

The Maximum of a Quasi-Smooth Function

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Abstract. Let Z be the class of real-valued functions, defined and continuous on the closed interval $I = [-1, 1]$, such that $f(\pm 1) = 0$ and $|f(\xi) - 2f\{(\xi + \eta)/2\} + f(\eta)| \leq |\xi - \eta|$ for all ξ and η in I . Let $K = \sup_{f \in Z} \max_{x \in I} |f(x)|$. We will prove that $13/10 \leq K \leq 1014/779 < 1.301669$.

1. Introduction. Zygmund [7] introduced the class of real valued functions $f(x)$, defined and continuous on the closed interval $I = [-1, 1]$, such that

$$(1) \quad |f(\xi) - 2f\{(\xi + \eta)/2\} + f(\eta)| \leq M|\xi - \eta|$$

for some constant M and all ξ and η in I . Timan [4], [5], [6] called such functions "quasi-smooth". Let Z be the class of quasi-smooth functions for which $M = 1$ and $f(-1) = f(+1) = 0$. Suppose that

$$K = \sup_{f \in Z} \max_{x \in I} |f(x)|.$$

Timan [6] proved that $K \leq 4/3$ and asserted that this was not the best result. Brudnyi [1] attributed to Abramov the claim that $K \leq 383/288$. He exhibited a function in Z whose maximum value is $5/4$; hence $K \geq 5/4$. Sokolova [3] proved that $K \leq 4/3 - 4/381 = 168/127 < 1.322835$, and conjectured that $K = 5/4$.

In this paper we will prove that

$$1.3 = 13/10 \leq K \leq 1014/779 < 1.301669,$$

thus disproving Sokolova's conjecture. (In no way discouraged by her failure, we conjecture that $K = 13/10$.)

The proof starts with the derivation in Section 2 of upper bounds for $|f(\xi)|$ as f ranges over Z when ξ is one of twenty-six particular numbers in I . Each of these upper bounds is the consequence of a particular identity in the functional values of f and the inequality (1). It is easy to see (cf. Section 5) that there is a function f_0 in Z and a point x_0 in I such that $f_0(x_0) = K$. In Sections 4 and 5 we derive two basic inequalities from which can be inferred upper bounds on K if it is known that x_0 lies in a subinterval (a, b) of I and if upper bounds for $f(a)$ and $f(b)$ as f ranges over Z are known. In Section 6 we apply these basic inequalities to each of the subintervals determined by adjacent values of ξ considered in Section 2, and thereby obtain the asserted upper bound on K . In Section 3 we show that the membership in Z of a piecewise linear continuous function $f(x)$ on I , whose corners lie in an arithmetic progression (with distance N^{-1} between successive numbers) can be decided in

Received June 24, 1982.

1980 *Mathematics Subject Classification*. Primary 26D20.

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0025-5718/83 \$1.00 + \$.25 per page

$2N(2N - 1)$ arithmetical tests, and then exhibit such a function with $N = 20$ that passes all of the tests, so that it is in Z , and that has the value $13/10$ when $x = 3/10$. This demonstrates the asserted lower bound on K . In Section 7 we describe the manner in which we used a computer to assist in the discovery of the functional identities used in Section 2 and of the piecewise linear function in Z recorded in Section 3. We emphasize, however, that the proofs of the identities and of the properties of the piecewise linear function are independent of the computer.

2. Some Preliminary Numerical Results. We will show in this section that, if $f \in Z$,

$$\begin{aligned}
 (2) \quad & |f(0)| \leq 1, \quad |f(1/2)| \leq 1, \quad |f(1/4)| \leq 5/4, \quad |f(1/3)| \leq 23/18, \\
 & |f(1/5)| \leq 32/25, \quad |f(1/6)| \leq 35/27, \quad |f(2/7)| \leq 127/98, \\
 & |f(2/9)| \leq 415/324, \quad |f(2/11)| \leq 2821/2178, \quad |f(3/10)| \leq 13/10, \\
 & |f(3/16)| \leq 373/288, \quad |f(3/20)| \leq 193/150, \quad |f(4/23)| \leq 686/529, \\
 & |f(5/17)| \leq 751/578, \quad |f(5/18)| \leq 419/324, \quad |f(5/31)| \leq 1244/961, \\
 & |f(5/32)| \leq 31/24, \quad |f(7/20)| \leq 127/100, \quad |f(7/22)| \leq 313/242, \\
 & |f(9/31)| \leq 161/124, \quad |f(11/35)| \leq 3179/2450, \quad |f(11/36)| \leq 421/324, \\
 & |f(13/42)| \leq 1145/882, \quad |f(19/64)| \leq 1331/1024, \quad |f(23/76)| \leq 1877/1444, \\
 & |f(25/84)| \leq 2293/1764.
 \end{aligned}$$

Consider (1) when $\xi = -1, \eta = 1, M = 1$. Because $f(\pm 1) = 0$, it follows that $|f(0)| \leq 1$. We infer from the identity

$$2f(1/2) = [2f(1/2) - f(0) - f(1)] + f(0) + f(1),$$

an application of (1) when $M = 1$, the just established upper bound for $f(0)$, and the vanishing of $f(1)$, that $|f(1/2)| \leq 1$. Because $f(-x) \in Z$ when $f(x) \in Z$, it follows that $|f(-1/2)| \leq 1$ also.

The remaining inequalities in (2) are consequences of analogous, but more complicated identities, recorded at the end of the paper, and similar reasoning.

3. Piecewise Linear Functions in Z . Let N be a positive integer, and $x_k = (k - N)/N, k = 0, 1, \dots, 2N$. Let y be a $2N + 1$ tuple $(y_0, y_1, \dots, y_{2n})$ such that $y_0 = y_{2N} = 0$, and suppose that $f(x)$ is a real-valued continuous function on I such that $f(x_k) = y_k$. A necessary condition that $f \in Z$ is that

$$(3) \quad |y_k - 2y_{k+l} + y_{k+2l}| \leq 2l/N$$

for all integers k and l such that $0 \leq k \leq 2N - 2, 1 \leq l \leq N - (k/2)$. It is also necessary that (3) hold when l is half of an odd integer such that $3 \leq 2l \leq 2N - k$, if y_{k+l} is defined as $f[(k + l - N)/N]$ for such values of l . If f is known to be linear on each interval (x_k, x_{k+1}) , the following lemma asserts that these conditions are sufficient.

