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Some Integrals Relating to the I_e-Function

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Abstract. Various integrals relating to the I_e-function

$$I_e(k,z)=\int_0^z e^{-x}I_0(kx)\,dx,$$

which finds a wide variety of applications in the fields of statistical communication theory and noise analysis, are evaluated in closed form.

1. Introduction. Rice [1] in his study of statistical properties of a sine wave influenced by Gaussian noise, presented a number of basic relations concerning the I_e-function. The above function yields the Nakagami-Hoyt probability distribution function [2], and hence it is often encountered in problems of statistical communication theory and noise analysis [3], [4], [5], [6]. In the present paper, we evaluate in closed form certain integrals involving the I_e-function, and also others reducible to it. Many of these results are believed to be new.

Short statements on derivations are presented in the Appendix. The notations for the special functions involved are consistent with those given in works by Erdelyi [7], [8] and Gradshteyn and Ryzhik [9]. Additional references [10], [11], [12], [13], [14] are also available for properties of integrals of Bessel functions.

2. Integrals Involving the I_e-Function.

$$(2.1)\quad \int_0^z e^{-px}I_e(k,x)\,dx=\frac{1}{p}\left\{\frac{1}{p+1}I_e\left(\frac{k}{p+1},[p+1]z\right)-e^{-pz}I_e(k,z)\right\}.$$

$$(2.2)\quad \int_0^z e^{-px}I_e(k,x)\,x\,dx$$

$$=\frac{1}{p}\left[\left\{\frac{1}{p(p+1)}+\frac{1}{(p+1)^2-k^2}\right\}\right.$$

$$\cdot I_e\left(\frac{k}{p+1},[p+1]z\right)-\left(z+\frac{1}{p}\right)e^{-pz}I_e(k,z)$$

$$\left.-\frac{z}{(p+1)^2-k^2}e^{-(p+1)z}\{kI_1(kz)+(p+1)I_0(kz)\}\right].$$

$$(2.3)\quad \int_0^z I_e(k,x)\,dx=\left(z-\frac{1}{1-k^2}\right)I_e(k,z)+\frac{ze^{-z}}{1-k^2}\{I_0(kz)+kI_1(kz)\}.$$

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(2.4) $$\int_0^z I_e(k,x)x\,dx = \frac{1}{2}\left[\left\{z^2 - \frac{2+k^2}{(1-k^2)^2}\right\}I_e(k,z) + \frac{ze^{-z}}{1-k^2}\cdot\left\{\left(z+\frac{2+k^2}{1-k^2}\right)I_0(kz) + k\left(z+\frac{3}{1-k^2}\right)I_1(kz)\right\}\right].$$

(2.5) $$\int_0^1 I_e(k,x)\,(1-x)^{\nu-1}\,dx = \frac{1}{\nu(\nu+1)}\Phi_2\left[\frac{1}{2},\frac{1}{2};\nu+2;-(1-k),-(1+k)\right], \qquad \nu>0.$$

(2.6) $$\int_0^1 I_e(k,x[1-x])\,x\,dx = \frac{1}{12}\Phi_2\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};-\frac{1-k}{4},-\frac{1+k}{4}\right).$$

(2.7) $$\int_0^\infty e^{-px}I_e(k,x)\,x^{\nu-2}\,dx = \frac{\Gamma(\nu)}{p^\nu}F_1\left(\nu;\frac{1}{2},\frac{1}{2};2;-\frac{1-k}{p},-\frac{1+k}{p}\right),$$

$p>k-1$ if $k\geqslant 1$, $p>0$ if $1>k>0$; $\nu>0$.

(2.8) $$\int_0^\infty e^{-px}I_e(k,x)\,dx = \frac{1}{p\sqrt{(p+1)^2-k^2}},$$

$p>k-1$ if $k\geqslant 1$, $p>0$ if $1>k>0$.

(2.9) $$\int_0^\infty e^{-px}I_e(k,x)x\,dx = \frac{(p+1)(2p+1)-k^2}{p^2\sqrt{\{(p+1)^2-k^2\}^3}},$$

$p>k-1$ if $k\geqslant 1$, $p>0$ if $1>k>0$.

