_\mu(x)$ and $J_\nu(x)$. If $j_{\nu,k} = j_{\mu,g}$ and $j_{\mu,g}$ is large enough, it is easy to see that $\lambda < \mu$ can be determined so that $j_{\lambda,i} = j_{\mu,g}$, but can it be found so that the second common zero is also a zero of $J_\lambda(x)$?

Another interesting question is the number of cases of pairs of Bessel functions with common pairs of zeros. It is obvious that the method outlined in this paper will produce at most a countable set of such functions. However, it is not proved that this method of obtaining the pairs of functions necessarily produces all of them.

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11. Tables of Those Bessel Functions Which Have Two Zeros in Common. In these tables u, v correspond to ν, μ in the above explanation, and a, b, c, d are used for the ranks of zeros. The calculations of the exact values were done by the large computer and entered on punch cards, which were then used to print the table. This avoids the difficulties of copying and proofreading.

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
1 OVER 2							
1	25,	2	27	71.87223635	1.33143271	79.83629187	86.12017291
1	24,	2	26	72.06767239	3.50186495	80.03849076	86.32714408
1	23,	2	25	72.47454645	5.85681372	80.45940642	86.75797004
1	22,	2	24	73.14649784	8.44339038	81.15443662	87.46929372
1	21,	2	23	74.15656752	11.32600274	82.19894227	88.53812215
1	20,	2	22	75.60672870	14.59497476	83.69801825	90.07177035
1	19,	2	21	77.64358658	18.38081725	85.80256850	92.22420770
1	18,	2	20	80.48542137	22.87894154	88.73695831	95.22415186
1	17,	2	19	84.47133368	28.39483094	92.84919401	99.42600082
1	16,	2	18	90.15660150	35.43223062	98.70813596	105.40839993
1	15,	2	17	98.51353133	44.88014463	107.30791651	114.18127176
2 OVER 3							
1	12,	3	15	30.99239611	0.14850988	37.15005920	46.57421695
1	11,	3	14	31.20313206	2.33692511	37.37323916	46.81207300
1	10,	3	13	31.93405948	4.99108417	38.14692369	47.63625954
1	9,	3	12	33.55728696	8.44729585	39.86295331	49.46229340
1	8,	3	11	36.89640309	13.45668958	43.38443891	53.20143381
1	7,	3	10	44.17960248	22.07559888	51.03197649	61.28908251
1	6,	3	9	63.97506249	42.33789479	71.65529954	82.93354995
2	20,	4	23	51.01306531	1.18274131	63.89531120	73.32124462
2	19,	4	22	51.25166448	3.39599769	64.15133462	73.58739209
2	18,	4	21	51.78370107	5.87176298	64.72204478	74.18058804
2	17,	4	20	52.71447351	8.70487222	65.71988844	75.21747778
2	16,	4	19	54.20528487	12.04127071	67.31661749	76.87599232
2	15,	4	18	56.51576514	16.11713072	69.78774188	79.44110873
2	14,	4	17	60.09079065	21.33965585	73.60348655	83.39830704
2	13,	4	16	65.75756352	28.47147580	79.63418431	89.64415888
2	12,	4	15	75.23563545	39.11126569	89.67940950	100.02751989
2	11,	4	14	92.72888373	57.20607243	108.11122386	119.02544055
3	28,	5	31	70.51372810	1.80947814	90.00471797	99.43108851
3	27,	5	30	70.73126732	4.00388550	90.23944324	99.67249383
3	26,	5	29	71.14728259	6.37577591	90.68823453	100.13404428
3	25,	5	28	71.80905336	8.96765542	91.40190196	100.86791936
3	24,	5	27	72.77986384	11.83659300	92.44831580	101.94380021
3	23,	5	26	74.14631672	15.06095269	93.92014097	103.45675044
3	22,	5	25	76.03001811	18.75119044	95.94716195	105.53980814
3	21,	5	24	78.60698812	23.06786265	98.71669945	108.38431718
3	20,	5	23	82.14142683	28.25308960	102.50902107	112.27851981
3	19,	5	22	87.04780300	34.68868023	107.76224715	117.66863811
3	18,	5	21	94.01306654	43.01111475	115.19967326	125.29334258
