

A Sinc Approximation for the Indefinite Integral

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Abstract. A method for computing $\int_0^1 f(t) dt$, $x = (0, 1)$ is outlined, where $f(t)$ may have singularities at $t = 0$ and $t = 1$. The method depends on the approximation properties of Whittaker cardinal, or sinc function expansions; the general technique may be used for semi-infinite or infinite intervals in addition to $(0, 1)$. Tables of numerical results are given.

1. Introduction and Summary. Here we present a basic method and experimental results for approximating $F(x) = \int_0^x f(t) dt$, $x \in (0, 1)$, where f is smooth, but may have singularities at $t = 0$ or $t = 1$. The method is based on approximation properties of sinc functions

$$(1.1) \quad S(k, h)(x) = \frac{\sin[(\pi/h)(x - kh)]}{(\pi/h)(x - kh)}.$$

With minor modifications, the method may also be employed to compute $\int_a^x f(t) dt$, $t \in (a, b)$, where a, b or both a and b are possibly infinite.

Our underlying formula is similar to formula (3.36) in [4], given there without proof. Our formula is considered in detail here, is proven, and is backed by numerical experiments. (See Section 4 for more discussion.)

In Section 2 we give the relevant properties of sinc functions, assumptions, and basic techniques. (Such techniques appear in more detail in [4] and elsewhere.) In Section 3 we present and verify our integral approximation formulas. Four tables of experimental results and conclusions appear in Section 4.

2. Underlying Properties and Assumptions. A general review of sinc functions and their uses has recently been given by Stenger in [4]. We therefore only outline properties important to our present goals, and refer the reader to [4] for further references.

Approximations on $(0, 1)$ are obtained from corresponding approximations on $\mathbf{R} = (-\infty, \infty)$ via a conformal map. To be approximable on \mathbf{R} , $f \in C^\infty(\mathbf{R})$ must obey certain analyticity and boundedness conditions in a strip in the complex plane \mathbf{C} which contains \mathbf{R} . Through the conformal map, we obtain a corresponding "eye-shaped" region (cf. Figure 4.1, p. 185, of [4]) containing the interval $(0, 1)$, in which our integrands must obey certain analyticity and boundedness conditions. With this in mind, we first list properties over \mathbf{R} .

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Definition 2.1. Let $d > 0$ and let \mathfrak{D}_d denote the open strip

$$(2.1) \quad \mathfrak{D}_d = \{z \in \mathbf{C} : |\operatorname{Im}(z)| < d\}.$$

Let $B_p(\mathfrak{D}_d)$ denote the family of all functions f that are analytic in \mathfrak{D}_d and which satisfy

$$(2.2) \quad \int_{-d}^d |f(x + iy)| dy \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty$$

and such that

$$(2.3) \quad N_p(f, \mathfrak{D}_d) = \lim_{N \rightarrow \infty} \left\{ \left(\int_{\mathbf{R}} |f(x + iy)|^p dx \right)^{1/p} + \left(\int_{\mathbf{R}} |f(x - iy)|^p dx \right)^{1/p} \right\} < \infty.$$

We denote $B_1(\mathfrak{D}_d)$ by $B(\mathfrak{D}_d)$; in all results below, $p = 1$ for simplicity unless otherwise stated, although most of these results generalize to arbitrary p ; cf. [4, Section 3.1].

If $f \in B(\mathfrak{D}_d)$, we approximate f by a truncated ‘‘Whittaker cardinal series’’

$$(2.4) \quad f(x) = C_N(f, h)(x) = \sum_{k=-N}^N f(kh)S(k, h)(x),$$

where $S(k, h)(x)$ is as in (1.1). We state the following without proof:

THEOREM 2.1. (See [2, pp. 237–238], and [4, pp. 177–178].) *Suppose $f \in B_p(\mathfrak{D}_d)$ for some $p \in [1, \infty)$, and suppose*

$$(2.5) \quad |f(x)| \leq Ce^{-\alpha|x|} \quad \text{for all } x \in \mathbf{R} \text{ (i.e., } |f(x)| = O(e^{-\alpha|x|})$$

for some C and $\alpha > 0$ dependent on f . Choose $h = [\pi d/(\alpha N)]^{1/2}$. Then there is a C' dependent on f , but not on N , such that

$$(2.6) \quad \|f(x) - C_N(f, h)(x)\|_\infty \leq C'N^{1/2}e^{-(\pi d\alpha N)^{1/2}},$$

where $\|\cdot\|_\infty$ is the supremum norm over \mathbf{R} .

Here we need not only to approximate functions f , but also their integrals. We have

THEOREM 2.2. (Trapezoidal Rule; cf. e.g. [2, p. 229], or [4, p. 178].) *Suppose $f \in B(\mathfrak{D}_d)$, suppose f satisfies (2.5), and choose $h' = [2\pi d/\alpha N]^{1/2}$. Then*

$$(2.7) \quad \left| \int_{-\infty}^{\infty} f(t) dt - h' \sum_{k=-N}^N f(kh') \right| \leq C_1 e^{-(2\pi d\alpha N)^{1/2}}$$

for some integral approximation constant C_1 dependent on f but not on N .

To obtain approximations over $(0, 1)$, we make use of the following conformal map:

$$(2.8) \quad \varphi(z) = \log\left(\frac{z}{1-z}\right), \quad \varphi'(z) = \frac{1}{z(1-z)}.$$

For any d such that $0 < d \leq \pi/2$, φ maps the region

$$(2.9) \quad \tilde{\mathfrak{D}}_d = \{z : |\arg[z/(1-z)]| < d\}$$

onto \mathfrak{D}_d where \mathfrak{D}_d is as in (2.1). (Note that $\partial\tilde{\mathfrak{D}}_d$ consists of two circular arcs intersecting symmetrically with an angle of $2d$ at 0 and 1); (cf. [4, p. 185].) Define

$$(2.10) \quad z = \psi(w) = \varphi^{-1}(w) = \frac{e^w}{1 + e^w}, \quad \psi'(w) = \psi(w)[1 - \psi(w)].$$

Suppose f is defined and analytic in the interior of $\tilde{\mathfrak{D}}_d$, and define

$$(2.11) \quad \tilde{f}(w) = f(\psi(w))\psi(w)[1 - \psi(w)]$$

for $w \in \mathfrak{D}_d$. Then $\tilde{f} \in B(\mathfrak{D}_d)$ provided

$$(2.12) \quad \liminf_{\mathcal{C} \subset \mathfrak{D}_d} \int_{\mathcal{C} \rightarrow \partial\tilde{\mathfrak{D}}_d} |f(z)| dz < \infty,$$

where \mathcal{C} is some curve; we obtain (2.12) from (2.2) and (2.3) by applying the change of variables $z = \psi(w)$ to the integrals in (2.2) and (2.3).