LEMMA 1. *Suppose that $x_k = (k - N)/N$, that $f(x)$ is linear on each interval (x_k, x_{k+1}) , and that $f(x_k) = y_k$. Then $f \in Z$ if, and only if, $y_0 = y_{2N} = 0$ and (3) holds whenever k and $2l$ are integers such that $0 \leq k \leq 2N - 2, 2 \leq 2l \leq 2N - k$, and the value $y_{k+1/2}$ is defined as $(y_k + y_{k+1})/2$.*

Consider the functions $Q_j(\xi, \eta) = (-1)^j\{f(\xi) - 2f[(\xi + \eta)/2] + f(\eta)\} - \eta + \xi$ ($j = 0, 1$). For each η in $(-1, 1)$ these functions are piecewise linear, continuous functions of ξ on $[-1, \eta]$ with corners in the set consisting of η and those points x_k and $2x_k - \eta$ that lie in $[-1, \eta]$. It follows that $Q_j(\xi, \eta) \leq 0$ when $-1 \leq \xi \leq \eta$ if, and only if, $Q_j(x_k, \eta) \leq 0$ and $Q_j(2x_k - \eta, \eta) \leq 0$ for all k such that $x_k < \eta$ and $-1 \leq 2x_k - \eta < \eta$, respectively. (Note that $Q_j(\eta, \eta) \equiv 0$.) For each j and k , the functions $Q_j(x_k, \eta)$ and $Q_j(2x_k - \eta, \eta)$ are piecewise linear, continuous functions of η on the respective intervals $[x_k, 1]$ and $[x_k, x'_k]$, in which $x'_k = \min(2x_k + 1, 1)$. The corners of $Q_j(x_k, \eta)$ lie in the set of those points x_n and $2x_n - x_k$ that are in $[x_k, 1]$, and the corners of $Q_j(2x_k - \eta, \eta)$ lie in the set of those points x_n and $2x_k - x_n$ that are in $[x_k, x'_k]$. Moreover $Q_j(x_k, \eta) \leq 0$ and $Q_j(2x_k - \eta, \eta) \leq 0$ for the appropriate values of k and η if, and only if, $Q_j(x_k, \eta_n) \leq 0$ and $Q_j(2x_k - \eta_n, \eta_n) \leq 0$ whenever η_n is an appropriate corner. Because $2x_h - x_k = x_{2h-k}$, it is now clear that $Q_j(\xi, \eta) \leq 0$ for all ξ, η such that $-1 \leq \xi \leq \eta \leq 1$ if, and only if, $Q_j(x_k, x_h) \leq 0$ for all k and h such that $0 \leq k < h \leq 2N$. If we define l so that $h = k + 2l$, then $2l$ is a positive integer and the lemma is now obvious, after we observe that (3) is trivial when $l = 1/2$, and that $1/2$ is the only permissible value of l when $k = 2N - 1$.

Consider the piecewise linear function $f_1(x)$ defined when $N = 20$ by the value $y = (0, 1/5, 2/5, 11/20, 3/5, 3/4, 17/20, 19/20, 19/20, 19/20, 1, 11/10, 6/5, 6/5, 6/5, 23/20, 11/10, 21/20, 1, 1, 1, 11/10, 6/5, 5/4, 5/4, 5/4, 13/10, 5/4, 23/20, 21/20, 1, 21/20, 1, 19/20, 7/8, 7/10, 3/5, 11/20, 2/5, 1/4, 0)$. There are $2N(2N - 1) = 1560$ tests (3) in Lemma 1, and all of them are passed, so $f_1(x) \in Z$. Because $f_1(3/10) = 13/10, f_1(0) = 1, f_1(1/2) = 1, f_1(1/4) = 5/4$, we conclude that the inequalities in (2) for $f(3/10), f(0), f(1/2)$ and $f(1/4)$ are best possible, and that $K \geq 13/10$. This example refutes Sokolova's conjecture [3] that $K = 3/4$.

It is also true that there exists, for each ξ in the set $\{1/3, 1/5, 1/6, 2/7, 2/9, 3/16, 3/20, 4/23, 5/31, 7/20, 7/22, 11/36, 19/64\}$, a piecewise linear function in Z for which the upper bound in (2) is attained. Because all of these upper bounds are less than $13/10$, this knowledge does not help us improve the lower bound $13/10$ for K , and we omit the proof of this assertion. We have been unable to decide whether the remaining inequalities in (2) are best possible.

4. The First Basic Inequality. For each f in Z let x_f be a point in I at which $|f(x)|$ attains its maximum value α_f . It follows from (1) that, if ξ and $\eta = 2x_f - \xi$ are both in I , then

$$(4) \quad 2\alpha_f \leq 2|x_f - \xi| + |f(\xi)| + |f(\eta)|, \quad \alpha_f \leq 2|x_f - \xi| + 2A(\xi),$$

if $A(\xi)$ is an upper bound on Z for $|f(\xi)|/2$. Therefore, $\alpha_f \leq K'$ for some K' if $\xi \in I, K' \geq 2A(\xi)$, and

$$(5) \quad \max[-(1 - \xi)/2, \xi + A(\xi) - (K'/2)] \leq x_f \leq \min[(1 + \xi)/2, \xi - A(\xi) + (K'/2)].$$

If a and b are any two points in I such that $a < b$, the set of values x_f that satisfy (5) either when $\xi = a$ or when $\xi = b$ surely contains the interval (a, b) if

$$\max[-(1 - b)/2, b + A(b) - (K'/2)] \leq \min[(1 + a)/2, a - A(a) + (K'/2)].$$

Thus we see that the following lemma is true.

LEMMA 2. *If $f \in Z$ and $-1 \leq a \leq x_f \leq b \leq 1$, then $\alpha_f \leq K_1$, in which*

$$K_1 = \max[A(a) + A(b) + b - a, 2A(b) + 2b - a - 1, 2A(a) - 2a + b - 1, 2A(a), 2A(b)].$$

Sokolova [3] had already proved that

$$\alpha_f \leq K_2 = 2 \min\{A(a), A(b)\} + 2(b - a)$$

if $f \in Z$, $a \leq x_f \leq b$, and $-1 \leq 2a - b \leq 2b - a \leq 1$. (This result follows easily from (4) when $\xi = a$ and $\xi = b$.) Under these circumstances it is easy to see that $K_1 \leq K_2$ if, and only if,

$$\Lambda \equiv |A(b) - A(a)| / (b - a) \leq 1,$$

and then $K_1 = A(a) + A(b) + b - a$. Thus we have proved the following corollary to Lemma 2.

COROLLARY. *Suppose that $-1 \leq 2a - b < 2b - a \leq 1$. If $f \in Z$, if $a \leq x_f \leq b$, and if $\Lambda \leq 1$, then*

$$(6) \quad \alpha_f \leq K_1 = A(a) + A(b) + b - a.$$

It follows from Lemma 2, but not from the corollary, when $a = 0$, $b = 1$, and when $a = -1$, $b = 0$, that $\alpha_f \leq 3/2$ (because $A(0) = 1/2$, $A(\pm 1) = 0$). Therefore, the set Z is uniformly bounded, and $K \leq 3/2$.

5. The Second Basic Inequality. Sokolova [3] asserted that the set Z is compact in the space of continuous functions on I . (She offered no proof, but the result is an immediate consequence of Ascoli's theorem [2], the fact that Z is uniformly bounded, and the existence of a modulus of continuity [6] that applies uniformly for f in Z , so that the functions in Z are equicontinuous.) It follows that there exists a function f_0 in Z and a point x_0 in I such that $f_0(x_0) = K$.