(2.10) $$\int_0^\infty e^{-px}I_e(k,x)\,x^2\,dx = \frac{3p+2-k^2}{p^3\sqrt{\{(p+1)^2-k^2\}^3}} - \frac{3(p+1)[(p+1)(2p+1)-k^2]}{p^2\sqrt{\{(p+1)^2-k^2\}^5}},$$

$p>k-1$ if $k\geqslant 1$, $p>0$ if $1>k>0$.

(2.11) $$\int_0^\infty e^{-px}I_e(k,x)\,x^{-1}\,dx$$

$$= \frac{1}{\sqrt{k^2-1}}\left\{\sin^{-1}\left(\frac{1}{k}\right) - \sin^{-1}\left(\frac{p+1-k^2}{pk}\right)\right\}, \qquad k>1, p>k-1;$$

$$= \sqrt{\frac{p+2}{p}} - 1, \qquad k=1, p>0;$$

$$= \frac{1}{\sqrt{1-k^2}}\ln\left[\frac{p+\sqrt{1-k^2}\left\{\sqrt{(p+1)^2-k^2}+\sqrt{1-k^2}\right\}}{p\left(1+\sqrt{1-k^2}\right)}\right],$$

$1>k>0, p>0$.

(2.12) $$\int_0^\infty I_e(k,x)\,x^{-(\nu+1)}\,dx = \frac{\Gamma(1-\nu)}{\nu}\,{}_2F_1\left(\frac{1-\nu}{2},\frac{2-\nu}{2};1;k^2\right),$$
$$1>k>0,\ 1>\nu>0.$$

(2.13) $$\int_0^\infty I_e(k,x^2)\,x^{-2}\,dx = \kappa\sqrt{\frac{2}{\pi k}}\,K(\kappa),\quad \text{where } \kappa^2=\frac{2k}{1+k},\ 1>k>0.$$

(2.14) $$\int_0^\infty I_e(k,x^2)J_1(2ax)\,dx = \frac{1}{2a\sqrt{1-k^2}}\exp\left(-\frac{a^2}{1-k^2}\right)I_0\left(\frac{ka^2}{1-k^2}\right),\qquad 1>k>0.$$

(2.15) $$\int_0^\infty I_e(k,x)J_{\nu+1}(ax)\,x^{-\nu}\,dx = \frac{a^{\nu-1}}{2^\nu\Gamma(\nu+1)}F_4\left(\frac{1}{2},1;\nu+1,1;-a^2,k^2\right),$$
$$1>k>0,\ \nu>-\tfrac{1}{2}.$$

(2.16) $$\int_0^\infty I_e(k,x)J_1(ax)\,dx = \frac{2}{\pi}\sqrt{\frac{\kappa\kappa'}{a^3k}}\,K(\kappa),$$
$$\text{where } \kappa^2=\frac{1}{2}\left\{1-\frac{1-k^2+a^2}{\sqrt{(1-k^2+a^2)^2+4a^2k^2}}\right\},$$
$$\kappa'^2=1-k^2,\ 1>k>0.$$

(2.17) $$\int_0^\infty I_e(k,x)K_1(kx)\,dx = \frac{2}{k(1+2k)}K(\kappa),$$
$$\text{where } \kappa^2=\left(\frac{1-2k}{1+2k}\right)^2,\ 1>k>0.$$

3. Integrals Reducible to the I_e-Function.

(3.1) $$\int_0^z e^{-px}I_0(ax)\,dx = \frac{1}{p}I_e(a/p,\,pz).$$

(3.2) $$\int_0^z e^{-px}I_0(ax)\,x\,dx = \frac{1}{p^2-a^2}\left[I_e(a/p,\,pz) - ze^{-pz}\{aI_1(az)+pI_0(az)\}\right].$$

(3.3) $$\int_0^z e^{-px}I_0(ax)\,x^2\,dx = \frac{1}{p(p^2-a^2)^2}\Big[(2p^2+a^2)I_e(a/p,\,pz)$$
$$-pze^{-pz}\{p(p^2-a^2)z+2p^2+a^2\}I_0(az)$$
$$-apze^{-pz}\{(p^2-a^2)z+3p\}I_1(az)\Big].$$