Suppose, for example, f has singularities at $z = 0$ and at $z = 1$, but f is analytic in $\tilde{\mathfrak{D}}_d$, and f is continuous on $\partial\tilde{\mathfrak{D}}_d \setminus \{0, 1\}$. Then, by examining the integrals over portions of the curves \mathcal{C} near $z = 0$ and $z = 1$ and the integrals over portions of \mathcal{C} away from $z = 0$ and $z = 1$ separately, one can show (2.12) holds if and only if

$$\int_{\partial\tilde{\mathfrak{D}}_d} |f(z)| dz < \infty.$$

Suppose further that f is such that we may take $d = \pi/2$, so $\partial\tilde{\mathfrak{D}}_d = \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$. Then ψ maps the line $\{z : \text{Im}(z) = \pi/2\}$ to the semicircle $\{|z - \frac{1}{2}| = \frac{1}{2}, \text{Im}(z) > 0\}$ and the line $\{z : \text{Im}(z) = -\pi/2\}$ to the semicircle $\{|z - \frac{1}{2}| = \frac{1}{2}, \text{Im}(z) < 0\}$. Condition (2.12) then becomes

$$(2.12)' \quad \int_{\mathcal{C}} |f(z)| dz = \frac{1}{2} \int_0^{2\pi} \left| f\left(\frac{1}{2} + \frac{1}{2}e^{it}\right) \right| dt \\ = \int_{\pi/4}^{3\pi/4} |f(e^{iu}\cos(u))| du + \int_{-\pi/4}^{\pi/4} |f(1 - e^{iv}\cos(v))| dv < \infty.$$

Condition (2.12)' then holds for those f which satisfy

$$(2.13) \quad f(z) = O(|z|^{\alpha_0}) \quad \text{as } z \rightarrow 0, \quad \text{and} \quad f(z) = O(|1 - z|^{\alpha_1}) \quad \text{as } z \rightarrow 1,$$

where $\alpha_0 > -1$ and $\alpha_1 > -1$, and which are continuous elsewhere on the closure of $\tilde{\mathfrak{D}}_d$ and which are analytic in $\tilde{\mathfrak{D}}_d$.

Corresponding to Theorem 2.1, we have

THEOREM 2.3. *Suppose f is analytic in some $\tilde{\mathfrak{D}}_d$ as in (2.9), suppose f satisfies (2.12) (or (2.12)'), suppose*

$$(2.5)' \quad |f(x)| \leq Cx^\beta(1 - x)^\beta, \quad 0 < x < 1,$$

for some $\beta > -1$ and some C dependent on f and set $h = [\pi d/\alpha N]^{1/2}$, where $\alpha = \beta + 1$. Then

$$(2.6)' \quad \left\| f(x) - \sum_{k=-N}^N f(\psi(kh))\psi(kh)[1 - \psi(kh)]S(k, h)(\varphi(x)) \right\|_\infty \\ \leq C'N^{1/2}e^{-(\pi d\alpha N)^{1/2}}$$

for some C' dependent on f but not on N .

Similarly, we have

THEOREM 2.4. *Suppose f is analytic in $\tilde{\mathcal{D}}_d$ for some d , suppose f satisfies (2.12) (or (2.12)'), suppose f satisfies (2.5)' for some $\beta > -1$ and some C dependent on f , and set $h' = [2\pi d/\alpha N]^{1/2}$. Then*

$$(2.7)' \quad \left| \int_0^1 f(t) dt - h' \sum_{k=-N}^N f(\psi(kh'))\psi(kh')[1 - \psi(kh')] \right| \leq C_1 e^{-(2\pi d\alpha N)^{1/2}}$$

for some C_1 dependent on f but not on N .

3. The Indefinite Integral Approximation Formulas. Analogously to Section 2, we will first obtain an approximation procedure for integrals of the form

$$(3.1) \quad F(w) = \int_{-\infty}^w f(t) dt,$$

where $f \in B(\mathcal{D}_d)$ (cf. Definition 2.1). We will then use the conformal mapping procedure to obtain approximation rules for integrals of the form

$$(3.1)' \quad F(x) = \int_{\psi(-\infty)}^{\psi(x)} f(\varphi(t)) dt,$$

where φ and ψ can be as in formulas (2.8) and (2.10), respectively. We begin with the following lemma, which is a direct consequence of Theorem 2.1.

LEMMA 3.1. *Define $F(w)$ as in (3.1), suppose $F \in B_p(\mathcal{D}_d)$, for some $p \in [1, \infty)$, suppose $|F(w)| \leq C e^{-\alpha|w|}$ for some C and $\alpha > 0$ and all $w \in \mathbf{R}$, and set $h = [\pi d/(\alpha N)]^{1/2}$. Then*

$$(3.2) \quad \left| \int_{-\infty}^w f(t) dt - \left\{ \sum_{k=-N}^N \left[\int_{-\infty}^{kh} f(t) dt \right] S(k, h)(w) \right\} \right| \leq C'_1 N^{1/2} e^{(\pi d\alpha N)^{1/2}}$$

for all $w \in \mathbf{R}$ and for some indefinite integral approximation constant C'_1 which does not depend on N .