Given two points a and b in I such that $a < b$, the function $f^*(y)$, defined so that

$$(7) \quad f_0(x) = \frac{(x - a)f_0(b) + (b - x)f_0(a)}{b - a} + \frac{b - a}{2} f^*(y),$$

$$y = \frac{2(x - a)}{b - a} - 1,$$

is easily seen to be in Z . Therefore, $|f^*(y)| \leq K$.

Because $K \leq 3/2$, there is no loss of generality in supposing that $K' \leq 3/2$ in (5). The existence of the function $f_1(x)$, described near the end of Section 3, shows that $A(\xi) \geq 1/2$ if $|\xi| \leq 1/2$. The argument leading to (5) thus shows that $K = \alpha_{f_0} \leq K'$ if $|\xi| \leq 1/2$, $2A(\xi) \leq K' \leq 3/2$, and

$$\xi + A(\xi) - (K'/2) \leq x_0 \leq \xi - A(\xi) + (K'/2).$$

In particular,

$$(8) \quad K \leq K' \quad \text{if } a \leq x_0 \leq a' \equiv a - A(a) + (K'/2),$$

$$K \leq K' \quad \text{if } b + A(b) - (K'/2) \equiv b' \leq x_0 \leq b,$$

and if $2 \max\{A(a), A(b)\} \leq K' \leq 3/2$, $0 \leq a < b \leq 1/2$. Note that in this case both $2a - b$ and $b - 2a$ are in I .

If $a' < b'$, so that K' is less than the right-hand side of (6), it does not follow from (8) that $K \leq K'$ if $a' \leq x_0 \leq b'$. It does, however, follow from (7) that, if $a' \leq x_0 \leq b'$,

$$(9) \quad K \leq 2 \max_{a' \leq x \leq b'} \left\{ \frac{(x - a)A(b) + (b - x)A(a)}{b - a} \right\} + \frac{(b - a)K}{2}.$$

The maximum in (9) occurs when $x = b'$ if $A(b) \geq A(a)$ and when $x = a'$ if $A(a) \geq A(b)$, so that

$$(10) \quad K \leq 2(1 + \Lambda)\max\{A(a), A(b)\} - \Lambda K' + (b - a)K/2.$$

Therefore, K is surely not greater than K' when $a' \leq x_0 \leq b'$, if K' is not less than the value K_3 at which (10) is an equality when $K' = K = K_3$. Thus we have proved the following lemma when $K_3 < K_1$ and $K_3 < 3/2$. (Its validity is obvious if $K_3 \geq K_1$ or $K_3 \geq 3/2$.)

LEMMA 3. *If $0 \leq a \leq x_0 \leq b \leq 1/2$ and $\Lambda \leq 1$, then*

$$(11) \quad K \leq K_3 = 2 \max\{A(a), A(b)\} / [1 - (b - a)/2(1 + \Lambda)].$$

The result in Lemma 3 is an improvement over an earlier result of Sokolova [3] that

$$K \leq K_4 = 2 \max\{A(a), A(b)\} / [1 - (b - a)/2]$$

if $a \leq x_0 \leq b$. We will refer to (6) and (11) as the basic inequalities.

6. Application of the Basic Inequalities. We arrange the values of x , for which we exhibited upper bounds on Z for $|f(x)|$ in (2), into an increasing sequence $\xi_0 = 0$, $\xi_1 = 3/20, \dots, \xi_{26} = 7/20, \xi_{27} = 1$, and calculate K_1 for each interval $(a, b) = (\xi_i, \xi_{i+1})$, verifying along the way that $\Lambda < 1$. It is obvious that $-1 \leq 2a - b \leq 2b - a \leq 1$, so that the corollary to Lemma 1 is applicable, except when $(a, b) = (\xi_{26}, \xi_{27}) = (7/20, 1)$ when we must use Lemma 1 itself. The values of K_1 are recorded in Table I as the smallest six-place decimal fraction that is not less than the actual rational number computed from (6). We conclude that

$$K \leq 3004633/2304324 < 1.303911.$$

We now calculate K_3 for each interval $(a, b) = (\xi_i, \xi_{i+1})$, starting with the interval for which K_1 is largest and continuing in descending order of K_1 until we reach a value K_1 that is less than the largest value of K_3 so far calculated. In this manner we conclude that

$$K \leq 1014/779 < 1.301669.$$

This completes the proof of the assertion that $1.3 \leq K < 1.301669$. The estimate $K = 1.300834$ is in error by less than 0.0642%.

TABLE I
Values of K_1 and K_3 calculated from the basic inequalities (6) and (11)

ξ	$A(\xi)$	K_1	K_3
0	1/2		
		1.293334	
3/20	193/300	1.295417	
5/32	31/48	1.298117	
5/31	622/961	1.300767	
1/6	35/54	1.303788	1.301348
4/23	343/529	1.303911	1.301469
2/11	2821/4356	1.300864	
3/16	373/576	1.300070	1.300201
1/5	16/25	1.302655	1.294979
2/9	415/648	1.293210	
1/4	5/8	1.299383	
5/18	419/648	1.302501	1.300327
2/7	127/196	1.301848	1.300752
9/31	161/248	1.302643	1.301511
5/17	751/1156	1.302314	1.301451
19/64	1331/2048	1.300590	
25/84	2293/3528	1.302325	1.301514
3/10	13/20	1.302563	1.301669
23/76	1877/2888	1.302547	1.301621
11/36	421/648	1.302753	1.301627
13/42	1145/1764	1.302631	1.301090
11/35	3179/4900	1.299366	
7/22	313/484	1.300735	
1/3	23/36	1.290556	
7/20	127/200	1.285000	
1	0		

7. A Related Linear Programming Problem. Although our paper is logically complete, it may be helpful to explain the origin of the various identities used in Section 2 to obtain upper bounds, and of the various piecewise linear functions in Z that were (or could have been) used in Section 3 to obtain lower bounds.

We say that the $2N + 1'$ tuple $y = (y_0, y_1, y_2, \dots, y_{2N-1}, y_{2N})$ is admissible if $y_0 = y_{2N} = 0$ and the inequalities (3) hold whenever k and l are integers such that $0 \leq k \leq 2N - 2, 1 \leq l \leq N - (k/2)$. Let P_{mN} be the (linear programming) problem of finding an admissible y for which y_m is a maximum. Let y_{mN} be this maximum value. Then $y_{mN} = y_{2N-m,N}$ because $f(-x) \in Z$ when $f(x) \in Z$. It is obvious that $f(x_m) \leq y_{mN}$ if $f \in Z$.