(3.4) $$\int_0^z e^{-px}I_1(ax)\,dx = \frac{1}{a}\left[I_e(a/p,\,pz)+e^{-pz}I_0(az)-1\right].$$

$$(3.5)\quad \int_0^z e^{-px} I_1(ax)\, x\, dx$$
$$= \frac{1}{p(p^2-a^2)}\left[a I_e(a/p,\, pz) - pz e^{-pz}\{ p I_1(az) + a I_0(az)\}\right].$$

$$(3.6)\quad \int_0^z e^{-pz} I_2(ax)\, x\, dx$$
$$= \frac{1}{a^2(p^2-a^2)}\left[(3a^2 - 2p^2) I_e(a/p,\, pz) \right.$$
$$- a^3 z e^{-pz} I_1(az) - \{a^2 pz + 2(p^2-a^2)\} e^{-pz} I_0(az)$$
$$\left. + 2(p^2 - a^2)\right].$$

$$(3.7)\quad \int_0^z e^{-px^2} I_0(ax)\, x\, dx = \frac{1}{4p}\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$
$$- \frac{1}{4p}\exp(-pz^2) I_0(az),$$

where in (3.7)–(3.13),

$$k = \frac{4apz}{a^2 + 4p^2z^2}, \quad Z = pz^2 + \frac{a^2}{4p}, \qquad p > 0.$$

$$(3.8)\quad \int_0^z e^{-px^2} I_0(ax)\, x^3\, dx$$
$$= \frac{1}{4p^2}\left(1 + \frac{a^2}{4p}\right)\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$
$$- \frac{1}{4p^2}\exp(-pz^2)\left\{\left(1 + \frac{a^2}{4p} + 2pz^2\right) I_0(az) + az I_1(az)\right\}.$$

$$(3.9)\quad \int_0^z e^{-px^2} I_1(ax)\, dx = \frac{1}{2a}\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$
$$+ \frac{1}{2a}\exp(-pz^2) I_0(az) - \frac{1}{a}.$$

$$(3.10)\quad \int_0^z e^{-px^2} I_1(ax)\, x^2\, dx = \frac{a}{8p^2}\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$
$$- \frac{a}{8p^2}\exp(-pz^2)\left\{I_0(az) + 4\frac{pz}{a} I_1(az)\right\}.$$

$$(3.11)\quad \int_0^z e^{-px^2} I_2(ax)\, x\, dx = \left(\frac{1}{4p} - \frac{1}{a^2}\right)\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$
$$- \left(\frac{1}{4p} + \frac{1}{a^2}\right)\exp(-pz^2) I_0(az) + \frac{2}{a^2}.$$

(3.12) $$\int_0^z e^{-px^2} I_2(ax)\, x^3\, dx$$

$$= \frac{a^2}{16p^3}\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$

$$- \frac{a^2}{16p^3}\exp(-pz^2)\left\{\left(1 + 8\frac{p^2z^2}{a^2}\right) I_0(az) + 4\frac{pz}{a}\left(1 - 4\frac{p}{a^2}\right) I_1(az)\right\}.$$

(3.13) $$\int_0^z e^{-px^2} I_3(ax)\, x^2\, dx$$

$$= \frac{a}{8p^2}\left(1 - 8\frac{p}{a^2} + 32\frac{p^2}{a^4}\right)\exp\left(\frac{a^2}{4p}\right)\left\{1 - \frac{a^2 - 4p^2z^2}{a^2 + 4p^2z^2} I_e(k, Z)\right\}$$

$$- \frac{a}{8p^2}\exp(-pz^2)\left\{\left(1 - 8\frac{p}{a^2} - 32\frac{p^2}{a^4}\right) I_0(az) + 4\frac{pz}{a} I_1(az)\right\} - \frac{8}{a^3}.$$

(3.14) $$\int_0^\infty J_0(ax) \ln\left(1 + \frac{b^2}{x^2}\right)\cos(cx)\, dx = \frac{\pi}{c} I_e\left(\frac{a}{c}, bc\right), \qquad c > a > 0, b > 0.$$

(3.15) $$\int_0^\infty J_1(ax) \ln\left(1 + \frac{b^2}{x^2}\right)\sin(cx)\, dx$$

$$= \frac{\pi}{a}\left\{I_e\left(\frac{a}{c}, bc\right) + e^{-bc} I_0(ab) - 1\right\}, \qquad c > a > 0, b > 0.$$