Lemma 3.1 does not apply to general integrals F as in (3.1), since in general $\int_{-\infty}^{\infty} f(t) dt \neq 0$, and formula (2.5) is not valid. For such F , we consider the related function

$$(3.3) \quad \begin{aligned} G(w) &= \int_{-\infty}^w f(t) dt - \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} \int_{-\infty}^{\infty} f(t) dt \\ &= \int_{-\infty}^w \left[f(t) - \frac{\alpha e^{\alpha t}}{(e^{\alpha t} + 1)^2} I_{\infty}(f) \right] dt = \int_{-\infty}^w g(t) dt, \end{aligned}$$

where

$$(3.4) \quad g(t) = f(t) - \frac{\alpha e^{\alpha t}}{(e^{\alpha t} + 1)^2} I_{\infty}(f), \quad I_{\infty}(f) = \int_{-\infty}^{\infty} f(t) dt.$$

We then have

LEMMA 3.2. *Suppose $|f(w)| \leq C e^{-\alpha|w|}$ for some C and $\alpha > 0$ and all $w \in \mathbf{R}$, and suppose G is as in (3.3). Then there is a C_G such that $|G(w)| \leq C_G e^{-\alpha|w|}$ for all $w \in \mathbf{R}$.*

Proof. We have

$$|G(w)| \leq \left| \frac{e^{\alpha w/2}}{e^{\alpha w/2} + e^{-\alpha w/2}} \left[\int_{-\infty}^w f(t) dt - I_{\infty}(f) \right] \right| + \left| \frac{e^{-\alpha w/2}}{e^{\alpha w/2} + e^{-\alpha w/2}} \int_{-\infty}^w f(t) dt \right|.$$

We will consider each factor in the two terms on the right as $w \rightarrow -\infty$ and as $w \rightarrow +\infty$.

Case 1, $w \rightarrow -\infty$. The second factor in the first term is bounded and approaches $-I_{\infty}(f)$, while the first factor in the first term is bounded by $e^{-\alpha|w|}$. The first factor in the second term is bounded by 1, while the second factor in the second term is bounded by $Ce^{-\alpha|w|}/\alpha$.

Case 2, $w \rightarrow \infty$. The first factor in the first term is bounded by 1, while the second factor in the first term is equal to $-\int_w^{\infty} f(t) dt$ and hence is bounded by $Ce^{-\alpha|w|}/\alpha$. The first factor in the second term is bounded by $e^{-\alpha|w|}$, while the second factor in the second term is bounded and approaches $I_{\infty}(f)$.

Combining the above, we may take

$$C_G = C/\alpha + \max \left\{ \sup_{w \leq 0} \left| \int_{-\infty}^w f(t) dt - I_{\infty}(f) \right|, \sup_{w \geq 0} \left| \int_{-\infty}^w f(t) dt \right| \right\} \sim C/\alpha + |I_{\infty}(f)|.$$

Combining Lemma 3.1 and Lemma 3.2 immediately gives

THEOREM 3.1. *Suppose $f \in B(\mathfrak{D}_d)$, suppose $|f(w)| \leq Ce^{-\alpha|w|}$ for some C and all $w \in \mathbf{R}$, and suppose $G \in B_p(\mathfrak{D}_d)$ for some p , where G is as in (3.3). Define*

$$(3.5) \quad F_N(w) = \sum_{k=-N}^N \left\{ \int_{-\infty}^{kh} \left[f(t) - \frac{\alpha e^{\alpha t}}{(e^{\alpha t} + 1)^2} \int_{-\infty}^{\infty} f(u) du \right] dt \right\} S(k, h)(w) + \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} \int_{-\infty}^{\infty} f(t) dt,$$

where $h = [\pi d / (\alpha N)]^{1/2}$. Then there is a C'_1 , independent of N such that

$$\left| \int_{-\infty}^w f(t) dt - F_N(w) \right| \leq C'_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$$

for all $w \in \mathbf{R}$.

In practice the quantities

$$\int_{-\infty}^{kh} \left[f(t) - \frac{\alpha e^{\alpha t}}{(e^{\alpha t} + 1)^2} \int_{-\infty}^{\infty} f(u) du \right] dt = \int_{-\infty}^{kh} g(t) dt = G(kh),$$

and the quantity $\int_{-\infty}^{\infty} f(u) du = I_{\infty}(f)$ must be approximated. Below, we show how to do this so that the order of approximation in (3.5) remains $O(N^2 e^{-(\pi d \alpha N)^{1/2}})$.

First observe that linear combinations of functions in $B(\mathfrak{D}_d)$ are also in $B(\mathfrak{D}_d)$; also linear combinations of functions satisfying (2.5) satisfy (2.5) (with a different C). Thus, we deduce

LEMMA 3.3. Suppose $f \in B(\mathbb{D}_d)$, suppose $|f(w)| \leq Ce^{-\alpha|w|}$ for some C and $\alpha > 0$ and all $w \in \mathbf{R}$, and suppose g is as in (3.4). Then $g \in B(\mathbb{D}_d)$, and there is a C_g such that $|g(w)| \leq C_g e^{-\alpha|w|}$ for all $w \in \mathbf{R}$. Thus, g and $\int_{-\infty}^{\infty} g(t) dt$ may be approximated as in (2.6) and (2.7), respectively.

Let $C_N(g, h)(t) = \sum_{j=-N}^N g(jh)S(j, h)(t)$ be the approximation to g . We use $C_N(g, h)(t)$ to approximate the quantities $G(kh)$ in (3.5).

LEMMA 3.4. Let $f \in B(\mathbb{D}_d)$, suppose $|f(t)| \leq Ce^{-\alpha|t|}$, $t \in \mathbf{R}$, and set $h = [\pi d/(\alpha N)]^{1/2}$. Suppose g and G are as in (3.3) and (3.4), and define

$$(3.6) \quad \hat{G}_N(w) = \int_{-Nh}^w C_N(g, h)(t) dt = \sum_{j=-N}^N g(jh) \int_{-Nh}^w S(j, h)(t) dt,$$

where $-Nh \leq w \leq Nh$. Then there is a constant \hat{C} dependent on f such that $|\hat{G}_N(w) - G(w)| \leq \hat{C}Ne^{-(\pi d \alpha N)^{1/2}}$ for all $w \in \mathbf{R}$.