4	37,	6	40	89.76208019	0.23048995	115.81643269	125.24114670
4	36,	6	39	89.83201307	2.29297600	115.89213060	125.31853367
4	35,	6	38	90.02913959	4.46912905	116.10549282	125.53665326
4	34,	6	37	90.37490655	6.77817046	116.47968542	125.91917724
4	33,	6	36	90.89577790	9.24386741	117.04325269	126.49526339
4	32,	6	35	91.62477769	11.89594616	117.83175844	127.30122683
4	31,	6	34	92.60363501	14.77206199	118.89006332	128.38285770
4	30,	6	33	93.88582197	17.92059730	120.27554382	129.79869564
4	29,	6	32	95.54094743	21.40472053	122.06273780	131.62475296
4	28,	6	31	97.66126296	25.30841526	124.35020817	133.96148890

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,c}$
2 OVER 4							
1	33,	3	37	95.23682922	0.66936858	103.93764840	115.50412177
1	32,	3	36	95.33758229	2.75724931	104.04130932	115.61148890
1	31,	3	35	95.57688391	4.96600993	104.28750950	116.36648238
1	30,	3	34	95.97900930	7.31691631	104.70120212	117.29492144
1	28,	3	32	97.40023755	12.55813985	106.16306230	118.80860313
1	29,	3	33	96.57413650	9.83647593	105.31339078	117.92886593
3 OVER 4							
1	8,	4	12	22.07191428	2.14975454	27.64512020	40.23622253
1	7,	4	11	23.09120867	5.07857406	28.73788535	41.44106274
1	6,	4	10	25.84764073	9.60051267	31.68344760	44.67857079
1	5,	4	9	33.74705322	18.89484760	40.06338287	53.81978567
1	4,	4	8	72.03933686	57.02819266	80.00917544	96.53401315
2	14,	5	18	33.88753510	1.38791384	45.35855381	57.92893249
2	13,	5	17	34.28195594	3.74660321	45.78986360	58.38874263
2	12,	5	16	35.20241108	6.58451988	46.79542308	59.46022965
2	11,	5	15	36.97938565	10.20537184	48.73297237	61.52282148
2	10,	5	14	40.28402717	15.23012116	52.32423385	65.33928528
2	9,	5	13	46.70413737	23.14036000	59.26266043	72.69091680
2	8,	5	12	61.02535337	38.45011276	74.59949930	88.85625862
3	19,	6	23	45.29053563	0.29787519	62.51564583	75.08180064
3	18,	6	22	45.85173033	4.80807591	63.13461185	75.73113237
3	17,	6	21	46.64813000	7.53268192	64.01227824	76.65159839
3	16,	6	20	47.98104329	10.74689878	65.47937641	78.18957744
3	15,	6	19	50.11001969	14.68993053	67.81813704	80.63966574
3	14,	6	18	53.48951171	19.78287361	71.51993934	84.51367524
3	13,	6	17	58.98950068	26.83697877	77.51923095	90.78233968
3	12,	6	16	68.49496262	37.62315994	87.82489305	101.52564439
3	11,	6	15	86.91853232	56.79815250	107.62399543	122.09155117
4	25,	7	29	56.58573612	1.12725887	79.51869463	92.08594098
4	24,	7	28	56.77472778	3.29902171	79.72822647	92.30368325
4	23,	7	27	57.18996578	5.67651392	80.18845252	92.78190647
4	22,	7	26	57.89611103	8.31880685	80.97066282	93.59458355
4	21,	7	25	58.98455983	11.30965133	82.17529142	94.84583843
4	20,	7	24	60.58866585	14.77167104	83.94832717	96.68626533
4	19,	7	23	62.91056942	18.89147565	86.51011574	99.34558903
4	18,	7	22	66.27133597	23.96678966	90.20893793	103.18175514
4	17,	7	21	71.21168502	30.50156787	95.62811359	108.79686755
4	16,	7	20	78.71383510	39.41652321	103.82060796	117.27453217
4	15,	7	19	90.75301477	52.57537773	116.88880215	130.77307441
5	30,	8	34	67.79794171	1.88975106	96.41357238	108.98192917
5	29,	8	33	68.01310046	4.08528807	96.65289195	109.22905664
5	28,	8	32	68.41900293	6.45426444	97.10425254	109.69511619
5	27,	8	31	69.05997702	9.03716694	97.81668961	110.43068317
5	26,	8	30	69.99498261	11.88798113	98.85524429	111.50280080
5	25,	8	29	71.30411889	15.08025285	100.30800457	113.00220590
5	24,	8	28	73.09890681	18.71669613	102.29719224	115.05469694
5	23,	8	27	75.53915029	22.94500103	104.99727514	117.83968684