(3.16) $$\int_0^\infty J_1(ax) \tan^{-1}(b/x)\cos(cx)\, dx$$

$$= \frac{\pi}{2a}\left\{1 - e^{-bc} I_0(ab) - I_e\left(\frac{a}{c}, bc\right)\right\}, \qquad c > a > 0, b > 0.$$

(3.17) $$\int_0^\infty J_0(ax) \tan^{-1}(b/x)\sin(cx)\, dx = \frac{\pi}{2c} I_e\left(\frac{a}{c}, bc\right), \qquad c > a > 0, b > 0.$$

(3.18) $$\int_0^\infty e^{-px^2} J_1(ax) J_0(bx)\, dx = \frac{1}{2a}\left\{1 - e^{-Z} I_0\left(\frac{ab}{2p}\right) + \frac{a^2 - b^2}{a^2 + b^2} I_e(k, Z)\right\},$$

where in (3.18)–(3.21),

$$k = \frac{2ab}{a^2 + b^2}, \quad Z = \frac{a^2 + b^2}{4p}, \qquad p > 0.$$

(3.19) $$\int_0^\infty e^{-px^2} J_1(ax) J_1(bx)\, x^{-1}\, dx$$

$$= \frac{1}{4}\left\{\frac{a}{b} + \frac{b}{a} + \left(\frac{b}{a} - \frac{a}{b}\right)\frac{a^2 - b^2}{a^2 + b^2} I_e(k, Z)\right\}$$

$$- \frac{1}{2} e^{-Z}\left\{\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) I_0\left(\frac{ab}{2p}\right) + I_1\left(\frac{ab}{2p}\right)\right\}.$$

(3.20) $$\int_0^\infty e^{-px^2} J_2(ax) J_0(bx)\, x\, dx$$

$$= \frac{1}{a^2}\left\{1 - \left(1 + \frac{a^2}{2p}\right) e^{-Z} I_0\left(\frac{ab}{2p}\right) + \frac{a^2 - b^2}{a^2 + b^2} I_e(k, Z)\right\}.$$

(3.21) $$\int_0^\infty e^{-px^2} J_2(ax)J_1(bx)\,dx$$
$$= \frac{b}{2a^2}\left[1 - e^{-Z}\left\{I_0\left(\frac{ab}{2p}\right) + 2\frac{a}{b}I_1\left(\frac{ab}{2p}\right)\right\} + \frac{a^2-b^2}{a^2+b^2}I_e(k,Z)\right].$$

(3.22) $$\int_0^\infty e^{-px^2} I_0(ax)\operatorname{erf}(bx)\,x\,dx$$
$$= \frac{b}{2p}\left[\frac{2\sqrt{p+b^2}}{p+2b^2}\exp\left(\frac{a^2}{4p}\right)I_e(k,Z)\right.$$
$$\left. + \frac{1}{\sqrt{p+b^2}}\exp\left\{\frac{a^2}{8(p+b^2)}\right\}I_0\left[\frac{a^2}{8(p+b^2)}\right]\right],$$

where in (3.22)–(3.24),

$$k = \frac{p}{p+2b^2}, \quad Z = \frac{a^2}{8p}\left(1 + \frac{b^2}{p+b^2}\right), \qquad p > 0.$$

(3.23) $$\int_0^\infty e^{-px^2} I_1(ax)\operatorname{erf}(bx)\,dx = \frac{2b\sqrt{p+b^2}}{a(p+2b^2)}\exp\left(\frac{a^2}{4p}\right)I_e(k,Z).$$

(3.24) $$\int_0^\infty e^{-px^2} I_2(ax)\operatorname{erf}(bx)\,x\,dx$$
$$= \frac{b}{p}\left[\frac{\sqrt{p+b^2}}{p+2b^2}\left(1 - 4\frac{p}{a^2}\right)\exp\left(\frac{a^2}{4p}\right)I_e(k,Z)\right.$$
$$\left. + \frac{1}{2\sqrt{p+b^2}}\exp\left\{\frac{a^2}{8(p+b^2)}\right\}I_0\left[\frac{a^2}{8(p+b^2)}\right]\right].$$