Proof.

$$|\hat{G}_N(w) - G(w)| \leq |G(-Nh)| + \int_{-Nh}^w |g(t) - C_N(g, h)(t)| dt.$$

But $|G(-Nh)| \leq C_G e^{-\alpha Nh} = C_G e^{-(\pi d \alpha N)^{1/2}}$, from Lemma 3.2. Also

$$\begin{aligned} \int_{-Nh}^w |g(t) - C_N(g, h)(t)| dt &\leq (Nh + w) C'_g N^{1/2} e^{-(\pi d \alpha N)^{1/2}} \leq 2N^{3/2} h C'_g e^{-(\pi d \alpha N)^{1/2}} \\ &= \tilde{C} N e^{-(\pi d \alpha N)^{1/2}}, \end{aligned}$$

from Lemma 3.3. Thus

$$|\hat{G}_N(w) - G(w)| \leq C_G e^{-(\pi d \alpha N)^{1/2}} + \tilde{C} N e^{-(\pi d \alpha N)^{1/2}} \leq \hat{C} N e^{-(\pi d \alpha N)^{1/2}}$$

for some appropriate \hat{C} .

We now replace $G(kh)$ by $\hat{G}(kh)$ in Theorem 3.1.

LEMMA 3.5. Suppose the assumptions of Theorem 3.1 hold, but

$$G(kh) = \int_{-\infty}^{kh} \left[f(t) - \frac{\alpha e^{\alpha t}}{(e^{\alpha t} + 1)^2} \int_{-\infty}^{\infty} f(u) du \right] dt$$

in (3.5) is replaced by $\hat{G}_N(kh)$ as in (3.6), to obtain

$$(3.5)' \quad \hat{F}_N(w) = \sum_{k=-N}^N \hat{G}_N(kh) S(k, h)(w) + \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} \int_{-\infty}^{\infty} f(t) dt.$$

Then there is a \hat{C}_F independent of N such that

$$\left| \int_{-\infty}^w f(t) dt - \hat{F}_N(w) \right| \leq \hat{C}_F N^2 e^{-(\pi d \alpha N)^{1/2}}.$$

Proof.

$$\left| \int_{-\infty}^w f(t) dt - \hat{F}_N(w) \right| \leq \left| \int_{-\infty}^w f(t) dt - F_N(w) \right| + |F_N(w) - \hat{F}_N(w)|,$$

where the first term on the right is bounded by $C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$, by Theorem 3.1. On the other hand,

$$\begin{aligned}
 |F_N(w) - \hat{F}_N(w)| &= \left| \sum_{k=-N}^N [G(kh) - \hat{G}_N(kh)] S(k, h)(w) \right| \\
 &\leq \sum_{k=-N}^N |G(kh) - \hat{G}_N(kh)| \leq 2N(\hat{C}_F N e^{-(\pi d \alpha N)^{1/2}}) \\
 &\leq \hat{C}_F N^2 e^{-(\pi d \alpha N)^{1/2}}
 \end{aligned}$$

for some constant \hat{C}_F dependent on f but not on N .

For the main result, we must replace $I_\infty(f) = \int_{-\infty}^\infty f(u) du$ in (3.5)' without changing the order of approximation.

LEMMA 3.6. *Suppose the assumptions of Theorem 3.1 hold, but in (3.5)' we replace $I_\infty(f) = \int_{-\infty}^\infty f(t) dt$ by $I_N(f) = h \sum_{k=-N}^N f(kh)$, where $h = [\pi d / (\alpha N)]^{1/2}$. Then the conclusion of Lemma 3.5 is still true.*

Proof. If there are a C and an $\alpha > 0$ such that $|f(t)| \leq C e^{-\alpha|t|}$, $t \in \mathbf{R}$, then, if $\alpha' = \alpha/2$, $|f(t)| \leq C e^{-\alpha'|t|}$, $t \in \mathbf{R}$. Thus from Theorem 2.2,

$$(3.7) \quad |I_\infty(f) - I_N(f)| \leq C_1 e^{-(\pi d \alpha N)^{1/2}}.$$

Define

$$(3.8) \quad \tilde{g}_N(t) = \sum_{j=-N}^N \left[f(jh) - \frac{\alpha e^{\alpha j h}}{(e^{\alpha j h} + 1)^2} I_N(f) \right] S(j, h)(t),$$

and define

$$\tilde{G}_N(w) = \int_{-N h}^w C_N(\tilde{g}_N, h)(t) dt$$

and

$$(3.9) \quad \tilde{F}_N(w) = \sum_{k=-N}^N \tilde{G}_N(kh) S(k, h)(w) + \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} I_N(f).$$

Then

$$(3.10) \quad \left| \int_{-\infty}^w f(t) dt - \tilde{F}_N(w) \right| \leq \left| \int_{-\infty}^w f(t) dt - \hat{F}_N(w) \right| + |\hat{F}_N(w) - \tilde{F}_N(w)|,$$

where \hat{F}_N is as in (3.5)'. By Lemma 3.5, the first term on the right of (3.10) is bounded by $\hat{C}_F N^2 e^{-(\pi d \alpha N)^{1/2}}$, while

$$(3.11) \quad |\hat{F}_N(w) - \tilde{F}_N(w)| \leq \left| \sum_{k=-N}^N [\hat{G}_N(kh) - \tilde{G}_N(kh)] S(k, h)(w) \right| + \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} |I_\infty(f) - I_N(f)|,$$

where \hat{G}_N is as in (3.6). The first term on the right of (3.11) equals

(3.12)

$$\left| \sum_{k=-N}^N \left\{ \sum_{j=-N}^N \frac{\alpha e^{\alpha j h}}{(e^{\alpha j h} + 1)^2} [I_N(f) - I_\infty(f)] \int_{-N h}^{k h} S(j, h)(t) dt \right\} S(k, h)(w) \right|,$$

while the inner sum in (3.12) equals

(3.13)
$$\Lambda_{k,N} = h \sum_{j=-N}^N \frac{\alpha e^{\alpha j h}}{(e^{\alpha j h} + 1)^2} [I_N(f) - I_\infty(f)] \int_{-N-j}^{k-j} \frac{\sin(\pi t)}{\pi t} dt.$$