We say that the $2N + 1'$ tuple y is *admissible if $y_0 = y_{2N} = 0$ and the inequalities (3) hold whenever k and $2l$ are integers such that $0 \leq k \leq 2N - 2, 2 \leq 2l \leq 2N - k$ and the value $y_{k+1/2}$ is defined as $(y_k + y_{k+1})/2$. Let P_{mN}^* be the linear programming problem of finding a *admissible y for which y_m is a maximum. Let y_{mN}^* be this maximum value. By virtue of Lemma 1 it is obvious that

$$\sup_{f \in Z} f(x_m) \geq y_{mN}^*.$$

We have written a computing program for the Cyber-176 that calculates the coefficients for both P_{mN} and P_{mN}^* , and stores them for use in the general purpose linear programming routine APEX III [8]. Each of the points ξ , for which an upper bound for $|f(\xi)|$ was furnished in (2), is a rational fraction of the form $x_m = (m - N)/N$ in which m and N are relatively prime integers and $N \leq m < 2N$. Accordingly, we can use the computer to find y_{mN} and y_{mN}^* .

The computer output for P_{mN} contains, among other things, not only y_{mN} and the values $y_k (k = 1, 2, \dots, 2N)$ corresponding to the maximizing solution, but also the values of the "marginal" variables (= dual variables = Lagrange multipliers) corresponding to each of the $2N^2$ inequalities (3). It is a well-known fact that the sum of the products of each marginal variable and the left-hand side of the corresponding inequality is identically equal to the function being maximized. These identities are the ones recorded in the appendix at the end of the paper after two modifications. All of the computer output is expressed as finite decimal fractions (mostly 5 place accuracy). We have guessed simple rational fractions to which the decimal fraction values for the marginal variables appeared to be reasonably accurate approximations. That our guesses were appropriate is confirmed by the observations that the identities in the appendix, written down with the guessed rational fractions (and multiplied by a least common denominator in order to use only integers as coefficients), are actually identities, verifiable ab initio without any reference to their computer origin. Next, the integers m and N can be multiplied by the same factor κ without changing x_m . It is obvious that $y_{\kappa m, \kappa N} \leq y_{mN}$ and $y_{\kappa m, \kappa N}^* \geq y_{mN}^*$ if κ is a positive integer. Accordingly, the identity actually recorded in the appendix corresponds to the smallest κ in the set of κ for which the problem $P_{\kappa m, \kappa N}$ can be handled by APEX-III on our computer and for which $y_{\kappa m, \kappa N}$ is least. It may happen that there is an integer κ' for which the problem $P_{\kappa' m, \kappa' N}$ can be handled by APEX-III on our computer and for which $y_{\kappa' m, \kappa' N}^*$ is equal to the least $y_{\kappa m, \kappa N}$. In this case,

$$\sup_{f \in Z} f(x_m) = y_{\kappa m, \kappa N},$$

so there is nothing to be gained by proceeding to larger values of κ or κ' . The numerical data show that it is possible that $y_{2m, 2N} = y_{m, N}$ (e.g., $y_{12,8} = y_{6,4} = y_{3,2} = 1$), that $y_{2m, 2N} < y_{m, N}$ (e.g., $y_{38,32} = 83/64 < y_{19,16} = 13/10$), that $y_{mN}^* = y_{mN}$ (e.g., $y_{5,4}^* = y_{5,4} = 5/4$), that $y_{mN}^* < y_{mN}$ (e.g., $y_{7,6}^* = 23/18 < y_{7,6} = 35/27$), that $\kappa' = \kappa > 1$

(e.g., $y_{17,13}^* < y_{34,26}^* < y_{51,39}^* < y_{68,52}^* = 439/338 = y_{68,52} < y_{34,26} < y_{51,39} < y_{17,13}$), that $\kappa' < \kappa$ (e.g., $y_{11,9}^* < y_{22,18}^* < y_{33,27}^* < y_{44,36}^* = 415/324 = y_{88,72} < y_{77,63} < y_{66,54} < y_{55,45} < y_{44,36} < y_{33,27} = y_{22,18} = y_{11,9}$), and that $\kappa' > \kappa$ (e.g., $y_{23,20}^* < y_{46,40}^* < y_{69,60}^* = 193/150 = y_{46,40} < y_{23,20}$). The fifth of these examples also shows that $y_{\kappa m, \kappa N}$ need not be monotone in κ .

The computer output for P_{mN}^* contains not only y_{mN}^* but also the values y_k^* ($k = 1, \dots, 2N - 1$) corresponding to the maximizing solutions. These decimal fractions satisfy conditions (3) as in Lemma 1, within some unknown round-off error. We have replaced the decimal fractions with simple rational fractions to which the decimal fraction appeared to be reasonably accurate approximations. The new values of y_k^* , recorded in the case $N = 20$, $m = 26$ and alluded to in other cases in Section 3, were then independently verified to satisfy (3) without any reference to their computer origin. This verification can be carried out by arithmetical operations in the ring of integers if each y_k^* and the tests (3) are first multiplied by the least common denominator of the numbers y_k^* . No round-off errors need occur and the arithmetic can be programmed for our computer.

A Set of Useful Identities

$$\begin{aligned}
 64f(19/64) &= 8[2f(19/64) - f(-13/32) - f(1)] + 8[2f(19/64) \\
 &\quad - f(17/64) - f(21/64)] + 10[2f(19/64) - f(3/32) \\
 &\quad - f(1/2)] + 4[2f(19/64) - f(7/64) - f(31/64)] \\
 &\quad + 2[2f(19/64) - f(9/64) - f(29/64)] + 4[2f(-13/32) \\
 &\quad - f(-33/64) - f(-19/64)] + 4[2f(17/64) - f(1/32) \\
 &\quad - f(1/2)] + 4[2f(21/64) - f(-11/32) - f(1)] \\
 &\quad + 10[f(-3/32) + f(3/32) - 2f(0)] + 4[f(-19/64) \\
 &\quad + f(7/64) - 2f(-3/32)] + 2[2f(31/64) - f(-1/32) \\
 &\quad - f(1)] + 2[f(-9/64) + f(9/64) - 2f(0)] \\
 &\quad + [2f(29/64) - f(-3/32) - f(1)] + 2[2f(-33/64) \\
 &\quad - f(-1) - f(-1/32)] + 4[f(-1/32) + f(1/32) \\
 &\quad - 2f(0)] + 2[2f(-11/32) - f(-35/64) - f(-9/64)] \\
 &\quad + [2f(-35/64) - f(-1) - f(-3/32)] + 3f(-1) \\
 &\quad + 32f(0) + 14f(1/2) + 15f(1),
 \end{aligned}$$

$$\begin{aligned}
 76f(23/76) &= 20[2f(23/76) - f(21/76) - f(25/76)] + 8[2f(23/76) \\
 &\quad - f(3/19) - f(17/38)] + 4[2f(23/76) - f(2/19) \\
 &\quad - f(1/2)] + 6[2f(23/76) - f(-15/38) - f(1)] \\
 &\quad + 10[2f(21/76) - f(1/19) - f(1/2)] + 10[2f(25/76) \\
 &\quad - f(-13/38) - f(1)] + 3[f(-5/19) + f(3/19) \\
 &\quad - 2f(-1/19)] + 5[f(-3/19) + f(3/19) - 2f(0)] \\
 &\quad + 4[2f(17/38) - f(-2/19) - f(1)] + 4[f(-2/19) \\
 &\quad + f(2/19) - 2f(0)] + 3[2f(-15/38) - f(-10/19)
 \end{aligned}$$