(3.25) $$\int_0^\infty e^{-px^2} I_0(ax^2)J_1(bx)\,dx = \frac{\sqrt{p^2-a^2}}{pb}I_e(k,Z),$$

where in (3.25)–(3.31),

$$k = \frac{a}{p}, \quad Z = \frac{pb^2}{4(p^2-a^2)}, \qquad p > a > 0.$$

(3.26) $$\int_0^\infty e^{-px^2} I_0(ax^2)J_2(bx)\,x\,dx$$
$$= \frac{2\sqrt{p^2-a^2}}{pb^2}I_e(k,Z) - \frac{1}{2\sqrt{p^2-a^2}}\exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\}I_0\left[\frac{ab^2}{4(p^2-a^2)}\right].$$

(3.27) $$\int_0^\infty e^{-px^2} I_1(ax^2) J_0(bx)\, x\, dx = \frac{1}{2a}\left[\frac{p}{\sqrt{p^2-a^2}}\exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\} I_0\left[\frac{ab^2}{4(p^2-a^2)}\right] + \frac{\sqrt{p^2-a^2}}{p} I_e(k, Z) - 1\right].$$

(3.28) $$\int_0^\infty e^{-px^2} I_1(ax^2) J_1(bx)\, dx = \frac{b}{4a}\left[\frac{1}{\sqrt{p^2-a^2}}\exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\} \cdot \left\{pI_0\left[\frac{ab^2}{4(p^2-a^2)}\right] + aI_1\left[\frac{ab^2}{4(p^2-a^2)}\right]\right\} + \frac{\sqrt{p^2-a^2}}{p} I_e(k, Z) - 1\right].$$

(3.29) $$\int_0^\infty e^{-px^2} I_1(ax^2) J_2(bx) x^{-1}\, dx = \frac{b^2}{16a}\left[\frac{1}{\sqrt{p^2-a^2}}\exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\} \cdot \left\{\left(p - 4\frac{a^2}{b^2}\right) I_0\left[\frac{ab^2}{4(p^2-a^2)}\right] + a\left(1 - 4\frac{p}{b^2}\right) I_1\left[\frac{ab^2}{4(p^2-a^2)}\right]\right\} + \frac{\sqrt{p^2-a^2}}{p}\left(1 + 16\frac{a^2}{b^4}\right) I_e(k, Z) - 1\right].$$

(3.30) $$\int_0^\infty e^{-px^2} I_1(ax^2) J_3(bx)\, dx = 4\frac{a}{b^3}\left[\frac{\sqrt{p^2-a^2}}{p} I_e(k, Z) - \frac{b^2}{4a\sqrt{p^2-a^2}}\exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\} \cdot \left\{pI_1\left[\frac{ab^2}{4(p^2-a^2)}\right] + aI_0\left[\frac{ab^2}{4(p^2-a^2)}\right]\right\}\right].$$

$$(3.31)\quad \int_0^\infty e^{-px^2} I_2(ax^2) J_0(bx)\, x^3\, dx$$

$$= \frac{1}{8a^2\sqrt{(p^2-a^2)^5}} \exp\left\{-\frac{pb^2}{4(p^2-a^2)}\right\}$$

$$\cdot \left[\left\{4a^2p(p^2-a^2) - a^2b^2(p^2+a^2) - 8p(p^2-a^2)^2\right\} \right.$$

$$\left. \cdot I_0\left[\frac{ab^2}{4(p^2-a^2)}\right] + 2a^3b^2pI_1\left[\frac{ab^2}{4(p^2-a^2)}\right]\right]$$

$$+ \frac{1}{a^2}\left\{1 - \frac{\sqrt{p^2-a^2}}{p} I_e(k, Z)\right\}.$$

Appendix—Derivations. In this appendix derivations of the above results are summarized:

(2.1) Integrate by parts, with $u = I_e(k, x)$ and $dv = e^{-px}\, dx$.

(2.2) Differentiate (2.1) with respect to p.

(2.3), (2.4) Integrate by parts. Use (3.2) and (3.3), respectively.

(2.5), (2.6) Give series expansions for $I_e(k, x)$ in powers of x. The results follow by use of the Beta integral. Refer to [7, art. 5.7.1 (21)]; [8, appendices]; or [9, art. 9.261 (2)] for definition of Φ_2.

(2.7) Use the series expansions in powers of x. Refer to [7, art. 5.7.1 (6)]; [8, appendices]; or [9, art. 9.180 (1)] for definition of F_1.