But $\Lambda_{k,N}$ is a trapezoidal rule approximation, as in Theorem 2.2, to $\int_{-\infty}^{\infty} \lambda_{k,N}(t) dt$, where

(3.14)
$$\begin{aligned} \lambda_{k,N}(w) &= [I_N(f) - I_\infty(f)] \left[\frac{\alpha e^{\alpha w}}{(e^{\alpha w} + 1)^2} \int_{-N-w/h}^{k-w/h} \frac{\sin(\pi t)}{\pi t} dt \right. \\ &= \frac{\alpha [I_N(f) - I(f)]}{4} \left[\int_{-N-w/h}^{k-w/h} \frac{\sin(\pi t)}{\pi t} dt \right] \operatorname{sech}^2 \left[\left(\frac{\alpha}{2} \right) w \right], \end{aligned}$$

where $h' = (\pi d / \alpha N)^{1/2} = (2\pi d / \alpha' N)^{1/2}$ for $\alpha' = \alpha/2$. But the magnitude of the integral in (3.13) is bounded by

$$\frac{2}{\pi} \sup_{a \geq 0} \int_0^a \frac{\sin(t)}{t} dt < 2;$$

thus

(3.15)
$$|\lambda_{k,N}(w)| \leq \frac{2\alpha |I_N(f) - I_\infty(f)|}{(e^{\alpha w/2} + e^{-\alpha w/2})^2} \leq 2\alpha |I_N(f) - I_\infty(f)| e^{-\alpha |w|}.$$

It is also possible to verify that $\lambda_{k,N}(w) \in B(\mathcal{D}_d)$. Thus, as in Theorem 2.2,

(3.16)
$$\left| \int_{-\infty}^{\infty} \lambda_{k,N}(t) dt - \Lambda_{k,N} \right| \leq C_I^\Lambda e^{-(\pi d \alpha N)^{1/2}}$$

for some constant C_I^Λ dependent on f but not on N . But, by (3.7)

(3.17)
$$\begin{aligned} \left| \int_{-\infty}^{\infty} \lambda_{k,N}(t) dt \right| &\leq \frac{\alpha}{4} |I_N(f) - I_\infty(f)| \left\{ \int_{-\infty}^{\infty} \operatorname{sech}^2 \left[\left(\frac{\alpha}{2} \right) t \right] dt \right\} \\ &= |I_N(f) - I_\infty(f)| \leq C_I e^{-(\pi d \alpha N)^{1/2}} \end{aligned}$$

Combining (3.17) and (3.16), we see that the quantity in (3.12) is bounded by

(3.18)
$$\sum_{k=-N}^N (C_I + C_I^\Lambda) e^{-(\pi d \alpha N)^{1/2}} = 2(C_I + C_I^\Lambda) N e^{-(\pi d \alpha N)^{1/2}}$$

Combining the bounds (3.18), (3.7), and (3.11) now gives

(3.19)
$$|\hat{F}_N(w) - \tilde{F}_N(w)| \leq 2(C_I + C_I^\Lambda) N e^{-(\pi d \alpha N)^{1/2}} + C_I e^{-(\pi d \alpha N)^{1/2}}$$

We obtain the conclusion in Lemma 3.6 from Lemma 3.5, 3.10, and 3.19.

We summarize the above in the following

THEOREM 3.2 (MAIN RESULT). *Suppose $f \in B(\mathfrak{D}_d)$, and $|f(w)| \leq Ce^{-\alpha|w|}$ for some $\alpha > 0$ and C , and all $w \in \mathbf{R}$. Suppose further $G \in B(\mathfrak{D}_d)$, where G is defined in terms of f by (3.3). Define σ_q, q an integer, by*

$$(3.20) \quad \sigma_q = \int_0^q \frac{\sin(\pi t)}{\pi t} dt.$$

Set $h = [\pi d/(\alpha N)]^{1/2}$, and approximate $\int_{-\infty}^w f(t) dt$ by

$$(3.21) \quad \begin{aligned} \tilde{F}_N(w) = h \left[\sum_{k=-N}^N \left\{ \sum_{j=-N}^N \left[f(jh) - \frac{\alpha e^{\alpha j h}}{(e^{\alpha j h} + 1)^2} I_N(f) \right] (\sigma_{N+j} + \sigma_{k-j}) \right\} S(k, h)(w) \right] \\ + \frac{e^{\alpha w/2}}{e^{-\alpha w/2} + e^{\alpha w/2}} I_N(f), \end{aligned}$$

where $I_N(f) = h \sum_{k=-N}^N f(kh)$. Then there is a constant C_F independent of N such that

$$\left| \int_{-\infty}^w f(t) dt - \tilde{F}_N(w) \right| \leq C_F N^2 e^{-(\pi d \alpha N)^{1/2}},$$

for all $w \in \mathbf{R}$.

Proof. Theorem 3.2 is a restatement of Lemma 3.6, once one observes

$$h(\sigma_{N+j} + \sigma_{k-j}) = \int_{-Nh}^{kh} S(j, h)(t) dt.$$

A table of $\sigma_q, q = 1, \dots, 100$, is found on p. 175 of [4]. Also, $\sigma_q = \text{Si}(q\pi)/\pi$, an asymptotic expansion for which is given on p. 244 of [1]; we are told this expansion gives 20 significant figures for $q \geq 20$.