5	22,	8	26	78.86181805	27.98500770	108.66586750	121.62182077
5	21,	8	25	83.43316575	34.17793405	113.69925591	126.80772278
5	20,	8	24	89.84897452	42.08143608	120.73869352	134.05448170
5	19,	8	23	99.14324407	52.66839439	130.89060800	144.49406442

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
3 OVER 4 (continued)							
6	36,	9	40	78.86830477	0.52820555	113.14151254	125.70789597
6	35,	9	39	78.96082055	2.61227306	113.24466801	125.81393841
6	34,	9	38	79.19218234	4.82253087	113.50260545	126.07908374
6	33,	9	37	79.58798283	7.18230477	113.94376493	126.53256518
6	32,	9	36	80.18040547	9.72098393	114.60383362	127.21101689
6	31,	9	35	81.01046974	12.47610558	115.52818861	128.16102377
6	30,	9	34	82.13125334	15.49635428	116.77539277	129.44267152
6	29,	9	33	83.61262197	18.84597604	118.42231074	131.13478049
6	28,	9	32	85.54835232	22.61144440	120.57178903	133.34274957
6	27,	9	31	88.06717773	26.91182781	123.36452562	136.21066987
6	26,	9	30	91.35050555	31.91547068	126.99804131	139.94067036
6	25,	9	29	95.66198024	37.86791887	131.75821605	144.82504767
7	41,	10	45	89.97423922	1.20201904	129.90342884	142.47020508
7	40,	10	44	90.09196393	3.30899231	130.03491939	142.60491660
7	39,	10	43	90.33404242	5.52898629	130.30527180	142.88188623
7	38,	10	42	90.72042528	7.88018594	130.73669141	143.32385073
7	37,	10	41	91.27549945	10.38485738	131.35626949	143.95854136
7	36,	10	40	92.02938159	13.07054854	132.19739282	144.82012224
7	35,	10	39	93.01968820	15.97173590	133.30166801	145.95115314
7	34,	10	38	94.29399955	19.13212184	134.72159295	147.40531301
7	33,	10	37	95.91335410	22.60790074	136.52433526	149.25125014
7	32,	10	36	97.95731019	26.47250147	138.79713934	151.57813965
3 OVER 5							
1	20,	4	25	59.27289816	3.05242296	66.77328974	82.49419377
1	19,	4	24	59.69968052	5.43002658	67.21676537	82.96605410
1	18,	4	23	60.44537337	8.09027686	67.99144259	83.79005309
1	17,	4	22	61.61736582	11.12920263	69.20852902	85.08396820
1	16,	4	21	63.37621372	14.69107258	71.03402881	87.02323078
1	15,	4	20	65.97265918	19.00244410	73.72572833	89.88063780
1	14,	4	19	69.82011862	24.43922613	77.71243629	94.10374469
1	13,	4	18	75.64919741	31.67054659	83.74191029	100.47897689
1	12,	4	17	84.87553135	42.00366140	93.26598465	110.52006732
2	34,	5	39	91.09105247	0.22973347	106.39054339	122.09838743
2	33,	5	38	91.16349396	2.29376941	106.46666918	122.17765878
2	32,	5	37	91.36814075	4.47469145	106.68171407	122.40158226
2	31,	5	36	91.72790663	6.79282410	107.05972393	122.79517340
2	30,	5	35	92.27121906	9.27344172	107.63050236	123.38941721
2	29,	5	34	93.03376624	11.94835577	108.43142612	124.22314377
2	28,	5	33	94.06094777	14.85814819	109.50998846	125.34565348
2	27,	5	32	95.41138098	18.05537716	110.92743322	126.82046440
2	26,	5	31	97.16203021	21.60928255	112.76406367	128.73078101
2	25,	5	30	99.41590125	25.61286980	115.12719467	131.18767662

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
4 OVER 5							
1	7,	5	12	18.09457160	1.39396589	23.35916924	39.08169370
1	6,	5	11	18.97199517	4.20341343	24.30799927	40.15782184
1	5,	5	10	21.81548227	8.83687798	27.36987614	43.61629967
1	4,	5	9	32.08534843	20.44780831	38.30698562	55.82262293
2	12,	6	17	26.76611125	0.38594987	37.52130831	53.22887455
2	11,	6	16	27.03039416	2.62926644	37.81405707	53.54790948
2	10,	6	15	27.86799005	5.40109415	38.74079843	54.55728113
2	9,	6	14	29.73332654	9.12350938	40.79911100	56.79602987
2	8,	6	13	33.71317320	14.81319752	45.16780205	61.53437136