$$\begin{aligned}
& - f(-5/19)] + 10[f(-1/19) + f(1/19) - 2f(0)] \\
& + 5[2f(-13/38) - f(-10/19) - f(-3/19)] + 4[2f(-10/19) \\
& - f(-1) - f(-1/19)] + 4f(-1) + 38f(0) \\
& + 14f(1/2) + 20f(1), \\
126f(25/84) = & 16[2f(25/84) - f(11/42) - f(1/3)] + 16[2f(25/84) \\
& - f(23/84) - f(9/28)] + 16[2f(25/84) - f(2/7) \\
& - f(13/42)] + 5[2f(25/84) - f(2/21) - f(1/2)] \\
& + 4[2f(25/84) - f(5/42) - f(10/21)] + 4[2f(25/84) \\
& - f(11/84) - f(13/28)] + 2[2f(25/84) - f(1/7) \\
& - f(19/42)] + 8[2f(11/42) - f(1/42) - f(1/2)] \\
& + 8[2f(1/3) - f(19/84) - f(37/84)] + 8[2f(23/84) \\
& - f(1/21) - f(1/2)] + 8[2f(9/28) - f(-5/14) \\
& - f(1)] + 8[2f(2/7) - f(1/14) - f(1/2)] \\
& + 8[2f(13/42) - f(5/21) - f(8/21)] + 2[f(-5/21) \\
& + f(2/21) - 2f(-1/14)] + 3[f(-2/21) + f(2/21) \\
& - 2f(0)] + 4[f(-5/42) + f(5/42) - 2f(0)] \\
& + 2[2f(10/21) - f(-1/21) - f(1)] + 4[f(-5/28) \\
& + f(11/84) - 2f(-1/42)] + 2[2f(13/28) - f(-1/14) \\
& - f(1)] + 2[f(-5/21) + f(1/7) - 2f(-1/21)] \\
& + 3[2f(19/42) - f(-2/21) - f(1)] + 4[2f(19/84) \\
& - f(0) - f(19/42)] + 8[f(-1/42) + f(1/42) \\
& - 2f(0)] + 4[2f(37/84) - f(-5/42) - f(1)] \\
& + 8[f(-1/21) + f(1/21) - 2f(0)] + 4[2f(-5/14) \\
& - f(-15/28) - f(-5/28)] + 8[f(-1/14) + f(1/14) \\
& - 2f(0)] + 4[2f(5/21) - f(-11/21) - f(1)] \\
& + 4[2f(8/21) - f(-5/21) - f(1)] + 2[2f(-15/28) \\
& - f(-1) - f(-1/14)] + 2[2f(-11/21) - f(-1) \\
& - f(-1/21)] + 4f(-1) + 66f(0) + 29f(1/2) + 27f(1). \\
2f(1/4) = & 2[f(1/4) - f(0) - f(1/2)] + f(0) + f(1/2), \\
6f(1/3) = & 2[2f(1/3) - f(-1/3) - f(1)] + 2f(1) \\
& + [2f(1/3) - f(1/6) - f(1/2)] + f(1/2) \\
& + [2f(-1/3) - f(-1/6) - f(-1/2)] + f(-1/2) \\
& + [f(-1/6) + f(1/6) - 2f(0)] + 2f(0), \\
5f(1/5) = & 2[2f(1/5) - f(-3/5) - f(1)] + 2f(1) \\
& + [f(-1/5) + f(1/5) - 2f(0)] + 2f(0) \\
& + [2f(-3/5) - f(-1) - f(-1/5)] + f(-1),
\end{aligned}$$

$$\begin{aligned}
9f(1/6) &= 4[2f(1/6) - f(-2/3) - f(1)] + 4f(1) \\
&\quad + [f(-1/6) + f(1/6) - 2f(0)] + 2f(0) \\
&\quad + 2[2f(-2/3) - f(-1) - f(-1/3)] + 2f(-1) \\
&\quad + [2f(-1/3) - f(-1/2) - f(-1/6)] + f(-1/2), \\
14f(2/7) &= 3[2f(2/7) - f(1/14) - f(1/2)] + 4[2f(2/7) \\
&\quad - f(-3/7) - f(1)] + [f(-3/14) - 2f(-1/14) \\
&\quad + f(1/14)] + 2[f(-1/14) - 2f(0) + f(1/14)] \\
&\quad + 2[2f(-3/7) - f(-1/2) - f(-5/14)] + [2f(-5/14) \\
&\quad - f(-1/2) - f(-3/14)] + 3f(1/2) + 4f(1) \\
&\quad + 4f(0) + 3f(-1/2), \\
36f(2/9) &= 10[2f(2/9) - f(0) - f(4/9)] + 4[2f(2/9) \\
&\quad - f(5/24) - f(17/72)] + 4[2f(2/9) - f(7/36) \\
&\quad - f(1/4)] + 5[2f(4/9) - f(-1/9) - f(1)] \\
&\quad + 2[2f(5/24) - f(1/9) - f(11/36)] + 2[2f(17/72) \\
&\quad - f(0) - f(17/36)] + 2[2f(7/36) - f(1/9) \\
&\quad - f(5/18)] + 5[f(-1/9) + f(1/9) - 2f(0)] \\
&\quad + [2f(11/36) - f(1/9) - f(1/2)] + [2f(17/36) \\
&\quad - f(-1/18) - f(1)] + [2f(5/18) - f(1/18) \\
&\quad - f(1/2)] + [f(-1/18) + f(1/18) - 2f(0)] \\
&\quad + 24f(0) + 4f(1/4) + 2f(1/2) + 6f(1), \\
264f(2/11) &= 28[2f(2/11) - f(-7/11) - f(1)] + 104[2f(2/11) \\
&\quad - f(5/33) - f(7/33)] + 4[2f(5/33) - f(1/33) \\
&\quad - f(3/11)] + 14[f(-3/11) - 2f(-2/33) + f(5/33)] \\
&\quad + 28[2f(5/33) - f(2/33) - f(8/33)] + 26[f(5/33) \\
&\quad - 2f(0) + f(-5/33)] + 52[2f(7/33) - f(-19/33) \\
&\quad - f(1)] + 14[2f(-7/11) - f(-1) - f(-3/11)] \\
&\quad + 2[2f(3/11) - f(1/33) - f(17/33)] + 7[f(-1/33) \\
&\quad - 2f(0) + f(1/33)] + [2f(17/33) - f(1/33) \\
&\quad - f(1)] + 28[f(-2/33) - 2f(0) + f(2/33)] \\
&\quad + 14[2f(8/33) - f(0) - f(16/33)] + 7[2f(16/33) \\
&\quad - f(-1/33) - f(1)] + 26[2f(-19/33) - f(-1) \\
&\quad - f(-5/33)] + 136f(0) + 88f(1) + 40f(-1), \\
10f(3/10) &= 4[2f(3/10) - f(-2/5) - f(1)] + 4f(1) \\
&\quad + [2f(3/10) - f(1/10) - f(1/2)] + f(1/2) \\
&\quad + 2[2f(-2/5) - f(-1/2) - f(-1/10)] + 3f(-1/2)
\end{aligned}$$