Theorem 3.2 is used in conjunction with conformal mapping techniques (explained in [4, Section 4], and the references therein) to evaluate singular integrals of the form $\int_a^b f(t) dt$, where a or b or both a and b are finite, and f possibly has singularities at a or b . We consider here $(a, b) = (0, 1)$, and the conformal maps φ and ψ defined in (2.8) and (2.10). From Theorem 3.2 and considerations in Section 2 of this paper, we obtain

THEOREM 3.3. *Suppose (as in (2.8)–(2.13)) that $f \in B(\tilde{\mathfrak{D}}_d)$, and suppose*

$$(3.22) \quad |f(x)| \leq C|x|^\beta(1-x)^\beta \quad \text{all } x \in (0, 1),$$

for some C and $\beta > -1$. Set $\alpha = \beta + 1$, and set $h = (\pi d/\alpha N)^{1/2}$. Define

$$(3.21)' \quad \begin{aligned} \tilde{F}_N(x) = h \left[\sum_{k=-N}^N \left\{ \sum_{j=-N}^N \left[f(x_j)(x_j)(1-x_j) - \frac{\alpha e^{\alpha j h}}{(e^{\alpha j h} + 1)^2} I_N(f) \right] (\sigma_{N+j} + \sigma_{k-j}) \right\} \right. \\ \left. \cdot S(k, h)(\varphi(x)) + \frac{x^\alpha}{x^\alpha + (1-x)^\alpha} I_N(f) \right], \end{aligned}$$

where $I_N(f) = h \sum_{k=-N}^N f(x_k)(x_k)(1 - x_k)$ and where $x_j = \psi(jh) = e^{jh}/(1 + e^{jh})$. Then, for $x \in (0, 1)$,

$$(3.23) \quad \left| \int_0^x f(t) dt - \tilde{F}_N(x) \right| \leq C_F N^2 e^{-(\pi d \alpha N)^{1/2}}$$

for some C dependent on f but independent of N . If $f(z) = O(|z|^\beta)$ as $z \rightarrow 0, z \in \mathbb{C}$, then $f \in B(\tilde{Q}_d)$, with $d = \pi/2$.

Note that (3.22) is equivalent to

$$|f(\psi(w))\psi(w)[1 - \psi(w)]| = |f(x)x(1 + x)| \leq ce^{-\alpha|w|} \quad \text{for } w \in \mathbb{R}.$$

Theorem 3.3. is otherwise obtained from Theorem 3.2 by making the transformation

$$(3.24) \quad \int_0^x f(t) dt = \int_{-\infty}^{\varphi(x)} f(\psi(u))\psi(u)[1 - \psi(u)] du,$$

then applying Theorem 3.2 to the right member of (3.24).

To comment on the power of the approximations in Theorem 3.2 and Theorem 3.3, we note that it is unnecessary to know precisely what the order α is; in particular, if $|f(t)| \leq ce^{-\alpha t}$ and $0 < \alpha' < \alpha$, then we may replace α by α' in any of the formulas. Note that the convergence order is not preserved in this way when we use, say, Newton-Cotes or Gauss type formulas derived with the method of moments for weighted integrands of the form $x^\alpha p(x)$, p a polynomial. Also, sinc function approximations are optimal, as explained in [3].

4. Experimental Results. To empirically test formula (3.21)', we tried it for various N on the following functions:

- (i) $f_1(x) = \frac{1}{3}x^{-2/3}$;
- (ii) $f_2(x) = \frac{4}{3}x^{1/3}$;
- (iii) $f_3(x) = \frac{1}{6}[x^{-2/3} + (1 - x)^{-2/3}]$; and
- (iv) $f_4(x) = \frac{3}{40}[x^{-9} + (1 - x)^{-7}]$.

In each case, the difference $\tilde{F}_N(x) - \int_0^x f(t) dt$ was computed at $x = .1i, i = 1, \dots, 10$, for $N = 4, 8, 16$, and 32 , corresponding to 9, 17, 33, and 65 evaluations of f , respectively. These results were compared to the errors $F_N^G(x) - \int_0^x f(t) dt$, where $F_N^G(x)$ is the result of applying a $2N$ -point Gauss-Legendre formula to compute $\int_0^x f(t) dt$, for $2N = 8, 16$, and 32 ; $F_{64}^G(x)$ was defined to be the result of applying the composite 32-point Gauss-Legendre formula with 2 subdivisions. These particular formulas were chosen for comparison since the Gauss formulas are commonly thought to be generally accurate, and since the 8-, 16-, and 32-point formulas were closest in number of points to 9, 17, and 33 of those formulas already programmed on our system.

The functions $f_i, i = 1, \dots, 4$, are normalized so that $\int_0^1 f_i(t) dt = 1$. The functions have singularities at $x = 0$ and $x = 1$ of varying orders; the singularities of f_3 are the same order at $x = 0$ and $x = 1$, while the singularities of $f_i, i \neq 3$, differ in order at $x = 0$ and $x = 1$.

Four tables appear, corresponding to each of the four functions. Table 3 has three parts, corresponding to various estimated orders β (cf. (3.22)).

The sinc function approximations compare favorably to the corresponding Gauss formulas, except for $f_2(x)$. We postulate that the worse the singularity, the better the merit of the sinc function approximation.

TABLE 1. $f(x) = \frac{1}{3}x^{-2/3}$

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1 | -0.690-01 | -0.560-01 | -0.440-01 | -0.220-01 | -0.280-01 | -0.500-02 | -0.220-01 | -0.630-03 |
| 0.2 | -0.870-01 | -0.520-01 | -0.560-01 | -0.210-01 | -0.350-01 | -0.510-02 | -0.280-01 | -0.630-03 |
| 0.3 | -0.990-01 | -0.520-01 | -0.640-01 | -0.200-01 | -0.410-01 | -0.500-02 | -0.320-01 | -0.630-03 |
| 0.4 | -0.110+00 | -0.550-01 | -0.700-01 | -0.210-01 | -0.450-01 | -0.490-02 | -0.350-01 | -0.620-03 |
| 0.5 | -0.120+00 | -0.570-01 | -0.760-01 | -0.210-01 | -0.480-01 | -0.500-02 | -0.380-01 | -0.630-03 |
| 0.6 | -0.120+00 | -0.590-01 | -0.800-01 | -0.220-01 | -0.510-01 | -0.510-02 | -0.410-01 | -0.630-03 |
| 0.7 | -0.130+00 | -0.590-01 | -0.850-01 | -0.220-01 | -0.540-01 | -0.500-02 | -0.430-01 | -0.630-03 |
| 0.8 | -0.140+00 | -0.560-01 | -0.880-01 | -0.210-01 | -0.560-01 | -0.490-02 | -0.450-01 | -0.630-03 |
| 0.9 | -0.140+00 | -0.510-01 | -0.920-01 | -0.200-01 | -0.590-01 | -0.510-02 | -0.460-01 | -0.630-03 |
| 1.0 | -0.150+00 | -0.550-01 | -0.950-01 | -0.210-01 | -0.610-01 | -0.500-02 | -0.480-01 | -0.630-03 |