2	7,	6	12	43.15136923	25.62870955	55.42887873	72.60107589
2	6,	6	11	74.84710729	57.65795798	89.26854371	108.65062763
3	16,	7	21	35.05276754	1.06198254	51.13966152	66.84964093
3	15,	7	20	35.35082919	3.33630345	51.47343414	67.20614901
3	14,	7	19	36.07274407	6.00113922	52.28112331	68.06858575
3	13,	7	18	37.44482289	9.26632980	53.81351137	69.70381121
3	12,	7	17	39.88485845	13.52117277	56.53035827	72.59975490
3	11,	7	16	44.25368966	19.57265418	61.37093439	77.74971838
3	10,	7	15	52.63179053	29.38571316	70.58160144	87.51790682
3	9,	7	14	71.51525742	49.15381390	91.08510265	109.14102367
4	20,	8	25	43.13608374	1.56584794	64.48901265	80.20031966
4	19,	8	24	43.43377771	3.83977961	64.82456182	80.55436390
4	18,	8	23	44.06172400	6.41787554	65.53187247	81.30052066
4	17,	8	22	45.15239490	9.42282453	66.75885697	82.59444621
4	16,	8	21	46.91844056	13.05230771	68.74165256	84.68424298
4	15,	8	20	49.72305104	17.64543320	71.88096836	87.99008775
4	14,	8	19	54.23877786	23.83340488	76.91322169	93.28224645
4	13,	8	18	61.85610384	32.92919270	85.34737296	102.13400137
4	12,	8	17	75.94791897	48.14088554	100.80499286	118.30558848
5	24,	9	29	51.11240993	1.97887429	77.69764366	93.40957474
5	23,	9	28	51.40067018	4.24409473	78.02399683	93.75106259
5	22,	9	27	51.95726642	6.75639136	78.65380258	94.40999517
5	21,	9	26	52.86827107	9.59541396	79.68367195	95.48726643
5	20,	9	25	54.26023977	12.87862579	81.25501718	97.13040027
5	19,	9	24	56.32682907	16.78638758	83.58311154	99.56368791
5	18,	9	23	59.37929554	21.60972524	87.01190969	103.14494382
5	17,	9	22	63.94990121	27.84854365	92.12560818	108.48077493
5	16,	9	21	71.02538649	36.43438010	99.99882373	116.68447490
5	15,	9	20	82.64740770	49.30614850	112.83517282	130.03249470
6	29,	10	34	58.92912030	0.24950528	90.71374553	106.42155609
6	28,	10	33	59.02543521	2.33810262	90.82316024	106.53535901
6	27,	10	32	59.30229746	4.59281634	91.13761116	106.86240813
6	26,	10	31	59.80415585	7.05466234	91.70735222	107.45492527
6	25,	10	30	60.59109399	9.77920649	92.60008541	108.38321749
6	24,	10	29	61.74628607	12.84356120	93.90917399	109.74416681
6	23,	10	28	63.38812376	16.35779819	95.76694176	111.67496285
6	22,	10	27	65.69079200	20.48437273	98.36713019	114.37626377
6	21,	10	26	68.92098386	25.47289948	102.00476331	118.15325502
6	20,	10	25	73.50754520	31.72635752	107.15134752	123.49298163
6	19,	10	24	80.18401151	39.93711846	114.60785052	131.22138834
6	18,	10	23	90.31008883	51.39505875	125.84747537	142.85448251

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
4 OVER 5 (continued)							
7	33,	11	38	66.79144503	0.56492844	103.77421379	119.48222085
7	32,	11	37	66.89789449	2.66284451	103.89543164	119.60771969
7	31,	11	36	67.16360746	4.90728741	104.19795077	119.92091286
7	30,	11	35	67.62253682	7.32958189	104.72025713	120.46161506
7	29,	11	34	68.31886131	9.97056531	105.51227958	121.28145319
7	28,	11	33	69.31110641	12.88445235	106.63994270	122.44855505
7	27,	11	32	70.67841383	16.14472942	108.19206620	124.05464918
7	26,	11	31	72.53037922	19.85342821	110.29114612	126.22614929
7	25,	11	30	75.02305213	24.15625084	113.11081161	129.14208455
7	24,	11	29	78.38607701	29.26830418	116.90529720	133.06432781
7	23,	11	28	82.97110078	35.52014679	122.06174242	138.39123858
7	22,	11	27	89.34352682	43.44537074	129.19878622	145.75845595
7	21,	11	26	98.47006830	53.96029470	139.36692943	156.24371347
8	37,	12	42	74.62960342	0.86004282	116.80238603	132.51059776