$$\begin{aligned}
& + [f(-1/10) - 2f(0) + f(1/10)] + 2f(0) \\
& + [2f(-3/10) - f(-1/2) - f(-1/10)], \\
24f(3/16) = & 8[2f(3/16) - f(1/6) - f(5/24)] + 2[2f(3/16) \\
& - f(-1/48) - f(19/48)] + 2[2f(3/16) - f(0) \\
& - f(3/8)] + 2[2f(1/6) - f(1/12) - f(1/4)] \\
& + [f(-1/4) + f(1/6) - 2f(-1/24)] + [f(-5/24) \\
& + f(1/6) - 2f(-1/48)] + 2[f(-1/6) + f(1/6) \\
& - 2f(0)] + 4[2f(5/24) - f(0) - f(5/12)] \\
& + 4[f(-1/48) + f(1/48) - 2f(0)] + [2f(19/48) \\
& - f(-5/24) - f(1)] + [2f(3/8) - f(-1/4) \\
& - f(1)] + 2[f(-1/24) + f(1/12) - 2f(1/48)] \\
& + 2[2f(5/12) - f(-1/6) - f(1)] + 18f(0) \\
& + 2f(1/4) + 4f(1), \\
60f(3/20) = & 14[2f(3/20) - f(1/20) - f(1/4)] + 12[2f(3/20) \\
& - f(1/8) - f(7/40)] + 4[2f(3/20) - f(1/10) \\
& - f(1/5)] + 14[f(-1/20) + f(1/20) - 2f(0)] \\
& + 6[2f(1/8) - f(-3/4) - f(1)] + 6[2f(7/40) \\
& - f(-1/20) - f(2/5)] + 4[f(-1/5) + f(1/10) \\
& - 2f(-1/20)] + 2[2f(1/5) - f(0) - f(2/5)] \\
& + 3[2f(-3/4) - f(-1) - f(-1/2)] + 4[2f(2/5) \\
& - f(-1/5) - f(1)] + 3f(-1) + 3f(-1/2) \\
& + 30f(0) + 14f(1/4) + 10f(1), \\
92f(4/23) = & 40[2f(4/23) - f(3/23) - f(5/23)] + 6[2f(4/23) \\
& - f(-1/23) - f(9/23)] + 12[2f(3/23) - f(1/23) \\
& - f(5/23)] + 13[f(-3/23) + f(3/23) - 2f(0)] \\
& + 3[f(-5/23) + f(3/23) - 2f(-1/23)] + 26[2f(5/23) \\
& - f(0) - f(10/23)] + 12[f(-1/23) + f(1/23) \\
& - 2f(0)] + 3[2f(9/23) - f(-5/23) - f(1)] \\
& + 13[2f(10/23) - f(-3/23) - f(1)] + 76f(0) + 16f(1), \\
68f(5/17) = & 14[2f(5/17) - f(3/34) - f(1/2)] + 8[2f(5/17) \\
& - f(-7/17) - f(1)] + 4[2f(5/17) - f(9/68) \\
& - f(31/68)] + 4[2f(5/17) - f(9/34) - f(11/34)] \\
& + 4[2f(5/17) - f(19/68) - f(21/68)] + 12[f(-3/34) \\
& + f(3/34) - 2f(0)] + [f(-7/34) + f(3/34) \\
& - 2f(-1/17)] + [f(-9/34) + f(3/34) - 2f(-3/34)]
\end{aligned}$$

$$\begin{aligned}
& + 4[2f(-7/17) - f(-35/68) - f(-21/68)] + 4[f(-21/68) \\
& + f(9/68) - 2f(-3/34)] + 2[2f(31/68) - f(-3/34) \\
& - f(1)] + 2[2f(9/34) - f(1/34) - f(1/2)] \\
& + 2[2f(11/34) - f(1/4) - f(27/68)] + 2[2f(19/68) \\
& - f(1/17) - f(1/2)] + 2[2f(21/68) - f(-13/34) \\
& - f(1)] + [2f(27/68) - f(-7/34) - f(1)] \\
& + 2[f(-1/17) + f(1/17) - 2f(0)] + [2f(-13/34) \\
& - f(-1/2) - f(-9/34)] + 2[2f(-35/68) - f(-1) \\
& - f(-1/34)] + 2[f(-1/34) + f(1/34) - 2f(0)] \\
& + 2f(-1) + f(-1/2) + 32f(0) + 2f(1/4) \\
& + 18f(1/2) + 13f(1).
\end{aligned}$$

$$\begin{aligned}
36f(5/18) = & 8[2f(5/18) - f(1/18) - f(1/2)] + 8[2f(5/18) \\
& - f(1/4) - f(11/36)] + 2[2f(5/18) - f(1/12) \\
& - f(17/36)] + [f(-1/6) + f(1/18) - 2f(-1/18)] \\
& + 7[f(-1/18) + f(1/18) - 2f(0)] + 4[2f(11/36) \\
& - f(-7/18) - f(1)] + 2[f(-7/36) + f(1/12) \\
& - 2f(-1/18)] + [2f(17/36) - f(-1/18) - f(1)] \\
& + [2f(-7/12) - f(-1) - f(-1/6)] + 2[2f(-7/18) \\
& - f(-7/12) - f(-7/36)] + f(-1) + 14f(0) \\
& + 8f(1/4) + 8f(1/2) + 5f(1),
\end{aligned}$$

$$\begin{aligned}
31f(5/31) = & 12[2f(5/31) - f(-21/31) - f(1)] + 2[2f(5/31) \\
& - f(3/31) - f(7/31)] + 3[f(-5/31) + f(5/31) \\
& - 2f(0)] + 6[2f(-21/31) - f(-1) - f(-11/31)] \\
& + 2[f(-3/31) + f(3/31) - 2f(0)] + [2f(7/31) \\
& - f(-17/31) - f(1)] + 3[2f(-11/31) - f(-17/31) \\
& - f(-5/31)] + 2[2f(-17/31) - f(-1) - f(-3/31)] \\
& + 8f(-1) + 10f(0) + 13f(1),
\end{aligned}$$

$$\begin{aligned}
48f(5/32) = & 8[2f(5/32) - f(1/16) - f(1/4)] + 16[2f(5/32) \\
& - f(1/8) - f(3/16)] + 8[f(-1/16) - 2f(0) \\
& + f(1/16)] + 4[f(-1/4) - 2f(-1/16) + f(1/8)] \\
& + 6[2f(1/8) - f(-3/4) - f(1)] + 8[2f(3/16) \\
& - f(0) - f(3/8)] + 4[2f(3/8) - f(-1/4) \\
& - f(1)] + 3[2f(-3/4) - f(-1) - f(-1/2)] \\
& + 8f(1/4) + 24f(0) + 10f(1) + 3f(-1) + 3f(-1/2)
\end{aligned}$$