assumed beta: -0.66670+00 d= 0.15710+01

TABLE 2. $f(x) = \frac{4}{3}x^{1/3}$

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| 0.1 | 0.220-04 | -0.280-02 | 0.370-05 | -0.450-03 | 0.610-06 | -0.240-04 | 0.240-06 | -0.370-06 |
| 0.2 | 0.550-04 | -0.300-02 | 0.930-05 | -0.430-03 | 0.150-05 | -0.250-04 | 0.610-06 | -0.380-06 |
| 0.3 | 0.940-04 | -0.270-02 | 0.160-04 | -0.460-03 | 0.260-05 | -0.260-04 | 0.100-05 | -0.370-06 |
| 0.4 | 0.140-03 | -0.260-02 | 0.230-04 | -0.410-03 | 0.390-05 | -0.250-04 | 0.150-05 | -0.410-06 |
| 0.5 | 0.190-03 | -0.270-02 | 0.320-04 | -0.420-03 | 0.520-05 | -0.240-04 | 0.210-05 | -0.380-06 |
| 0.6 | 0.240-03 | -0.310-02 | 0.400-04 | -0.470-03 | 0.660-05 | -0.260-04 | 0.260-05 | -0.380-06 |
| 0.7 | 0.290-03 | -0.340-02 | 0.500-04 | -0.430-03 | 0.810-05 | -0.230-04 | 0.320-05 | -0.410-06 |
| 0.8 | 0.350-03 | -0.270-02 | 0.590-04 | -0.410-03 | 0.970-05 | -0.260-04 | 0.390-05 | -0.380-06 |
| 0.9 | 0.410-03 | -0.280-02 | 0.690-04 | -0.430-03 | 0.110-04 | -0.270-04 | 0.450-05 | -0.420-06 |
| 1.0 | 0.470-03 | -0.200-01 | 0.800-04 | -0.450-02 | 0.130-04 | -0.500-03 | 0.520-05 | -0.210-04 |

assumed beta: 0.33330+00 d= 0.15710+01

TABLE 3. $f(x) = \int_0^1 [x^{-2/3} + (1-x)^{-2/3}]$

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1 | -0.340-01 | -0.300-01 | -0.220-01 | -0.110-01 | -0.140-01 | -0.240-02 | -0.110-01 | -0.320-03 |
| 0.2 | -0.430-01 | -0.250-01 | -0.280-01 | -0.110-01 | -0.180-01 | -0.260-02 | -0.140-01 | -0.310-03 |
| 0.3 | -0.500-01 | -0.240-01 | -0.320-01 | -0.990-02 | -0.200-01 | -0.250-02 | -0.160-01 | -0.320-03 |
| 0.4 | -0.550-01 | -0.250-01 | -0.350-01 | -0.990-02 | -0.220-01 | -0.240-02 | -0.180-01 | -0.310-03 |
| 0.5 | -0.590-01 | -0.270-01 | -0.380-01 | -0.110-01 | -0.240-01 | -0.250-02 | -0.190-01 | -0.310-03 |
| 0.6 | -0.620-01 | -0.290-01 | -0.400-01 | -0.110-01 | -0.260-01 | -0.260-02 | -0.200-01 | -0.320-03 |
| 0.7 | -0.660-01 | -0.310-01 | -0.420-01 | -0.110-01 | -0.270-01 | -0.250-02 | -0.210-01 | -0.310-03 |
| 0.8 | -0.690-01 | -0.300-01 | -0.440-01 | -0.100-01 | -0.280-01 | -0.240-02 | -0.220-01 | -0.320-03 |
| 0.9 | -0.720-01 | -0.250-01 | -0.460-01 | -0.980-02 | -0.290-01 | -0.260-02 | -0.230-01 | -0.310-03 |
| 1.0 | -0.150+00 | -0.550-01 | -0.950-01 | -0.210-01 | -0.610-01 | -0.500-02 | -0.480-01 | -0.630-03 |

assumed beta: -0.66670+00 d= 0.15710+01

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1 | -0.340-01 | -0.530-01 | -0.220-01 | -0.240-01 | -0.140-01 | -0.760-02 | -0.110-01 | -0.150-02 |
| 0.2 | -0.430-01 | -0.500-01 | -0.280-01 | -0.240-01 | -0.180-01 | -0.760-02 | -0.140-01 | -0.150-02 |
| 0.3 | -0.500-01 | -0.490-01 | -0.320-01 | -0.230-01 | -0.200-01 | -0.760-02 | -0.160-01 | -0.150-02 |
| 0.4 | -0.550-01 | -0.490-01 | -0.350-01 | -0.230-01 | -0.220-01 | -0.760-02 | -0.180-01 | -0.150-02 |
| 0.5 | -0.590-01 | -0.510-01 | -0.380-01 | -0.240-01 | -0.240-01 | -0.760-02 | -0.190-01 | -0.150-02 |
| 0.6 | -0.620-01 | -0.520-01 | -0.400-01 | -0.240-01 | -0.260-01 | -0.760-02 | -0.200-01 | -0.150-02 |
| 0.7 | -0.660-01 | -0.520-01 | -0.420-01 | -0.240-01 | -0.270-01 | -0.760-02 | -0.210-01 | -0.150-02 |
| 0.8 | -0.690-01 | -0.510-01 | -0.440-01 | -0.230-01 | -0.280-01 | -0.760-02 | -0.220-01 | -0.150-02 |
| 0.9 | -0.720-01 | -0.490-01 | -0.460-01 | -0.240-01 | -0.290-01 | -0.760-02 | -0.230-01 | -0.150-02 |
| 1.0 | -0.150+00 | -0.100+00 | -0.950-01 | -0.470-01 | -0.610-01 | -0.150-01 | -0.480-01 | -0.290-02 |