(2.8) This is a limiting case of (2.1) as $z \to \infty$.

(2.9)–(2.11) Differentiate and integrate (2.8), respectively, with respect to p.

(2.12) Integrate by parts, with $u = I_e(k, x)$ and $dv = x^{-(\nu+1)}\, dx$.

(2.13) Let $\nu = \frac{1}{2}$ in (2.12). Refer to [15, art. 1.13 (13.5)].

(2.14) Integrate by parts, with $u = I_e(k, x^2)$ and $dv = J_1(ax)\, dx$. Employ [16, (71)].

(2.15) Integrate by parts, with $u = I_e(k, x)$ and $dv = J_{\nu+1}(ax)\, x^{-\nu}\, dx$. Employ [8, art. 4.16 (13)] after replacing α by $i\alpha$. Refer to [7, art. 5.7.1 (9)]; [8, appendices]; or [9, art. 9.180 (4)] for definition of F_4.

(2.16) This integral is a special case of (2.15) with $\nu = 0$, in which [18, art. 4.1 (1)] is useful.

(2.17) Integrate by parts, with $u = I_e(k, x)$ and $dv = K_1(kx)\, dx$. Employ [17, art. 4.3 (1)].

(3.1) Substitute $x = pt$ into the integral representation of the I_e-function in the Abstract and reidentify k as a/p.

(3.2), (3.3) Differentiate (3.1) with respect to p.

(3.4) Integrate by parts, with $u = e^{-px}$ and $dv = I_1(ax)\, dx$.

(3.5) Differentiate (3.4) with respect to p.

(3.6) Combine (3.2) and (3.4) via [9, art. 8.486 (1)].

(3.7) Use the relation between the Marcum's Q-function and the I_e-function. Refer to [16, (25)].

(3.8) Differentiate (3.7) with respect to p.

(3.9) Integrate by parts, with $u = e^{-px^2}$ and $dv = I_1(ax)\,dx$.

(3.10) Integrate by parts, with $u = xI_1(ax)$ and $dv = xe^{-px^2}\,dx$.

(3.11) Combine (3.7) and (3.9) via [9, art. 8.486 (1)].

(3.12) Combine (3.8) and (3.10) via [9, art. 8.486 (1)].

(3.13) Combine (3.10) and (3.11) via [9, art. 8.486 (1)].

(3.14) Multiply [9, art. 6.695 (2)] by β, and then integrate with respect to β.

(3.15) Employ [9, art. 6.718 (3)] with $\nu = 1$. Multiply both sides by β, and then integrate with respect to β.

(3.16) Employ [9, art. 6.718 (4)] with $\nu = 1$. Multiply both sides by β^2, and then integrate with respect to β.

(3.17) Integrate [9, art. 6.695 (3)] with respect to β.

(3.18) Combine [16, (21) and (25)].

(3.19) Combine (3.18) and (3.21) via [9, art. 8.486 (1)].

(3.20) Employ [9, art. 6.633 (2)] with $p = 0$, and then combine this result and (3.18) via [9, art. 8.486 (1)].

(3.21) Use a procedure similar to that for (3.20). Here let $p = 2$ and then combine it with (3.20).

(3.22) Integrate by parts, with $u = \text{erf}(bx)$ and $dv = e^{-px^2}I_0(ax)\,x\,dx$. Employ [16, (10), (62) and (25)].

(3.23) Integrate by parts, with $u = \text{erf}(bx)$ and $dv = e^{-px^2}I_1(ax)\,dx$. Employ [16, (14), (63) and (25)].

(3.24) Combine (3.22) and (3.23) via [9, art. 8.486 (1)].

(3.25) Multiply [16, (71)] by b, and then integrate with respect to b.

(3.26) Combine (3.25) and [16, (71)] via [9, art. 8.473 (1)].

(3.27) Combine [16, (72) and (25)].

(3.28) Combine [16, (80) and (25)].

(3.29) Combine (3.28) and (3.30) via [7, art. 7.2.8 (56)].

(3.30) Employ [9, art. 6.651 (6)] after letting $\nu = 2$ and replacing β by $i\beta$. Multiply this equation by γ^3, and then integrate it with respect to γ.

(3.31) Differentiate [16, (71)] with respect to q. Combine this result and (3.27) via [9, art. 8.486 (1)].

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