$$\begin{aligned}
10f(7/20) = & [2f(7/20) - f(1/5) - f(1/2)] + 2[2f(7/20) \\
& - f(3/10) - f(2/5)] + 2[2f(7/20) - f(-3/10) \\
& - f(1)] + [f(-1/5) - 2f(0) + f(1/5)] \\
& + [2f(2/5) - f(-1/5) - f(1)] + [2f(3/10) \\
& - f(1/10) - f(1/2)] + [2f(-3/10) - f(-1/2) \\
& - f(-1/10)] + [f(-1/10) - 2f(0) + f(1/10)] \\
& + 2f(1/2) + 3f(1) + 4f(0) + f(-1/2)
\end{aligned}$$

$$\begin{aligned}
22f(7/22) = & 4[2f(7/22) - f(13/44) - f(15/44)] + 4[2f(7/22) \\
& - f(5/22) - f(9/22)] + 2[2f(7/22) - f(2/11) \\
& - f(5/11)] + [2f(7/22) - f(3/22) - f(1/2)] \\
& + 2[2f(13/44) - f(1/11) - f(1/2)] + 2[2f(15/44) \\
& - f(-7/22) - f(1)] + 2[2f(5/22) - f(-6/11) \\
& - f(1)] + 2[2f(9/22) - f(-2/11) - f(1)] \\
& + 2[f(-2/11) + f(2/11) - 2f(0)] + [2f(5/11) \\
& - f(-1/11) - f(1)] + [f(-3/22) + f(3/22) \\
& - 2f(0)] + 2[f(-1/11) + f(1/11) - 2f(0)] \\
& + [2f(-7/22) - f(-1/2) - f(-3/22)] + [2f(-6/11) \\
& - f(-1) - f(-1/11)] + f(-1) + f(-1/2) \\
& + 10f(0) + 3f(1/2) + 7f(1),
\end{aligned}$$

$$\begin{aligned}
248f(9/31) = & 40[2f(9/31) - f(-13/31) - f(1)] + 40[2f(9/31) \\
& - f(17/62) - f(19/62)] + 34[2f(9/31) - f(5/62) \\
& - f(1/2)] + 8[2f(9/31) - f(7/62) - f(29/62)] \\
& + 2[2f(9/31) - f(2/31) - f(16/31)] + 20[2f(-13/31) \\
& - f(-1/2) - f(-21/62)] + 20[2f(17/62) - f(3/62) \\
& - f(1/2)] + 16[2f(19/62) - f(-12/31) - f(1)] \\
& + 4[2f(19/62) - f(15/62) - f(23/62)] + 20[f(-5/62) \\
& + f(5/62) - 2f(0)] + 10[f(-11/62) + f(5/62) \\
& - 2f(-3/62)] + 2[2f(5/62) - f(2/31) - f(3/31)] \\
& + 8[f(-17/62) + f(7/62) - 2f(-5/62)] + 4[2f(29/62) \\
& - f(-2/31) - f(1)] + 4[f(-2/31) + f(2/31) \\
& - 2f(0)] + [2f(16/31) - f(1/31) - f(1)] \\
& + 10[2f(-21/62) - f(-1/2) - f(-11/62)] + 20[f(-3/62) \\
& + f(3/62) - 2f(0)] + 8[2f(-12/31) - f(-1/2) \\
& - f(-17/62)] + 2[2f(15/62) - f(0) - f(15/31)] \\
& + 2[2f(23/62) - f(-8/31) - f(1)] + 2[f(-8/31)
\end{aligned}$$

$$\begin{aligned}
& + f(3/31) - 2f(-5/62)] + [f(-1/31) + f(1/31) \\
& - 2f(0)] + [2f(15/31) - f(-1/31) - f(1)] \\
& + 38f(-1/2) + 92f(0) + 54f(1/2) + 64f(1), \\
700f(11/35) = & 144[2f(11/35) - f(-13/35) - f(1)] + 72[2f(11/35) \\
& - f(4/35) - f(18/35)] + 72[2f(11/35) - f(13/70) \\
& - f(31/70)] + 28[2f(11/35) - f(8/35) - f(2/5)] \\
& + 14[2f(11/35) - f(1/7) - f(17/35)] + 8[2f(11/35) \\
& - f(3/10) - f(23/70)] + 8[2f(11/35) - f(2/7) \\
& - f(12/35)] + 2[2f(11/35) - f(6/35) - f(16/35)] \\
& + 2[2f(11/35) - f(9/70) - f(1/2)] + 72[2f(-13/35) \\
& - f(-39/70) - f(-13/70)] + 72[f(-4/35) + f(4/35) \\
& - 2f(0)] + 36[2f(18/35) - f(1/35) - f(1)] \\
& + 72[f(-13/70) + f(13/70) - 2f(0)] + 36[2f(31/70) \\
& - f(-4/35) - f(1)] + 14[2f(8/35) - f(-19/35) \\
& - f(1)] + 14[f(-1/5) + f(1/7) - 2f(-1/35)] \\
& + 14[2f(2/5) - f(-1/5) - f(1)] + 8[2f(17/35) \\
& - f(-1/35) - f(1)] + 2[2f(17/70) - f(0) \\
& - f(17/35)] + 4[2f(3/10) - f(1/10) - f(1/2)] \\
& + 4[2f(23/70) - f(17/70) - f(29/70)] + 4[2f(2/7) \\
& - f(1/14) - f(1/2)] + 4[2f(12/35) - f(-11/35) \\
& - f(1)] + 2[f(-6/35) + f(6/35) - 2f(0)] \\
& + [2f(16/35) - f(-3/35) - f(1)] + 2[f(-9/70) \\
& + f(9/70) - 2f(0)] + 36[2f(-39/70) - f(-1) \\
& - f(-4/35)] + 36[f(-1/35) + f(1/35) - 2f(0)] \\
& + 7[2f(-19/35) - f(-1) - f(-3/35)] + 4[f(1/14) \\
& + f(1/10) - 2f(3/35)] + 8[f(-3/35) + f(3/35) \\
& - 2f(0)] + 2[2f(29/70) - f(-6/35) - f(1)] \\
& + 2[2f(-11/35) - f(-1/2) - f(-9/70)] + 43f(-1) \\
& + 2f(-1/2) + 386f(0) + 10f(1/2) + 259f(1), \\
36f(11/36) = & 12[2f(11/36) - f(5/18) - f(1/3)] + 4[2f(11/36) \\
& - f(5/36) - f(17/36)] + 2[2f(11/36) - f(-7/18) \\
& - f(1)] + 6[2f(5/18) - f(1/18) - f(1/2)] \\
& + 6[2f(1/3) - f(-1/3) - f(1)] + 3[f(-5/36) \\
& + f(5/36) - 2f(0)] + [f(-1/4) + f(5/36) \\
& - 2f(-1/18)] + 2[2f(17/36) - f(-1/18) - f(1)]
\end{aligned}$$