assumed beta: -0.46700+00 d= 0.15710+01

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1 | -0.340-01 | -0.110-02 | -0.220-01 | -0.250-02 | -0.140-01 | -0.740-03 | -0.110-01 | 0.130-04 |
| 0.2 | -0.430-01 | 0.500-02 | -0.280-01 | 0.900-03 | -0.180-01 | -0.520-04 | -0.140-01 | -0.750-04 |
| 0.3 | -0.500-01 | 0.460-02 | -0.320-01 | 0.160-02 | -0.200-01 | 0.450-03 | -0.160-01 | 0.450-04 |
| 0.4 | -0.550-01 | 0.160-02 | -0.350-01 | 0.680-03 | -0.220-01 | 0.340-03 | -0.180-01 | 0.750-04 |
| 0.5 | -0.590-01 | -0.250-02 | -0.380-01 | -0.910-03 | -0.240-01 | -0.110-03 | -0.190-01 | -0.420-05 |
| 0.6 | -0.620-01 | -0.650-02 | -0.400-01 | -0.250-02 | -0.260-01 | -0.560-03 | -0.200-01 | -0.850-04 |
| 0.7 | -0.660-01 | -0.950-02 | -0.420-01 | -0.340-02 | -0.270-01 | -0.670-03 | -0.210-01 | -0.540-04 |
| 0.8 | -0.690-01 | -0.990-02 | -0.440-01 | -0.270-02 | -0.280-01 | -0.170-03 | -0.220-01 | 0.660-04 |
| 0.9 | -0.720-01 | -0.380-02 | -0.460-01 | -0.650-03 | -0.290-01 | 0.520-03 | -0.230-01 | -0.220-04 |
| 1.0 | -0.150+00 | -0.790-02 | -0.950-01 | -0.520-02 | -0.610-01 | -0.330-02 | -0.480-01 | -0.330-02 |

assumed beta: -0.86700+00 d= 0.15710+01

TABLE 4. $f(x) \equiv \int_{40}^3 [x^{-9} + (1-x)^{-7}]$

| x | 8-point | n=4 | 16-point | n=8 | 32-point | n=16 | 64-point | n=32 |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1 | -0.340+00 | -0.140+00 | -0.300+00 | -0.900-01 | -0.280+00 | -0.420-01 | -0.250+00 | -0.130-01 |
| 0.2 | -0.370+00 | -0.140+00 | -0.320+00 | -0.870-01 | -0.280+00 | -0.410-01 | -0.260+00 | -0.130-01 |
| 0.3 | -0.380+00 | -0.150+00 | -0.340+00 | -0.870-01 | -0.290+00 | -0.400-01 | -0.270+00 | -0.130-01 |
| 0.4 | -0.390+00 | -0.150+00 | -0.350+00 | -0.890-01 | -0.300+00 | -0.410-01 | -0.280+00 | -0.130-01 |
| 0.5 | -0.400+00 | -0.150+00 | -0.350+00 | -0.910-01 | -0.310+00 | -0.410-01 | -0.290+00 | -0.130-01 |
| 0.6 | -0.410+00 | -0.160+00 | -0.360+00 | -0.920-01 | -0.310+00 | -0.420-01 | -0.290+00 | -0.130-01 |
| 0.7 | -0.420+00 | -0.160+00 | -0.370+00 | -0.940-01 | -0.320+00 | -0.420-01 | -0.300+00 | -0.130-01 |
| 0.8 | -0.420+00 | -0.160+00 | -0.370+00 | -0.940-01 | -0.320+00 | -0.420-01 | -0.300+00 | -0.130-01 |
| 0.9 | -0.430+00 | -0.160+00 | -0.370+00 | -0.910-01 | -0.330+00 | -0.410-01 | -0.310+00 | -0.130-01 |
| 1.0 | -0.480+00 | -0.160+00 | -0.410+00 | -0.100+00 | -0.350+00 | -0.530-01 | -0.320+00 | -0.250-01 |

assumed beta: -0.90000+00 d= 0.15710+01

For Table 1, $\beta = -2/3$ ($\alpha = 1/3$) was used, while $\beta = 1/3$ in Table 2. The values $\beta = -2/3$, $\beta = -.467$, and $\beta = -.867$ were used in Tables 3a, 3b, and 3c, respectively, while the value $\beta = -.9$ was used in Table 4. From these and other experiments, it appears it is better to underestimate β than to overestimate it.

In all cases, d was taken to be $\pi/2$.

Stenger has proven a formula analogous to (3.5)', namely

$$(4.1) \quad \int_{-\infty}^x g(t) dt = h \sum_{k=-N}^N \sum_{l=-N}^N \sigma_{k-l} g(lh) S(k, h)(x) + O(Ne^{-(\pi d \alpha N)^{1/2}}),$$

using formulas (2.17), (3.7), and (3.11) in [4] and the uniqueness of the coefficients in the sinc function expansion [5]. (This formula appears without proof as formula (3.36) in [4]*.) It is then possible, using the same argument as in the proof of Lemma 3.6, to obtain an analogue to Lemma 3.6 from (4.1). This analogue was also tried in the same cases as above; in all cases, the tables of errors were the same to at least two significant digits. Since (4.1) is more elegant and requires less additions, it is perhaps to be preferred.

The formula resulting from (3.21) when I_N is replaced by I_∞ was also tried. From the resulting error tables, it appeared that the approximation for I_∞ accounted for approximately half of the total error. Thus, it seems unnecessary to use a separate stepsize h in computing I_N , as suggested by (2.7). Also, taking d other than $\pi/2$ in no case improved the performance.

The computations were done on a Honeywell 6880 (Multics system) in double precision (63-bit mantissas). To get values at $x = 1$, $\tilde{x} = 1 - 2^{-62}$ was used.

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* Formula (3.36) in [4] was attributed to Kearfott, Sikorski, and Stenger; that paper never existed. Also, the derived formula (4.58) in [4] was mistakenly attributed to John Lund; that formula corresponds to (3.23) of this paper.