8	36,	12	41	74.74279513	2.96415786	116.93152410	132.64331126
8	35,	12	40	74.99827511	5.19917733	117.22294602	132.94442254
8	34,	12	39	75.42278614	7.58977009	117.70702683	133.44374430
8	33,	12	38	76.05011126	10.16715219	118.42203692	134.18121488
8	32,	12	37	76.92352975	12.97138899	119.41686165	135.20718589
8	31,	12	36	78.09936938	16.05473231	120.75492119	136.53694969
8	30,	12	35	79.65226677	19.48657361	122.51995410	138.40666792
8	29,	12	34	81.68316961	23.36099735	124.82478105	140.78235250
8	28,	12	33	84.33189024	27.80866120	127.82499678	143.87388449
8	27,	12	32	87.79751538	33.01616393	131.74113313	147.90768592
8	26,	12	31	92.37300776	39.25897674	136.89605947	153.21494020
8	25,	12	30	98.50686880	46.96031140	143.78130450	160.29929265
9	41,	13	46	82.45029586	1.14045757	129.80728036	145.51568047
9	40,	13	45	82.56811891	3.24883045	129.94190612	145.65414703
9	39,	13	44	82.81442030	5.47539933	130.22328908	145.94355184
9	38,	13	43	83.21078826	7.84007564	130.67599033	146.40914239
9	37,	13	42	83.78385804	10.36746312	131.33024103	147.08198477
9	36,	13	41	84.56686384	13.08831199	132.22366411	148.00072742
9	35,	13	40	85.60179611	16.04154430	133.40366017	149.21404383
9	34,	13	39	86.94245686	19.27712833	134.93077538	150.78407896
9	33,	13	38	88.65887807	22.86024427	136.88355524	152.79141312
9	32,	13	37	90.84386375	26.87746610	139.36570474	155.34238002
9	31,	13	36	93.62293857	31.44618504	142.51693551	158.58014521
9	30,	13	35	97.16994855	36.72942388	146.52990646	162.70199483
10	45,	14	50	90.25795187	1.40988860	142.79433428	158.50340057
10	44,	14	49	90.37903603	3.52125938	142.93335906	158.64553177
10	43,	14	48	90.61716766	5.74030765	143.20575238	158.92501205
10	42,	14	47	90.99012814	8.08343183	143.63227212	159.36261631
10	41,	14	46	91.51943700	10.57050501	144.23738196	159.98342604
10	40,	14	45	92.23137944	13.22583743	145.05089312	160.81799646
10	39,	14	44	93.15838978	16.07947462	146.10949713	161.90392294
10	38,	14	43	94.34094243	19.16897520	147.45885493	163.28797480
10	37,	14	42	95.83018034	22.54188655	149.15649135	165.02905053
10	36,	14	41	97.69163689	26.25925767	151.27588050	167.20234665

a b	c d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
4 OVER 5 (continued)					
11 49,	15 54	98.05561831	1.67089593	155.76911998	171.47783067
11 48,	15 53	98.17904310	3.78441828	155.91047040	171.62256545
11 47,	15 52	98.40991085	5.99678061	156.17483449	171.89325546
11 46,	15 51	98.76311461	8.32171632	156.57919927	172.30728670
11 45,	15 50	99.25639160	10.77560202	157.14375569	172.88531983
11 44,	15 49	99.91102993	13.37811932	157.89268496	173.65209266
4 OVER 6					
1 16,	5 22	44.99027332	1.54136579	51.88077103	70.73578650
1 15,	5 21	45.31895612	3.83549419	52.22478637	71.11113461
1 14,	5 20	46.02512667	6.46789491	52.96366091	71.91692552
1 13,	5 19	47.27605324	9.58962019	54.27174443	73.34223025
1 12,	5 18	49.35231739	13.45575589	56.44078586	75.70225996
1 11,	5 17	52.76135789	18.53020944	59.99690000	79.56287008
1 10,	5 16	58.51876303	25.74926631	65.98946767	86.04643891
1 9,	5 15	68.97303085	37.29858381	76.83533993	97.72007262
1 8,	5 14	90.92474163	59.58088671	99.49919211	121.91638427
2 25,	6 31	66.43692800	1.66612446	80.35584366	99.20838603
2 24,	6 30	66.65684420	3.86275415	80.58939309	99.45622067
2 23,	6 29	67.08961457	6.24982508	81.04891016	99.94377716
2 22,	6 28	67.78983644	8.87645522	81.79218038	100.73221402
2 21,	6 27	68.83213765	11.81000141	82.89804767	101.90485930
2 20,	6 26	70.32117128	15.14526633	84.47685409	103.57814264