$$\begin{aligned}
& + [2f(-7/18) - f(-19/36) - f(-1/4)] + 6[f(-1/18) \\
& + f(1/18) - 2f(0)] + 3[2f(-1/3) - f(-19/36) \\
& - f(-5/36)] + 2[2f(-19/36) - f(-1) - f(-1/18)] \\
& + 2f(-1) + 18f(0) + 6f(1/2) + 10f(1),
\end{aligned}$$

$$\begin{aligned}
294f(13/42) = & 64[2f(13/42) - f(2/7) - f(1/3)] + 33[2f(13/42) \\
& - f(5/42) - f(1/2)] + 30[2f(13/42) - f(1/4) \\
& - f(31/84)] + 16[2f(13/42) - f(25/84) - f(9/28)] \\
& + 4[2f(13/42) - f(13/84) - f(13/28)] + 32[2f(2/7) \\
& - f(1/14) - f(1/2)] + 32[2f(1/3) - f(-1/3) \\
& - f(1)] + 18[f(-5/42) + f(5/42) - 2f(0)] \\
& + 15[f(-11/42) + f(5/42) - 2f(-1/14)] + 15[2f(31/84) \\
& - f(-11/42) - f(1)] + 8[2f(25/84) - f(2/21) \\
& - f(1/2)] + 8[2f(9/28) - f(-5/14) - f(1)] \\
& + 4[f(-13/84) + f(13/84) - 2f(0)] + 2[2f(13/28) \\
& - f(-1/14) - f(1)] + 32[f(-1/14) + f(1/14) \\
& - 2f(0)] + 16[2f(-1/3) - f(-23/42) - f(-5/42)] \\
& + 2[2f(-47/84) - f(-1) - f(-5/42)] + 8[f(-2/21) \\
& + f(2/21) - 2f(0)] + 4[2f(-5/14) - f(-47/84) \\
& - f(-13/84)] + 8[2f(-23/42) - f(-1) - f(-2/21)] \\
& + 10f(-1) + 124f(0) + 30f(1/4) + 73f(1/2) + 57f(1),
\end{aligned}$$

$$\begin{aligned}
64f(19/64) = & 8[2f(19/64) - f(-13/32) - f(1)] + 8[2f(19/64) \\
& - f(17/64) - f(21/64)] + 10[2f(19/64) - f(3/32) \\
& - f(1/2)] + 4[2f(19/64) - f(7/64) - f(31/64)] \\
& + 2[2f(19/64) - f(9/64) - f(29/64)] + 4[2f(-13/32) \\
& - f(-33/64) - f(-19/64)] + 4[2f(17/64) - f(1/32) \\
& - f(1/2)] + 4[2f(21/64) - f(-11/32) - f(1)] \\
& + 10[f(-3/32) + f(3/32) - 2f(0)] + 4[f(-19/64) \\
& + f(7/64) - 2f(-3/32)] + 2[2f(31/64) - f(-1/32) \\
& - f(1)] + 2[f(-9/64) + f(9/64) - 2f(0)] \\
& + [2f(29/64) - f(-3/32) - f(1)] + 2[2f(-33/64) \\
& - f(-1) - f(-1/32)] + 4[f(-1/32) + f(1/32) \\
& - 2f(0)] + 2[2f(-11/32) - f(-35/64) - f(-9/64)] \\
& + [2f(-35/64) - f(-1) - f(-3/32)] + 3f(-1) \\
& + 32f(0) + 14f(1/2) + 15f(1),
\end{aligned}$$

$$\begin{aligned}
76f(23/76) = & 20[2f(23/76) - f(21/76) - f(25/76)] + 8[2f(23/76) \\
& - f(3/19) - f(17/38)] + 4[2f(23/76) - f(2/19) \\
& - f(1/2)] + 6[2f(23/76) - f(-15/38) - f(1)] \\
& + 10[2f(21/76) - f(1/19) - f(1/2)] + 10[2f(25/76) \\
& - f(-13/38) - f(1)] + 3[f(-5/19) + f(3/19) \\
& - 2f(-1/19)] + 5[f(-3/19) + f(3/19) - 2f(0)] \\
& + 4[2f(17/38) - f(-2/19) - f(1)] + 4[f(-2/19) \\
& + f(2/19) - 2f(0)] + 3[2f(-15/38) - f(-10/19) \\
& - f(-5/19)] + 10[f(-1/19) + f(1/19) - 2f(0)] \\
& + 5[2f(-13/38) - f(-10/19) - f(-3/19)] + 4[2f(-10/19) \\
& - f(-1) - f(-1/19)] + 4f(-1) + 38f(0) \\
& + 14f(1/2) + 20f(1),
\end{aligned}$$

$$\begin{aligned}
126f(25/84) = & 16[2f(25/84) - f(11/42) - f(1/3)] + 16[2f(25/84) \\
& - f(23/84) - f(9/28)] + 16[2f(25/84) - f(2/7) \\
& - f(13/42)] + 5[2f(25/84) - f(2/21) - f(1/2)] \\
& + 4[2f(25/84) - f(5/42) - f(10/21)] + 4[2f(25/84) \\
& - f(11/84) - f(13/28)] + 2[2f(25/84) - f(1/7) \\
& - f(19/42)] + 8[2f(11/42) - f(1/42) - f(1/2)] \\
& + 8[2f(1/3) - f(19/84) - f(37/84)] + 8[2f(23/84) \\
& - f(1/21) - f(1/2)] + 8[2f(9/28) - f(-5/14) \\
& - f(1)] + 8[2f(2/7) - f(1/14) - f(1/2)] \\
& + 8[2f(13/42) - f(5/21) - f(8/21)] + 2[f(-5/21) \\
& + f(2/21) - 2f(-1/14)] + 3[f(-2/21) + f(2/21) \\
& - 2f(0)] + 4[f(-5/42) + f(5/42) - 2f(0)] \\
& + 2[2f(10/21) - f(-1/21) - f(1)] + 4[f(-5/28) \\
& + f(11/84) - 2f(-1/42)] + 2[2f(13/28) - f(-1/14) \\
& - f(1)] + 2[f(-5/21) + f(1/7) - 2f(-1/21)] \\
& + 3[2f(19/42) - f(-2/21) - f(1)] + 4[2f(19/84) \\
& - f(0) - f(19/42)] + 8[f(-1/42) + f(1/42) \\
& - 2f(0)] + 4[2f(37/84) - f(-5/42) - f(1)] \\
& + 8[f(-1/21) + f(1/21) - 2f(0)] + 4[2f(-5/14) \\
& - f(-15/28) - f(-5/28)] + 8[f(-1/14) + f(1/14) \\
& - 2f(0)] + 4[2f(5/21) - f(-11/21) - f(1)] \\
& + 4[2f(8/21) - f(-5/21) - f(1)] + 2[2f(-15/28) \\
& - f(-1) - f(-1/14)] + 2[2f(-11/21) - f(-1) \\
& - f(-1/21)] + 4f(-1) + 66f(0) + 29f(1/2) + 27f(1).
\end{aligned}$$

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