2 19,	6 25	72.40819115	19.01982975	86.68771430	105.91964844
2 18,	6 24	75.31983148	23.64078473	89.76843663	109.17931829
2 17,	6 23	79.41091106	29.33402384	94.09027982	113.74641697
2 16,	6 22	85.26810248	36.64149362	100.26515836	120.26077083
2 15,	6 21	93.93156303	46.53025575	109.37415139	129.84918452
3 34,	7 40	86.66187600	0.84216536	107.34948339	126.19935877
3 33,	7 39	86.77234967	2.94090616	107.46764678	126.32303565
3 32,	7 38	87.02200378	5.16407673	107.73465620	126.60249115
3 31,	7 37	87.43573220	7.53401062	108.17707808	127.06549741
3 30,	7 36	88.04462904	10.07866582	108.82805115	127.74667197
3 29,	7 35	88.88802658	12.83349366	109.72943223	128.68970428
3 28,	7 34	90.01638667	15.84409800	110.93483525	129.95050590
3 27,	7 33	91.49549053	19.17010008	112.51402692	131.60176186
3 26,	7 32	93.41265667	22.89089248	114.55943755	133.73966421
3 25,	7 31	95.88622232	27.11444239	117.19606891	136.49414589
3 24,	7 30	99.08045886	31.99118885	120.59704696	140.04492348
4 OVER 7					
1 29,	5 36	84.51035458	1.64364997	92.88943233	114.88310703
1 28,	5 35	84.68932625	3.80130500	93.07398308	115.07972144
1 27,	5 34	85.03482978	6.10573091	93.43023545	115.45922119
1 26,	5 33	85.58035940	8.58659867	93.99267947	116.05826099
1 25,	5 32	86.36885322	11.28205466	94.80549804	116.92373739
1 24,	5 31	87.45623649	14.24194095	95.92619206	118.11659709
1 23,	5 30	88.91667488	17.53258716	97.43095009	119.71746844
1 22,	5 29	90.85057692	21.24413799	99.42282032	121.83521616
1 21,	5 28	93.39716888	25.50211473	102.04453883	124.62034436
1 20,	5 27	96.75498075	30.48633149	105.49940605	128.28676274

a	b	c	d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
5 OVER 6							
1	6,	6	12	16.15956282	2.09581664	21.25886395	40.15423803
1	5,	6	11	17.61375201	5.45160970	22.83832076	41.98413332
1	4,	6	10	22.85481249	12.36728227	28.48463108	48.47359538
1	3,	6	9	58.02952913	47.98237712	65.48084340	89.69121075
2	10,	7	16	23.12034791	1.83390284	33.46471157	52.33102951
2	9,	7	15	23.83277420	4.49646133	34.26025340	53.21381752
2	8,	7	14	25.62572121	8.16703856	36.25614721	55.42520478
2	7,	7	13	29.85399737	14.12257125	40.93201461	60.58780300
2	6,	7	12	41.54204675	27.14617804	53.68759611	74.55831602
3	14,	8	20	29.59134969	1.14917536	44.99012362	63.84319155
3	13,	8	19	29.95574243	3.48773714	45.40259238	64.29071627
3	12,	8	18	30.85705843	6.32581573	46.42140659	65.39559599
3	11,	8	17	32.65760017	10.00270304	48.45091625	67.59435928
3	10,	8	16	36.12910310	15.24505089	52.34413646	71.80434528
3	9,	8	15	43.23611296	23.91554590	60.24602975	80.31930574
3	8,	8	14	60.71283358	42.45804181	79.39317599	100.81096220
4	17,	9	23	36.08638409	2.49975641	56.49520933	75.35805456
4	16,	9	22	36.59178106	4.96930461	57.07159672	75.97449310
4	15,	9	21	37.58537317	7.89310552	58.20315904	77.18422534
4	14,	9	20	39.32980870	11.51718890	60.18492219	79.30144558
4	13,	9	19	42.31210429	16.29975963	63.55946617	82.90250549
4	12,	9	18	47.54176847	23.19571849	69.44035270	89.16620559
4	11,	9	17	57.48000809	34.54852469	80.50980457	100.91839029
4	10,	9	16	79.92313004	57.84319171	105.13740677	126.91642482
5	21,	10	27	42.22588956	1.52692242	67.57113056	86.42404582
5	20,	10	26	42.51021416	3.79096795	67.89728701	86.76946874
5	19,	10	25	43.11161095	6.34969051	68.58665452	87.49943902
5	18,	10	24	44.15403769	9.31884090	69.77995544	88.76264381
5	17,	10	23	45.83437956	12.88285387	71.69935474	90.79348205
5	16,	10	22	48.48439146	17.35366664	74.71655222	93.98340638
5	15,	10	21	52.70754836	23.30119691	79.50206427	99.03691912
5	14,	10	20	59.72098066	31.88085168	87.39501745	107.35675256
5	13,	10	19	72.36980450	45.81014604	101.48946187	122.17084383
5	12,	10	18	99.03135949	73.15077259	130.76869445	152.79952687
6	25,	11	31	48.37120113	0.56352015	78.63916406	97.48880302
6	24,	11	30	48.51516734	2.69719142	78.80499225	97.66315485
6	23,	11	29	48.88725001	5.04273645	79.23341180	98.11356281
6	22,	11	28	49.55742169	7.66536130	80.00445806	98.92406455
6	21,	11	27	50.62700741	10.65971935	81.23348273	100.21566801
6	20,	11	26	52.24822148	14.16840230	83.09283142	102.16896728
6	19,	11	25	54.66020344	18.41615546	85.85158197	105.06553327
6	18,	11	24	58.26025650	23.77775268	89.95357305	109.36902393
6	17,	11	23	63.75755651	30.92501065	96.18451964	115.89853456
6	16,	11	22	72.54501731	41.18397591	106.07300955	126.24329554

a b c d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
5 OVER 6 (continued)				
7 28, 12 34	54.56740026	1.64997145	89.75719298	108.60913931
7 27, 12 33	54.78982592	3.85649024	90.01413354	108.87776497
7 26, 12 32	55.23167059	6.26683081	90.52431351	109.41110780
7 25, 12 31	55.95311791	8.93710548	91.35669505	110.28116941
7 24, 12 30	57.03819188	11.94582840	92.60714045	111.58796192
7 23, 12 29	58.60770848	15.40617718	94.41280923	113.47445509
7 22, 12 28	60.84159964	19.48718713	96.97683627	116.15216404
7 21, 12 27	64.01951472	24.45241565	100.61294489	119.94733824
7 20, 12 26	68.59967679	30.73528513	105.83153214	125.38993141
7 19, 12 25	75.38517902	39.09854526	113.51992345	133.39951924
7 18, 12 24	85.91411822	51.00986184	125.36197492	145.71705241
8 32, 13 38	60.64814709	0.63102892	100.73604913	119.58572098
8 31, 13 37	60.76551653	2.73999240	100.87196962	119.72719106
8 30, 13 36	61.05383179	5.00769270	101.20577198	120.07460992
8 29, 13 35	61.55238146	7.47070894	101.78270031	120.67503032
8 28, 13 34	62.31329722	10.17766591	102.66257379	121.59062930
8 27, 13 33	63.40721028	13.19457777	103.92611350	122.90525744
8 26, 13 32	64.93215655	16.61328058	105.68485596	124.73469136
8 25, 13 31	67.02811690	20.56523566	108.09725826	127.24327332
8 24, 13 30	69.90179882	25.24511927	111.39599826	130.67208037
8 23, 13 29	73.87109277	30.95326393	115.93666427	135.38910087
8 22, 13 28	79.44994835	38.17694292	122.29013127	141.98427852
8 21, 13 27	87.52348170	47.75864012	131.43185148	151.46392290
9 35, 14 41	66.78704559	1.66736102	111.77811207	130.62930112
9 34, 14 40	66.96538693	3.83293025	111.98502225	130.84387303
9 33, 14 39	67.30978769	6.15271400	112.38447596	131.25810229
9 32, 14 38	67.85529017	8.65935276	113.01686400	131.91383914
9 31, 14 37	68.64738685	11.39529282	113.93444584	132.86520654
9 30, 14 36	69.74618764	14.41672459	115.20601462	134.18341339
9 29, 14 35	71.23273516	17.79956366	116.92393603	135.96401142
9 28, 14 34	73.21886966	21.64881519	119.21509256	138.33816878
9 27, 14 33	75.86318884	26.11376090	122.25850222	141.49080138
9 26, 14 32	79.39795173	31.41362708	126.31486283	145.69094326
9 25, 14 31	84.17670269	37.88314772	131.77855576	151.34513882
9 24, 14 30	90.76370747	46.05838636	139.27471907	159.09696791
10 39, 15 45	72.83914847	0.62407731	122.71644542	141.56607700
10 38, 15 44	72.93499736	2.71305567	122.82783470	141.68124246
10 37, 15 43	73.16678988	4.92825158	123.09716286	141.95969541
10 36, 15 42	73.55949022	7.29289357	123.55330836	142.43127686
10 35, 15 41	74.14448126	9.83622474	124.23247027	143.13337933
10 34, 15 40	74.96174876	12.59556374	125.18062454	144.11347575
10 33, 15 39	76.06301514	15.61926717	126.45703437	145.43273165
10 32, 15 38	77.51633811	18.97108554	128.13938092	147.17128107
10 31, 15 37	79.41303382	22.73673465	130.33145426	149.43612959
10 30, 15 36	81.87840870	27.03410498	133.17502164	152.37333991
10 29, 15 35	85.08896644	32.02966965	136.86876497	156.18745833
10 28, 15 34	89.30110615	37.96591840	141.69970419	161.17371689
10 27, 15 33	94.90127200	45.20942649	148.09781100	167.77393811

a b	c d	u	v	$j_{u,a} = j_{v,b}$	$j_{u,c} = j_{v,d}$
5 OVER 6 (continued)					
11 42,	16 48	78.94439362	1.63085975	133.71423130	152.56490051
11 41,	16 47	79.09049152	3.76648606	133.88423466	152.74021642
11 40,	16 46	79.36950888	6.02553142	134.20884012	153.07495821
11 39,	16 45	79.80422745	8.42920229	134.71441356	153.59629943
11 38,	16 44	80.42287359	11.00380915	135.43353249	154.33780523
11 37,	16 43	81.26083228	13.78238447	136.40691417	155.34141284
11 36,	16 42	82.36304689	16.80695181	137.68610538	156.66019300
11 35,	16 41	83.78744495	20.13177116	139.33730769	158.36227505
11 34,	16 40	85.60993705	23.82808363	141.44693730	160.53654809
11 33,	16 39	87.93189398	27.99122289	144.12990614	163.30114711
11 32,	16 38	90.89165082	32.75158021	147.54230675	166.81644188
11 31,	16 37	94.68279133	38.29207185	151.90147929	171.30556787
11 30,	16 36	99.58432897	44.87704092	157.51897096	177.08811665
12 46,	17 52	84.98221900	0.57563175	144.63178290	163.48137125
12 45,	17 51	85.06100775	2.64876991	144.72357368	163.57582367
12 44,	17 50	85.25231884	4.82637130	144.94642858	163.80513786
12 43,	17 49	85.57335213	7.12443305	145.32030860	164.18984558
12 42,	17 48	86.04495065	9.56236383	145.86934475	164.75476174
12 41,	17 47	86.69261034	12.16393585	146.62297779	165.53015323
12 40,	17 46	87.54784272	14.95857151	147.61749157	166.55331075
12 39,	17 45	88.65004238	17.98310966	148.89810716	167.87069468
12 38,	17 44	90.04909017	21.28427185	150.52189181	169.54091421
12 37,	17 43	91.80905136	24.92217230	152.56187602	171.63894179
12 36,	17 42	94.01354207	28.97542082	155.11300478	174.26220126
12 35,	17 41	96.77370544	33.54872242	158.30094647	177.53957396
13 49,	18 55	91.06562178	1.56317030	155.60101564	174.45133316
13 48,	18 54	91.18730896	3.67613098	155.74291839	174.59707299
13 47,	18 53	91.41950614	5.89169272	156.01364946	174.87512082
13 46,	18 52	91.77815606	8.22468886	156.43171479	175.30447471
13 45,	18 51	92.28237952	10.69292997	157.01925808	175.90786227
13 44,	18 50	92.95530155	13.31798302	157.80299357	176.71269545
13 43,	18 49	93.82514768	16.12620687	158.81543974	177.75233318
13 42,	18 48	94.92671825	19.15014682	160.09657339	179.06777326
13 41,	18 47	96.30340120	22.43044175	161.69608012	180.70995369
13 40,	18 46	98.00996580	26.01847554	163.67646731	182.74293419
14 53,	19 59	97.09582760	0.50169993	166.50707578	185.35663222
14 52,	19 58	97.16095968	2.56215927	166.58309669	185.43457944
14 51,	19 57	97.32192402	4.71158227	166.77095430	185.62719573
14 50,	19 56	97.59126445	6.96163185	167.08524147	185.94943936
14 49,	19 55	97.98378335	9.32608485	167.54314312	186.41892243
14 48,	19 54	98.51706956	11.82132943	168.16503392	187.05652186
14 47,	19 53	99.21218058	14.46700867	168.97524838	187.88716662

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2. G. N. Watson, *Treatise on Bessel Functions*, Cambridge Univ. Press, Oxford, 1945.

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